Self-similarity in kinetic models of information-exchange processes

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Outline

1. Revisit models for ***"percolation" of information*** *of common interests through a large multi agent environment and reveal information to each other over time.*
2. (according to *Duffie, Giraux, Malamud and Manso, 2007, 2009, 2010 Market information percolation models*)
   - ***Model***: Time evolution of the cross-sectional distribution of posterior beliefs of the various agents $:= \mu_\tau(\theta)$ (i.e. acquired knowledge).
   - ***Phase space:*** Baysian rules: *type* of informative signals $:= \theta$
   - ***Interactions:*** pre to posterior *types* of m-multi agents signals:= aggregation of *types*
   - ***Results:*** Explicit solutions, convergence and convergence rates by means of Wild sums (formula) representations.

2. Connections to the **Boltzmann equation**: **dynamics of Kac Master equation for m-particle interactions and Maxwell type of interactions**. (Bobylev, Cercignani, I.M.G. 2006, CMP 2010, F. Bolley, I.M.G and with R. Srinivasan, in progress)
   - Phase space interactions beyond aggregation: examples-Wild sums representations
   - existence, stability and self similarity as attracting states –stable laws –martingales limit theorems.

3. **Generalizations to FMIE (Finite Markov Information Exchange processes)** type of interactions

Part I : Revisit Information aggregation model

(Duffie and Manso 07, Duffie, Giroux, Manso, Giroux 09, Duffie Malamud and Manso, 2010)

- A random variable X measuring **potential concern to all agents** has 2 possible outcomes,

H ("high") with probability ν , and L ("low") $1-\nu$.

- Each agent is initially endowed with a sequence of signals $\{s_1, \ldots, s_n\}$ that may be informative about X.

- The signals $\{s_1, \ldots, s_n\}$ observed by a particular agent are, conditional on X, independent with outcomes 0 and 1 (Bernoulli trials).

- w.l.g assume $P(s_i = 1|H) > P(s_i = 1|L)$.

Definition: A signal **' i ' is informative** if

$$P(s_i = 1|H) > P(s_i = 1|L).$$

Basic probability by **Bayes' rule**: the logarithm of the likelihood ratio between states H and L conditional on signals $\{s_1, \ldots, s_n\}$

$$\log \frac{\mathrm{P}(X = H \mid s_1, \ldots, s_n)}{\mathrm{P}(X = L \mid s_1, \ldots, s_n)} = \log \frac{\nu}{1-\nu} + \theta,$$

with

$$\theta = \sum_{i=1}^{n} \log \frac{\mathrm{P}(s_i \mid H)}{\mathrm{P}(s_i \mid L)}.$$

"type" θ of the set of signals

$\theta \in \mathbb{R}$

1. The higher the type θ of the set of signals, the higher is the likelihood ratio between states H and L and the higher the posterior probability that X is high. (well defined phase space condition)

2. Any particular agent is matched to other agents at each of a sequence of Poisson arrival times with a mean arrival rate λ ,(intensity interaction frequency) ***which is assumed the same across agents***. *(this condition may be relaxed to time moments dependency)*

3. At each meeting time, m−1 other agents **are randomly selected** from the population of agents

Interaction law :

Aggregation model

*The meeting group size **m is a parameter of the information** model that varies*

• **Binary (m=2):** for almost every pair of agents, the matching times and counterparties of one agent are independent of those of the other:

whenever an agent of **type θ meets an agent with type ϕ** and they communicate to each other their posterior distributions of X, **they both attain the posterior type $\theta+\phi$**

• **m-ary :** whenever **m** agents of respective types $\theta_1, \ldots, \theta_m$ share their beliefs, they attain the common posterior type $\theta_1 + \cdots + \theta_m$

Equivalently, from the phase space definition

$$\theta_j = \log \prod_{ij}^{nj} \frac{P(s_{j,i} \mid H)}{P(s_{j,i} \mid L)}$$

it follows that

$$\theta_1 + \cdots + \theta_m = \log \left(\prod_i^{n1} \frac{P(s_{1,i} \mid H)}{P(s_{1,i} \mid L)} \cdots \prod_k^{qm} \frac{P(s_{m,k} \mid H)}{P(s_{m,k} \mid L)} \right)$$

Statistical equation: *(Duffie, Manso 07, Duffie, Giraux, Manso 2009, Duffie Malamud Manso 10)*

$\mu_t(\theta)$ denotes the cross-sectional distribution of **posterior types** in the population **at t:**

- The initial distribution μ_0 of types induced by an initial allocation of signals to agents.
- Assume that there is a positive mass of agents that has at least one informative signal.
- The first moment $m_1(\mu_0(\theta)) > 0$ if $X = H$, and $m_1(\mu_0(\theta)) < 0$ if $X = L$.
- Assume that the initial law μ_0 **has a moment generating function** $\varphi(k)$, finite on a neighborhood of $k = 0$, defined by

$$\varphi(k) = \int e^{ik\theta} d(\mu_0(\theta))$$

- We also define $f(t, \cdot)$ be the probability distribution of $\mu_t(.)$ that is

$$\mu_t(\theta) = \int_0^t f_\tau(\tau,\theta)\, d\tau \qquad \text{temporal cumulative of } \mathbf{f}$$

Then the model stipulates that **the rate density $f(t,\theta)$ of agents of type θ:**

• ***is reduced*** at the rate λ $f(t,\theta)$ at which agents of type θ meet other agents and change type,

and

• ***is increased*** at the **aggregate rate** $\lambda\int f(t,\theta-y)f(t,y)\,dy$ at which an agent of some type y meets an agent of type $\theta-y$, and therefore becomes an agent of type θ.

Or equivalently, one obtains an *associated to* ***Forward Kolmogorov equation*** *for birth-death rates of aggregation type*

Aggregation model **(Smolukowski type eq.)**

$$\partial_t f(t,\theta) = \lambda \left(\int f(t,(\theta - y)\, f(t,y)\, dy \quad - \quad f(t,\theta) \right)$$

$$\text{with} \quad \int f(t,\theta)\, d\theta \quad = 1$$

aggregation model

$$\text{for } \partial_t f = f_t\,; \qquad f_t(t,\theta) = \lambda\left[\int f(t,(\theta - y)\, f(t,y)\, dy \;-\; f(t,\theta)\right], \qquad \text{with} \quad \int f(t,\theta)\, d\theta = 1$$

Equivalently, the evolution equation in integral form is (time cumulative)

$$\mu(t,\theta) = \mu_0(\theta) - \lambda \int_0^t (\mu * \mu - \mu)(s,\theta)\, ds$$

Binary agent model

or

$$\mu(t,\theta) = \mu_0(\theta) - \lambda \int_0^t (\mu^{*m} - \mu)(s,\theta)\, ds$$

"m-ary" Multi-agent

Existence by 'Wild sums' methods: explicit solution for the cross-sectional type distribution, in the form of a Wild summation:

1- The unique solution of the **binary model** is given by the well known sum

$$\mu(t,\theta) = \Sigma_{n\geq 1}\; e^{-\lambda t}\,(1 - e^{-\lambda t})^{n-1}\; \mu_o^{*n}\,,$$

where ρ^{*n} is the n-fold convolution of a measure ρ

In this summation, the term $\mathbf{e^{-t}\,(1-e^{-t})^{n-1}}$ associated with the **n-th** convolution of μ_0 represents the fraction of agents that has been involved in **(n − 1) direct or indirect meetings up to time t.**

Moreover, writing the evolution of the binary equation in terms of the Fourier transform $\varphi(\cdot, t)$ of μ_t, yields the local *ODE* $\varphi_t(s, t) = \lambda \varphi^2(s, t) - \lambda \varphi(s, t)$

with $\varphi(s, 0)$ ***positive, which has the explicit solution***

$$\varphi(s,t) = \frac{\varphi(s,0)}{e^{\lambda t}(1-\varphi(s,0)) + \varphi(s,0)}$$

that can be expanded as

$$\varphi_t(s,t) = \Sigma_{n \geq 1} \; e^{-\lambda t} (1 - e^{-\lambda t})^{n-1} \; \varphi^n(s,0)$$

which is identical to the Fourier transform of the right-hand side of Wild representation of the binary aggregation eq.

In the case of the m-ary interaction (*argument follows Duffie, Giraux,Manso '09*)

2 - The unique solution of the **m**-aggregation model is given by

$$\mu_t = \sum_{n \geq 1} a_{[(m-1)(n-1)+1]} e^{-\lambda t} (1 - e^{-(m-1)\lambda t})^{n-1} \mu_0^{*[(m-1)(n-1)+1]},$$

where $a_1 = 1$ and, for $n > 1$

$$a_{(m-1)(n-1)+1} = \frac{1}{m-1} \left(1 - \sum_{\left\{ \substack{i_1, \ldots, i_{(m-1)} < n \\ \sum i_k = n+m-2} \right\}} \prod_{k=1}^{m-1} a_{[(m-1)(i_k - 1)+1]} \right)$$

Cross-sectional distribution π_t of posterior probabilities that X = H is defined by the cumulative function of $\mu_t(s)$ with respect to s as follows:

$$\pi_t(0,b) = \mu_t\left(-\infty,\ \log\frac{b}{(1-b)} - \log\frac{\nu}{(1-\nu)}\right)$$

→the **beliefs distribution** π_t has an outcome that differs depending on whether X = H or X = L.

which converges to a common posterior distribution π_∞ **if, almost surely,**
π_t **converges in distribution to** π_∞ **,** **with unique exponential convergence rate $\lambda > 0$,**
such that for any b in (0, 1), there are constants κ_0 and κ_1 such that

$$\mathbf{e^{-\lambda t}\ \kappa_0 \leq |\pi_t(0, b) - \pi_\infty(0, b)| \leq e^{-\lambda t}\ \kappa_1.}$$

The proof (Duffie, Giraux and Manso'09) uses estimates of the Wild summation formula to estimate the cumulating function by the one of the initial state: $\mu_t(-\infty, a) \geq e^{-\lambda t}\mu_0(-\infty, a)$
to obtain uniform time estimates depending on the cumulative function of the initial state

$$\mu_0(-\infty,a)e^{-\lambda t} \leq \mu_t(-\infty,a) \leq \left(\beta + e^{ac}\frac{\gamma}{1-\gamma}\right)e^{-\lambda t},$$

where β and γ_0 depend on $\mu_0(-\infty, a)$ and $\boldsymbol{\nu}$, β
with $\boldsymbol{\nu}$ *=P(X=H)*

Analogous results are obtained for the solution of the m-ary interaction model.

- *However the authors did not analyzed any possible existence of dynamically scaled states such as* ***self-similarity that can produce additional stable laws and corresponding asymptotic limits → gives raise to stable laws with Pareto or more general power law tails***

Part II:

Connection between the kinetic Boltzmann equations and Kac probabilistic interpretation of statistical mechanics

Part II: Connection between the kinetic Boltzmann equations and Kac probabilistic interpretation of statistical mechanics *(Bobylev, Cercignani and IMG, arXiv.org'06, 09, CMP'09)*

I.1 Generalized interacting model of "Maxwell type":

Take a spatially homogeneous d-dimensional ($d \geq 2$) "rarefied gas of particles" with unit mass. Let *f(v, t)*, where $v \in R^d$ and $t \in R^+$, be a one-point **pdf** with the usual normalization

$$\int_{\mathbb{R}^d} f(v,t)\, dv = 1$$

Assumptions:

I – interaction (collision) frequency is independent of the phase-space variable (Maxwell-type)

II - the total "scattering cross section" (interaction frequency w.r.t. directions) is finite.

III- Choose such units of time such that the corresponding classical Boltzmann eq. reads as a birth-death rate equation for **pdf**s

$$f_t = Q_+(f) - f$$

with

$$\int_{\mathbb{R}^d} [Q_+(f)](v)\, dv = 1$$

$Q^+(f)$ is the gain term of the collision integral which Q^+ transforms f into another probability density

The same stochastic model admits other possible generalizations.

For example we can also include multiple interactions and interactions with a background (thermostat). This type of model will formally correspond to a version of the kinetic equation for some $\boldsymbol{Q_+(f)}$.

$$Q_+(f) = \alpha_1 Q_+^{(1)}(f) + \alpha_2 Q_+^{(2)}(f) + \cdots + \alpha_M Q_+^{(M)}(f)$$

where $Q^{(j)}_+$, $j = 1, \ldots, M$, are j-linear positive operators describing interactions of $j \geq 1$ particles, and $\alpha_j \geq 0$ are relative probabilities of such interactions, where

$$\text{each} \quad \int_{\mathbb{R}^d} [Q_+^{(j)}(f)](v)\, dv = 1 \; ; \qquad \text{and that} \quad \sum_{j=1}^{M} \alpha_j = 1$$

Assumption: ***Temporal evolution of the system is invariant under scaling transformations in phase space:*** if S_t is the evolution operator for the given N-particle system such that

$$S_t\{v_1(0), \ldots, v_M(0)\} = \{v_1(t), \ldots, v_M(t)\}, \quad t \geq 0,$$

then

$$S_t\{\lambda v_1(0), \ldots, \lambda v_M(0)\} = \{\lambda v_1(t), \ldots, \lambda v_M(t)\} \qquad \text{for any constant } \lambda > 0$$

which leads to the property

$$Q_+^{(j)}(A_\lambda f) = A_\lambda Q_+^{(j)}(f), \quad A_\lambda f(v) = \lambda^d f(\lambda v), \quad \lambda > 0, \; (j = 1, 2, .., M)$$

Note that the transformation A_λ is consistent with the normalization of f with respect to v.

Note: this property on ***Q(j)+*** **is needed to make the consistent with the classical BTE for Maxwell-type interactions**

Assumption II: ***Temporal evolution of the system is invariant under scaling transformations of phase space:*** Makes the use of the Fourier Transform a natural tool

$$\hat f(k,t) = \mathcal{F}(f) = \int_{\mathbb{R}^d} f(v,t) e^{-ik\cdot v}\, dv \;, \qquad k \in \mathbb{R}^d \,.$$

so the evolution eq. is transformed into an ***evolution eq. for characteristic functions***

$$\hat f_t = \hat Q_+(\hat f) - \hat f \;, \qquad \hat Q_+(\hat f) = \sum_{j=1}^{M} \alpha_j \hat Q_+^{(j)}(\hat f) \;,$$

which is also invariant under scaling transformations $\boldsymbol{k \to \lambda\, k}$, $\boldsymbol{k \in R^d}$

If solutions are isotropic $\hat f(k,t) = u(|k|^2, t)$ then

$$\hat Q_+^{(j)}(u) = \int_{-\infty}^{\infty} da_1 \dots \int_{-\infty}^{\infty} da_j Q_j(a_1,\dots,a_j) \prod_{i=1}^{j} u(a_i x)$$

pointwise in x

where $\boldsymbol{Q_j(a_1, \dots, a_j)}$ can be a mass distribution function of j-non-negative variables a_j **(interaction laws and kernels).**

or equivalently, $\hat Q_+^{(j)}(u) = \mathbf{E}[u(a_1 x), \dots, u(a_j x)]$ w.r.t. the density $Q_j(a_1,..a_j)$

All these considerations remain valid for ***d = 1,*** *the only two differences are:*
The evolving Boltzmann Eq should be considered as the one-dimensional Kac master equation *and one uses the* ***Laplace transform***

$$u(x,t) = \int_0^{\infty} f(v,t) e^{-xv}\, dv \;, \qquad x \geq 0$$

Connection of the Kac Master approach to the Boltzmann equation

The structure of this eq. follows from the well-known probabilistic interpretation by M. Kac:

Consider stochastic dynamics of N particles with phase coordinates (velocities)

$V_N = \{v_i(t)\}$, $i = 1..N$, ***with each*** $v_i(t) \in \Omega^d$ ***and*** $\Omega = R$ ***or*** R_+

A simplified Kac rules of binary dynamics is: on each time-step **t = 2/N** , choose randomly a pair of integers $1 \le i < l \le N$ and perform a transformation $(v_i, v_l) \rightarrow (v'_i, v'_l)$ which corresponds to an interaction of two particles with ***'pre-collisional'*** **velocities** v_i **and** v_l.

Then introduce N-particle distribution function $F(V_N, t)$ and consider a weak form of the **Kac Master equation** ***(we have assumed that*** $V'_{Nj} = V'_{Nj}(V_{Nj}, U_{Nj} \cdot \sigma)$ ***for pairs*** $j = i,l$ ***with*** σ ***in a compact set) with*** $U_{Nj} = V_{ni} - V_{Nl}$

$$\frac{d}{dt}\int_{\Omega^{dN}} F(V_N,t)\Phi(V_N)\,dV_N = \frac{N}{2}\int_{\Omega^{dN} \times S^{d-1}} F(V_N,t)\langle\Phi(V'_N) - \Phi(V_N)\; B(\underline{U}_{Nj}\cdot\sigma)\,dV_N\,d\sigma$$

$\underline{U}_{Nj} = U_{Nj}/|U_{Nj}|$

Introducing a one-particle distribution function (by setting $v_1 = v$) and the hierarchy reduction

$$f(v,t) = \int_\Omega \int_\Omega F(V_N,t)\, dv_2, \dots dv_N \ , \qquad \int_\Omega f(v,t)\, dv = 1$$

The assumed rules lead (formally, under additional assumptions) **to molecular chaos**, that is **"Stosszahlansatz"**

$$F(V_N,t) \approx \prod_{k=1}^{N} f(v_k,t) \ , \qquad N \to \infty$$

collision number hypothesis

(Duffie&Yin'07, Durret & Reminik'10 for multi-agent modeling)

The corresponding ***"weak formulation"*** *for* ***f(v,t)*** *for any test function* ***φ(v)*** *where the the* **RHS** *has a bilinear structure 'birth/death' process from evaluating* ***f(v,t) f(v*, t)*** →

$$\int_{\mathbb{R}^d} Q(f,f)\,\varphi\, dv = \frac{1}{2}\int_{\mathbb{R}^{2d}} \int_{S_+^{d-1}} f f_* \left(\varphi' + \varphi'_* - \varphi - \varphi_*\right) B(\underline{u}\cdot\sigma)\; d\sigma\, dv_*\, dv$$

M. Kac showed that ***yields the classical Boltzmann equation in weak form***

$\underline{u} = (v - v_*)/|v - v_*|$

where $B(u \cdot \sigma)$ is the interaction kernel: density of transition of state v → v'.

The angular integration corresponds to a 'mixing' of compactly supported positive measures

In Strong Form: Boltzmann equation for conservative or dissipative interactions

$$f_t + v\cdot\nabla_x f \;=\; C\, a^{d-1} G(x|\rho) \int_{\mathbb{R}^d} \int_{S_+^{d-1}} \Big[\frac{1}{{}'e\, {}'J_e}\, {}'f\, {}'f_* - f\, f_*\Big]\, K(\underline{u}\cdot\eta)\, d\eta\, dv_*$$

A *general form* *statistical transport* : The space-homogenous BTE with external heating sources

Important examples from mathematical physics and social sciences:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q_{\beta,\gamma,d}(f)(x,v,t) + \mathcal{G}(f)(x,v,t)$$

where the interacting integral is written in **weak form** as

$$\int_{v\in\mathbb{R}^d} Q_{\beta,\gamma,d}(f)(\cdot,t)\phi dv = c_d \int_{v,v_*\in\mathbb{R}^{2d};\sigma\in S^{d-1}} f f_* (\phi(v') - \phi(v)) B_{\beta,\gamma,d}(|u|, \tfrac{u\cdot\sigma}{|u|}) d\sigma dv_* dv$$

$$v' = v + \frac{\beta}{2}(|u|\sigma - u), \qquad v'_* = v_* - \frac{\beta}{2}(|u|\sigma - u) \text{ interaction law}$$

The term $\mathcal{G}(f)(v,t)$ **models external heating sources:**

- background thermostat (linear collisions),
- thermal bath (diffusion)
- shear flow (friction),
- dynamically scaled long time limits (self-similar solutions).

$$u = v - v_* \text{ (relative velocity)}$$

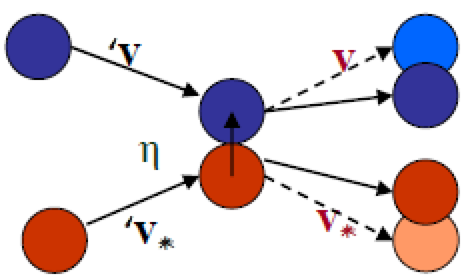
$$B_{\beta,\gamma,d}(|u|, \sigma(\theta)) \qquad \text{(collisional kernel)}$$

$$\cos\theta = \frac{(u\cdot\sigma)}{|u|} \text{ cosine of scattering angle,}$$

$$\beta = \frac{1+e}{2}, \qquad e = \text{restitution coefficient}$$

$$\beta = e = 1 \text{ elastic interaction}, \ \beta < 1 \text{ dissipative interaction}$$

$$J_\beta = \frac{\partial(v, v_*)}{\partial('v, 'v_*)} \text{ post-precollision Jacobian}$$



Inelastic Collision

u'= (1-β) u + β |u| σ , with **σ** the direction of elastic post-collisional relative velocity

Energy dissipation implies the appearance of **Non-Equilibrium Stationary Statistical States**

$$(\frac{\partial f}{\partial t}, \varphi)(t) = g(\rho,\theta)\left[\int_{\mathbb{R}^{2d}\times S^{d-1}} f\, f_*[\varphi(v') - \varphi(v)]\,|u|^\gamma\; b_{\gamma,d,\beta}(\tfrac{u\cdot\sigma}{|u|})d\sigma dv_* dv\right](t) + (\mathcal{G}(f),\varphi)(t)$$

NESS satisfies : $\int_{\mathbb{R}^d} f_\infty(v)\mathcal{M}_\gamma^{-1} dv$

β	γ	$\mathcal{G}(f)$	$\mathcal{M}_\gamma$ = NESS tail asymptotics
$\beta = 1$	$0 \le \gamma \le 1\ (VHP)$	0	$C\exp(-r\,\|v\|^2)$
$\frac{1}{2} \le \beta < 1$	$0 \le \gamma \le 1\ (VHP)$	$\Delta_v f$	$C\exp(-r\|v\|^{\frac{\gamma+2}{2}})$
$\frac{1}{2} \le \beta < 1$	$\gamma = 1\ (HS)$	$\Delta_v f + \tau\nabla\cdot(vf)$	$C\exp(-r\,\|v\|^2)$
$\frac{1}{2} \le \beta < 1$	$\gamma = 1\ (HS)$	$v_2\frac{\partial f}{\partial v_3}$	at least $C\exp(-r\,\|v\|^1)$
$\frac{1}{2} \le \beta < 1$	$0 < \gamma \le 1\ (VHP)$	$Q(f, M_{aT}) - \mu\ v\cdot\nabla f$ $a = 0$ or 1	$C((1-a)\exp(-r\,\|v\|^\gamma)+$ $+aC\exp(-r\,\|v\|^2)$
$\frac{1}{2} \le \beta \le 1$	$\gamma = 0$ (MM)	$\theta_b\, Q(f, M_{aT}) - \mu\, v\cdot\nabla f$ $a = 0$ or 1	$(1-a)C(c_1 + c_2\|v\|^{\mathrm{k}})^{-1}+$ $+aC\exp(-r\,\|v\|^2)$

for $C = C_{(\gamma,\beta,\theta,d)}$ and $r = r_{(\gamma,\beta,\theta,d)}$. Also C, c_1, c_2 and k in the last case depend on $\beta, \theta, \theta_b, T, d$

Rigourously worked in Bobylev, Carrillo &IMG, JSP'00, Bobylev, Cercignani &Toscani JSP'02,03 IMG, Panferov&Villani, CMP'04, Bobylev, Cercignani &IMG 06 and CMP'10, Bassetti and Ladelli AP'11 And with Toscani,JSP'11, among many more references in the last decade.

The approach extends to more general Information Percolation models where the signal type do not necessarily aggregate but "distributes" itself between the posterior types as in the framework of Finite Markov Information-Exchange (FMIE) processes *popularized recently by D. Aldous (Berkeley, lecture notes, 2011):*

Let $\Theta'_m = \mathbf{G}\,\Theta_m$; $\Theta_m = (\theta_1; \ldots; \theta_m)$; $\Theta'_m = (\theta'_1, \ldots ; \theta'_m)$; where $G = g_{ij}$ is a square $m \times m$ matrix of randomly distributed numbers independent of the numeration of identical agent types

- **Binary (m=2):** for almost every pair of agents, the matching times and counterparties of one agent are independent of those of the other:

whenever an agent of **type θ meets an agent with type ϕ** and they communicate to each other their posterior distributions of X,

θ' and ϕ' attain the posterior types $\theta' = g_{11}\theta + g_{12}\,\phi$ **and** $\phi' = g_{21}\,\theta + g_{22}\,\phi$

- **m-ary :** whenever **m** agents of respective types $\theta_1, \ldots, \theta_m$ share their beliefs, they attain the corresponding posterior type $\theta_i' = g_{i1}\,\theta_1 + \cdots + g_{im}\,\theta_m$

•Equivalently , from the phase space definition it follows that

For $\theta_i = \log\left(\prod_{ji}^{ni} \frac{P(s_{j,i}\,|\,H)}{P(s_{j,i}\,|\,L)}\right)$,

$$\theta_i' = \log\left(\prod_{jl}^{nl}\left(\frac{P(s_{1,ij}|H)}{P(s_{1,ij}|L)}\right)^{g_{i1}} \cdots \prod_{km}^{qm}\left(\frac{P(s_{m,ik}\,|\,H)}{P(s_{m,ik}\,|\,L)}\right)^{g_{im}}\right)$$

plus constrains from conserved properties (like the mean) that gives constitutive laws to the g_{ij}

Extention to m-ary interactions model the Kac Master Equation formulation

Let the type signals V_m *and its posterior* V'_{m}:

with $V'_m = \mathbf{G}\, V_m$; $V_m = (v_1; \ldots; v_m)$; $V'_m = (v'_1, \ldots ; v'_m)$; where

G is a square m x m matrix with entries $\mathbf{G} = \{\mathbf{g}_{ik} = 1 \text{ , for all } i, k = 1, \ldots, m\}$,

Then the m-particle distribution function F(V_N, t) and the weak form of the **Kac Master eq.**

for N=m

$$\frac{d}{dt}\int_{\mathbb{R}^N_+} F(V_N,t)\Phi(V_N)\,dV_N = \frac{N}{\lambda^{-1}}\int_{\mathbb{R}^N_+} F(V_N,t)\langle\Phi(V'_N)-\Phi(V_N)\rangle\; dV_N\,,$$

Introducing a one-particle distribution function (by setting $v_1 = v$) and the hierarchy reduction

$$f(v,t) = \int_0^\infty \int_0^\infty F(V_N,t)\; dv_2, \ldots dv_N\;, \qquad \int_0^\infty f(v,t)\,dv \;=1$$

The assumed rules lead (formally, under additional assumptions) **to molecular chaos**, that is

$$F(V_N,t) \approx \prod_{k=1}^{N} f(v_k,t)\;, \qquad N\to\infty$$

Then, an extension of the BTE for FMIE $f(V_m, t)$ holds for either binary or multi-agent interacting forms

Interacting models of Maxwell type (as originally studied for binary elastic or inelastic interactions)

$$f_t = Q^+(f,f)(t,v) - f(v) \qquad \int f\,dv = 1 = \int Q^+(f,f)\,dv$$

so $Q^+(f,f)(t,v)$ *is also a probability distribution function in v.*

Then: work in the space of "characteristic functions" associated to Probabilities: "positive probability measures in v-space are continuous bounded functions in Fourier transformed k-space"

The Fourier transformed problem: For $\varphi(t,k) = \mathcal{F}_{v\to k}[f(t,v)], \quad \varphi(t,0) = \int f_0 dv = 1, \ \forall t > 0$

$$\widehat{Q^+(f,g)} = \Gamma_\beta(\hat{f},\hat{g}) = \mathbf{E}_\beta[\boldsymbol{\varphi}(a_-(|k|,t)), \boldsymbol{\varphi}(a_+(|k|,t))]$$ --→ **Fourier transformed operator**

$$\frac{\partial \varphi}{\partial t} = \frac{1}{|S^{n-1}|}\int_{S^{n-1}} \{\varphi(t,k_-)\varphi(t,k_+) - \varphi(t,0)\varphi(t,k)\}\, b\left(\frac{k\cdot\sigma}{|k|}\right) d\sigma = \Gamma(\varphi,\varphi) - \varphi(t,k)$$

characterized by $k_- = \beta\frac{(k-|k|\sigma)}{2} = \beta|k|\frac{1}{2}\left(\frac{k}{|k|} - \sigma\right), \qquad k_+ = k - k_- \qquad \beta = \frac{1+e}{2}$

One may think of this model as the generalization original Kac ('59) probabilistic interpretation of rules of dynamics on each time step $\Delta t=2/N$ of N particles associated to system of vectors randomly interchanging velocities pairwise while preserving momentum and local energy, independently of their relative velocities.

Bobylev, '75-80, for the elastic, energy conservative case.

Drawing from Kac's models and Mc Kean work in the 60's : ***Connections to Probability*** *- Carlen, Carvalho, Gabetta, Toscani, 80-90's , Bassetti, Ladelli Regazzini '08 – '11 -* ***For inelastic interactions****: Bobylev,Carrillo, I.M.G. JSP'00, Bobylev, Cercignani,Toscani, 03, Bobylev, Cercignani, I.M.G'06 and 09, for general non-conservative problem.* ***For wealth distribution models****: A.Pulvirenti, Toscani, Bissi, Toscani, Spiga, Pareschi '06-11*

From Fourier transform: ***n^{th} moments of f(., v) are n^{th} derivatives of $\varphi(.,k)|_{k=0}$***

$$\varphi(t,0) = 1, \qquad \nabla_k \varphi(t,0) = 0 \ , \qquad \Theta(t) = -\tfrac{\mu}{d}\Delta_k \varphi(t,0)$$

For isotropic ($x = |k|^2/2$) or

self similar solutions by $x = |k|^2/2 \; e^{\mu t}$, ***μ is the energy dissipation rate, that is: $\Theta_t = -\mu\,\Theta$, and***

$$|k_-|^2 = \beta^2 s \frac{|k|^2}{2} \ , \qquad |k_+|^2 = \frac{|k|^2}{2}\,[1 - s\beta(2-\beta)] \qquad \text{with} \quad s = \frac{1}{2}\left(1 - \frac{k\cdot n}{|k|}\right)$$

the Fourier transformed collisional ***gain*** *operator is written*

$$\mathbf{E}_\beta[\varphi(a_-(x,t)),\, \varphi(a_+(x,t))] = c_d \int_0^1 \varphi(\beta^2 s\, x)\varphi((1-\beta(2-\beta)s)\,x)\, G(s)\, ds = c_d \int_0^1 \varphi(a_\beta(s)\,x)\varphi(b_\beta(s)\,x)\, G(s)\, ds$$

$$K_d = \frac{1}{2}\int_0^1 G(s)\, ds \qquad \text{and} \qquad 0 \le a_\beta(s), b_\beta(s) \le 1$$

accounts for the integrability of the function ***$b(1-2s)(s-s^2)^{(N-3)/2}$***

For isotropic solutions the equation becomes (after rescaling in time the dimensional constant)

$$\varphi_t + \varphi = \mathbf{E}_\beta[\varphi(a_-(x,t)),\, \varphi(a_+(x,t))] = \Gamma(\varphi, \varphi) \ ; \qquad \varphi(t,0)=1, \quad \varphi(0,k)=F(f_0)(k), \quad \Theta(t) = -\varphi'(0)$$

In this case, using the linearization of $\Gamma(\varphi,\varphi)$ about the stationary state $\varphi=1$, we can inferred the energy rate of change by looking at *$\gamma_{\beta,1}$ defined by*

$$\gamma_{\beta,1} := \int_0^1 (a_\beta(s) + b_\beta(s))\, G(s)\, ds \quad \begin{cases} < 1 & \text{kinetic energy is dissipated (inelastic)} \\ = 1 & \text{kinetic energy is conserved (elastic)} \\ > 1 & \text{kinetic energy is generated (aggregation)} \end{cases}$$

Classical Existence approach : Wild's sum in the Fourier representation.

The existence theorems for the classical elastic case ($\beta = e = 1$) of Maxwell type of interactions were proved by *Morgenstern, ,Wild 1950s, Bobylev 70s* ***and for inelastic ($\beta<1$)*** *by Bobylev,Carrillo, I.M.G.JSP'00* ***using the Fourier transform***

- rescale time $t \to \tau$ $\quad \tau = 1 - \exp(-t)$, $\quad \varphi(t,k) = \exp(-t)\Phi(\tau,k)$,

and solve the initial value problem

$$\frac{\partial \Phi}{\partial \tau} = \Gamma(\Phi, \Phi) = \mathbf{E}_\beta[\varphi(a_-(x,t)), \varphi(a_+(x,t)] \ , \qquad \Phi(k,0) = \varphi_0(k)$$

by a power series expansion in time where the phase-space dependence is in the coefficients

$$\Phi(\tau,k) = \sum_{n=0}^{\infty} \Phi_n(k)\tau^n$$

$$\Phi_0 = \varphi_0$$

$$\Phi_{n+1} = \frac{1}{n+1}\sum_{k=0}^{n} \Gamma(\Phi_k, \Phi_{n-k}) \ , \qquad n \geq 0$$

Wild's sum in the Fourier representation for **non conservative problem :** analog to binary trees dynamics representation by McKean 60s)

Note that if the initial coefficient $|\varphi_0| \leq 1$, then $|\Phi_n| \leq 1$ for any $n \geq 0$.

→ ***the series converges uniformly for $\tau \in [0; 1)$.***

Classical Examples from rarefied molecular states

Existence, asymptotic behavior - self-similar solutions and power like tails:

From a unified point of energy dissipative Maxwell type models: λ_1 energy dissipation rate (Bobylev, I.M.G.JSP'06, Bobylev,Cercignani,I.G. arXiv.org'06- CMP'09)

$$\frac{\partial\varphi}{\partial t} = \int_0^1 ds\, G(s)\,\{\varphi(a(s)x)\,\varphi[(b(s)x] - \varphi(x)\,\varphi(0)\} + \theta \int_0^1 ds\, H(s)\,\{\varphi[c(s)x] - \varphi(x)\} =$$

$$= I_{a,b,\lambda_1}(\varphi,\varphi) + \theta\, I_{c,1,\lambda_1}\;;$$

$\varphi_0(x) = 1 - x^p \quad p \le 1$ initial state ,

$G(s), H(s)$ non-negative, integrable on $[0,1]$; $0 \le a(s), b(s), c(s) \le 1,\ \ s \in [0,1]$.

- Classical **elastic** Maxwell gas with infinite initial energy:
  $a(s) = s,\ \ b(s) = 1 - s$, and $\boxed{\varphi_t = I_{a,b,0}(\varphi,\varphi)}$
- Gas of **inelastic** Maxwell particles with finite or infinite initial energy, with constant restitution coefficient $\beta = (1+\alpha)/2$:
  $a(s) = \beta^2 s,\ \ b(s) = 1 - \beta(2-\beta)s$ and $\boxed{\varphi_t = I_{a,b,\lambda_1}(\varphi,\varphi)}$
- Classical **elastic** Maxwell gas with finite or infinite energy in the presence of an equilibrium background gas of particles with mass M, density n_1 and temperature T_1,
  $a(s) = s;\ \ b(s) = 1 - s;\ \ c(s) = 1 - 4M/(1+M)^2 s < 1;$
  and $\boxed{\varphi_t = I_{a,b,0}(\varphi,\varphi) + \theta I_{c,1,\lambda_1}(\varphi, e^{T_1 x})}$ Energy non-conservative

Study of j-ary interactions for Maxwell type interacting models

Existence, uniqueness, stability

(Bobylev, Cercignani, I.M.G.;.arXig.org '06 – CMP '10)

Self-similar asymptotics and Power-like Tails

For $\phi(k,t) = \mathcal{F}_{v\to k}[f(v,t)]$, let $\boxed{\Gamma(\phi) = \mathcal{F}_{v\to k}[Q^+]}$ be the Fourier Transform of the contribution from the gain operator $Q^+(f,f)$ associated to a generalized BTE equation of Maxwell type.

In the case of isotropic solutions $f(|v|^2,t) \longrightarrow \phi(|k|^2,t) = u(x,t)$.

$\int f(v,t)|v|^2 dv = \Delta_k \phi(k,t)\,|_{k=0} = \Theta(t) = u_x(0,t)$ **is the kinetic energy** ***(or variance)***

The initial value problem:

For initial states $u(x,0) = u_o(x) = 1 + O(x^p) \in U$, $\|u_o\| = 1$, *with* $0 < p < 1$ *infinity energy, or* $p \geq 1$ *finite energy*

U the unit sphere in $(C_B(\Re^d), \|\cdot\|_\infty)$, take

$$\boxed{u_t + u = \Gamma(u) = \sum_{j=1}^{M} \alpha_j \Gamma^{(j)}(u)} \qquad \sum_{j=1}^{M} \alpha_j = 1\,,\ \alpha_j \geq 0\,,$$

$$\Gamma^{(j)}(u) = \int_0^\infty \dots \int_0^\infty A_j(a_1,\dots,a_j) \prod_{k=1}^{j} u(a_k x)\, da_1 \dots da_j, \quad j = 1,\dots,M\ .$$

$$A_j(a) = A_j(a_1,\dots,a_j) \geq 0\,, \qquad \int_0^\infty da_1 \dots \int_0^\infty da_j\ A(a_1,\dots,a_j) = 1\,,$$

where $\Gamma(0) = 0$ and $\Gamma(1) = 1$ **are trivial solutions**

Theorem: The Γ-operator satisfies three fundamental properties

Fundamental properties of the generalized model for m-ary interactions:

Theorem: The Γ-operator satisfies

- Preserves the unit sphere U in $(C_B(\Re^d), \|\cdot\|_\infty)$
- It has *L-Lipschitz condition*: there exists a linear bounded operator L from $(C_B(\Re^d), \|\cdot\|_\infty)$ into itself, such that, for $x = \frac{k^2}{2}$

$$|\Gamma(u_1) - \Gamma(u_2)|(x,t) \le L(|u_1 - u_2|(x,t)), \qquad \text{for} \|u_i\|_\infty \le 1; i = 1, 2.$$

- Invariance under dilations:
  $e^{\tau \mathbf{D}}\Gamma(u) = \Gamma(e^{\tau \mathbf{D}}u)$, $\mathbf{D} = x\frac{\partial}{\partial x}$, $e^{\tau \mathbf{D}}u(x) = u(xe^{\tau})$, $\tau \in \mathrm{R}^+$

- L-**Lipschitz** condition on the operator Γ is a point-wise condition $\Rightarrow$ classical Lipschitz condition on $\mathcal{B}$.
- $\Gamma(u)$ is L-Lipschitz, where L is the linearization of $\Gamma(u)(x,t) = \mathcal{F}_{v\to k}[Q(|v|,t)]$ about the state $u = 1$

- relation to the contractive property of the Wasserstein distance between two probabilities: ***for initial data with finite energy , i.e. p ≥ 1, :***
  - **For Maxwell type of interactions that conserve momentum the 2^{nd}-Wasserstein distance from $W_2(f(v,t), \delta_{\langle v\ f\rangle(t)}) = \int f(v,t)|v|^2$ is the kinetic energy.**
  - **The eigenvalue of L for $u = x$ is the energy dissipation rate $\mu(1)$ so $\Theta' = -\mu(1)\Theta \ \Rightarrow$ for bounded initial energy, long time asymptotics and decay rates in Fourier space yield the same qualitatively properties in W_2 metrics, since this metric is equivalent to the usual weak convergence of measures plus convergence of second moments.**

Relates to the work of Toscani, Gabetta,Wennberg, Villani,Carlen, Carvallo,…..

Existence-uniqueness and Stability *(for frequency =λ → rescale time by $\rho = \lambda t$)*

Uses the first two properties of the operator Γ)

Take the integral form of the equation

$$u(t) = u_0 e^{-t} + \int_0^t e^{-(t-\tau)} \Gamma[u(\tau)]\, d\tau$$

and apply the standard ***Picard iteration scheme***

$$u^{(n+1)}(t) = u_0 e^{-t} + \int_0^t e^{-(t-\tau)} \Gamma[u^{(n)}(\tau)]\, d\tau\,, \qquad u^{(0)} = u_0$$

Generalized Wild Sum

(for multi-linear operators)

Then, on any finite interval $0 \le \tau \le t$, set $|||u|||_t = \sup_{[0,t]} ||u||_\tau$ *, and initial* u_0 *in the unit ball* U

$$|| u^{(n+1)}(t)|| \le ||u_0|| e^{-\lambda t} + (1-e^{-\lambda t})||| \Gamma(u^{(n)}(t)|||_t \le \quad \Sigma_n\, e^{-\lambda t} (1- e^{-\lambda t})^{n-1} C^{n-1} || u_0||^n \text{ , with } C=||L||$$

so, for T such that $(1-e^{-\lambda T})C < 1$, then the estimate hold with uniform control in [0,T],

In addition,

1- $||u^{(n)}(t)|| \le 1$ for all n = 1, 2, . . . , and t ∈ [0, T], since $||u_0|| \le 1$ and $\Gamma(u)$ preserves the unit ball

2- $u(t) = \lim_{n\to\infty} u^{(n)}(t)$, $0 \le t \le T$, $T > 0$

Pointwise stability in the limit: *using the* ***L-Lipschitz condition any two solutions* u(t) *and* w(t)** *of the problem with initial data in the unit ball* U *satisfy the* ***pointwise in* x** *inequalities*

$$|u(t)-w(t)|(x) \le \exp\{t(L-1)\}(|u_0-w_0|)(x) \le e^{-\lambda t}\sum_{n=0}^{\infty}\frac{t^n}{n!}L^n(|u_0-w_0|)(x)$$

p-Kantorovich/Wasserstein distance stability: *When the initial data differ in the same transformed moments of order* **p,** the estimate is

$$|u(t)-w(t)|(x) \le e^{-\lambda t}\sum_{n=0}^{\infty}\frac{t^n}{n!}L^n(x^p)\sup\left(\frac{|u_0-w_0|}{(x^p)}\right) \le C\, e^{-\lambda t(\gamma(p)-1)}\, O(x^p)$$

⇓

$$\sup_x \frac{|u(t)-w(t)|(x)}{(x^p)} \le \sup_x \left(\frac{|u_0-w_0|}{(x^p)}\right) e^{-\lambda t(\gamma(p)-1)}$$

or equivalently, this is a stability estimate **in Kantorovich/Wasserstein distance** of order **p** > 0 between two the two probability measures ***f* = $\mathcal{F}^{-1}$ *u* and *g* = $\mathcal{F}^{-1}$ *w*** ,

$$\mathbf{W_p(f, g) := \inf_{(X', Y')} E(|X' - Y'|^p)^{(1/\max(p,1))}}$$

where the infimun is taken over all pairs **(X′, Y′)** of real random variables whose marginal probability distributions are ***f*** *and* ***g*** respectively. (also related to Zolotarev metrics) (Bassetti &Ladelli, AP'10)

$$\mathbf{W_p(f, g)(t)} \le e^{-\lambda t(\gamma(p)-1)}\ \mathbf{W_p(f, g)(0)}$$

Wild series and probabilistic representation of the solutions using the N-ary trees

Extension of the McKean binary tree representation of the Wild sums for each $\Gamma^{(N)}(\varphi)$:
(Bassetti and Ladelli, 2010)

Write the Wild series expansion of $\varphi(\cdot, t)$ solution to

$$\varphi_t(\xi, t) = \hat{Q}(\varphi_1(\xi, t), \ldots \varphi_N(\xi, t)) - \varphi(\xi, t) \quad \text{multi-linear structure}$$

with $\varphi(\xi, 0)$ positive, which has the explicit solution

$$\varphi(\xi, t) = \Sigma_{k\geq 0}\, \zeta(t, k)\; q_k(\xi)$$

and

$$\zeta(t,k) := b_k e^{-t}\left(1 - e^{-(N-1)t}\right)^k.$$

$\zeta(t, \cdot)$ is the prob. density of a Negative-Binomial r.v. of parameters $(1/(N-1), e^{-(N-1)t})$

and

$$q_k(\xi) = \Sigma_{\underline{i}}\, p_k(\underline{i})\; \hat{Q}\,(q_1(\xi, t), \ldots q_N(\xi, t))$$

defined recursively

with

$$p_k(\underline{i}) := \binom{k-1}{i_1, \ldots, i_N} \frac{\prod_{l=1}^{N} \prod_{m=0}^{i_l-1} f_m}{\prod_{r=0}^{k-1} f_r},$$

For $\underline{i} = i_1, .. i_N \;\epsilon\; I_\kappa$ an indexation for the k-level of the N-ary tree

This is enough to show that the posterior beliefs distributions π_t converges in distribution to π_∞ , with unique exponential convergence rate $\lambda > 0$, (in the rescaled time by λ)

(K_i depending on the initial state, mean &potential concern n)

$$e^{-\lambda t}\,\kappa_0 \leq |\pi_t(0, b) - \pi_\infty(0, b)| \leq e^{-\lambda t}\,\kappa_1 .$$

Finite Markov Information-Exchange (FMIE) processes (Aldous, lecture notes, 2011)

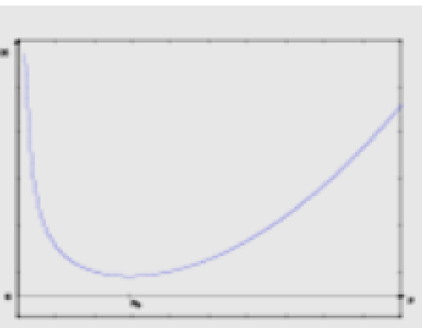
Examples: binary interactions: $v_* = g_{11}\, v + g_{12}\, w$

$w_* = g_{21}\, v + g_{22}\, w$

mean conservation: $g_{11} + g_{21} = g_{12} + g_{22} = 1$

Conserved mean models

Wealth distribution f(t,v), λ 'saving propensity'



$g_{11} = 1 - g_{21} = (1-\epsilon)\lambda$

$g_{12} = 1 - g_{22} = (1-\epsilon)(1-\lambda)$

$v_* = \lambda\, v + \epsilon(1-\lambda)(v+w),$

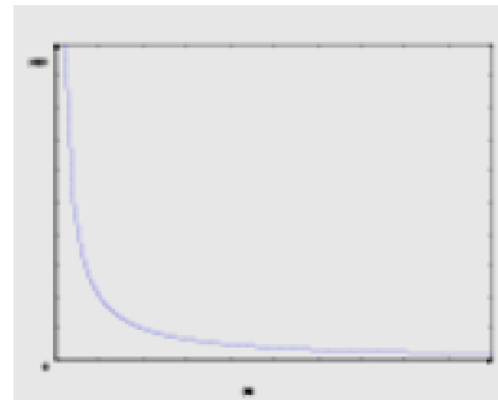
$w_* = \lambda\, w + (1-\epsilon)(1-\lambda)(v+w).$

$\mu(p) = 1/(2p)\ \left(\Sigma_{ij} (g_{ij}^{\,p}) - 1 \right)$

(Lux &Marchesi, Toscani &Pareschi, Chakraborti & Chakrabarti)

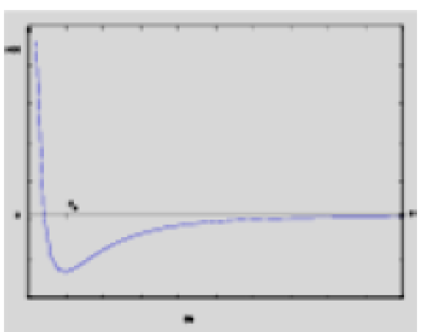
not conserved mean models

Aggregation models



$g_{ij} = 1$

$\mu(p) = 1/(2p)\ \ \Sigma\, (1^2 + 1^2) - 1 = 1/(2p)$



Opinion dynamics interaction (Ben Naim et al, 2003 – 2006)

(or classical elastic/inelastic Boltzmann dynamics)

$g_{ij} = 1/2$ *with* $\mu(p) = 1/(2p)\ [\ \Sigma\, [(1/2)^p + (1/2)^p] - 1\ ]$

also, with affine trans, by adding a random variable as added diffusion *(Ernst&Van Noije08, Bobylev &Cercignani JSP'02, IMG, Panferov and Villani CMP'04, Bassetti, Ladelli & Toscani,JSP'11)*

4- Consensus seeking model (Aldous, Ben Naim et al):

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \lambda & 1-\lambda \end{pmatrix} \quad \text{with } \lambda \in [0,1].$$

5- Randomized public goods 'games': *(Bobylev, Cercignani and I.M.G, Bobylev and Windfall)*

$$a_{ij} = \left[\delta_{ij}\left(1 - \frac{(N-1)\lambda}{N}\right) + (1-\delta_{ij})\frac{\lambda}{N}\right]\kappa$$

with δ_{ij} the Kronecker delta, $\kappa \in [0,\infty)$ random, and $\lambda \in [0,1]$

6- ordered consensus seeking (non-linear interaction law) (*Aldous, Chatterjie & Durret)*

$$\begin{aligned} \theta_i' &= a_{11}\min\{\theta_i,\theta_j\} + a_{12}\max\{\theta_i,\theta_j\} \\ \theta_j' &= a_{21}\min\{\theta_i,\theta_j\} + a_{22}\max\{\theta_i,\theta_j\} \end{aligned}, \qquad a_{ij} \in \mathbb{R}.$$

7- Subjective assessment of an average (multidimensional states):

$$\begin{aligned} \left(\theta_i^{(1)},\theta_i^{(2)}\right)' &= \left(\theta_i^{(1)}, \lambda\theta_i^{(2)} + (1-\lambda)\left[\kappa\theta_j^{(1)} + (1-\kappa)\,\theta_j^{(2)}\right]\right) \\ \left(\theta_j^{(1)},\theta_j^{(2)}\right)' &= \left(\theta_j^{(1)}, \lambda\theta_j^{(2)} + (1-\lambda)\left[\kappa\theta_i^{(1)} + (1-\kappa)\,\theta_i^{(2)}\right]\right) \end{aligned}, \quad \overline{\theta^{(1)}} = 0, \quad \kappa,\lambda \in \mathbb{R}.$$

or, equivalently

$$\theta_i' = L\theta_i + R\theta_j, \qquad \theta_j' = L_*\theta_j + R_*\theta_i$$

$$L = L_* = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \quad R = R_* = \begin{pmatrix} 0 & 0 \\ (1-\lambda)\kappa & (1-\lambda)(1-\kappa) \end{pmatrix}.$$

Self-similar asymptotics - spectral properties

Spectral Properties of L :

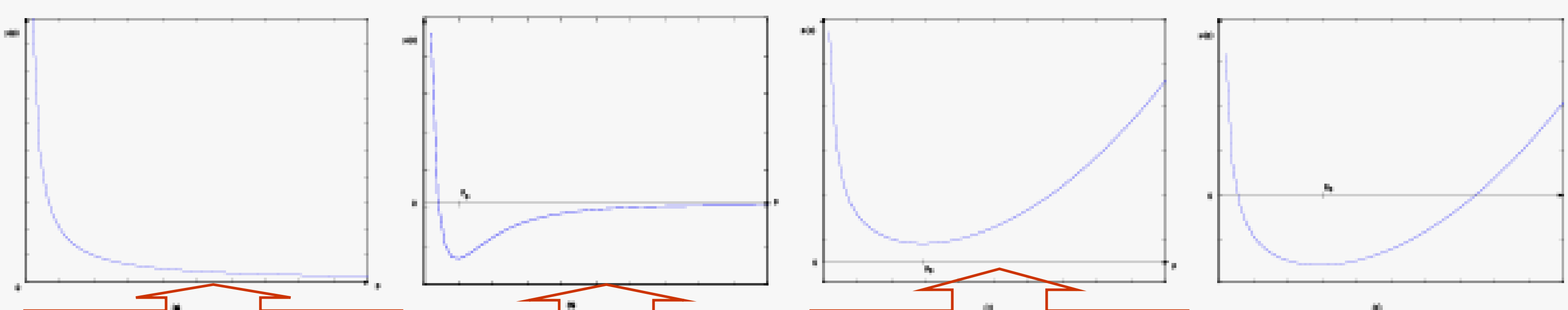
$Lu = \int_0^\infty K(a) u(a\,x) da$; $\qquad K(a) = \sum_{n=1}^{M} n\alpha_n K_n(a),$

where $K_n(a) = \int_0^\infty da_2 \ldots \int_0^\infty da_n \, A_n(a_1, a_2, \ldots, a_n)$ and $\sum_{n=1}^{M} \alpha_n = 1$, and satisfies:

- x^p is the ei-function with ei-value $\gamma(p)$ of the linear operator L associated to Γ

$$Lx^p = \gamma(p)\, x^p, \quad \gamma(p) = \int_0^\infty K(a) a^p da$$

- $\gamma(1) - 1$ is the energy dissipation rate.
- we call $\mu(p) = \frac{\gamma(p) - 1}{p}$ **the spectral function associated to** Γ.
- $\mu(0+) = +\infty$ and $0 < p_0$, such that $\mu(p_0) = \min_{p>0} \mu(p)$ is the **unique minima** .



Aggregation Spectrum

Boltzmann Spectrum

Wealth distribution Spectrum

Existence of Self-Similar Solutions and long time dynamics

For $x = \frac{|k^2|}{2}$ and $\boxed{\eta = xe^{-\mu_* t}}$, with initial conditions $u_o(x) = 1 + O(x^p)$, $u_o(0) = 1$ and $\|u_o\| \leq 1$

if $0 < p \leq 1 < p_0$ and $\mu_* = \mu(p)$ ($\Rightarrow$ one can take $u_o = e^{-x}$ to fulfill the conditions),

then, there exists a non-trivial self-similar solution $u(t,x) = \Psi(\eta)$ to

$$\mu_* \eta \Psi' + \Psi = \Gamma(\Psi), \qquad \text{with initial state} \qquad \Psi_{|\eta=x} = u_o(x) \quad \text{such that}$$

$$\Psi(\eta) = u_o(\eta) + O(\eta^{p+\varepsilon}) = 1 - \eta^p + O(\eta^{p+\varepsilon}) \quad \text{for } \eta \geq 0, \text{ and}$$

time decay rate to self-similarity

$$|\, u(xe^{-\mu_* t}, t) - \Psi(x)\,| \leq C e^{-t(p+\varepsilon)\overbrace{(\mu_* - \mu(p+\varepsilon))}} O(x^{p+\varepsilon}) \qquad \text{for } 0 < p < p+\varepsilon < p_0\,,$$

This estimate is equivalent to **the p-Kantorovich/Wasserstein distance estimate:**

$$e^{d\mu(p)t}\, W_p(f(ve^{-\mu(p)t}),\ F_p(v)) \leq W_p(f,\ F_p(v))\ e^{-t(p+\varepsilon)(\mu_* - \mu(p+\varepsilon))}$$

Remarks:

1- ***Existence proof can be rewritten as existence of martingales whose weak limit is a scale mixture of p–stable laws (Bassetti &Ladelli, AP'11),***

2- The transformation $\overline{x} = \beta x^p$, for $p > 0$ transforms the study of the initial value problem to $u_o(x) = 1 + x^p$ and $||u_o|| \leq 1$ **so it is enough to study the case** p=1

Further, by making a different rescaling choice one obtains convergence to trivial states (with decay rates in corresponding p-Kantorovich distances)

for $\mu_* > \mu(p)$ then $e^{d\,\mu_* t} f(|v|e^{-\mu_* t}, t) \to \delta_0$ as $t \to \infty$

for $\mu(p_0) < \mu(p+\delta) < \mu_* < \mu(p)$ then $e^{d\,\mu_* t} f(|v|e^{-\mu_* t}, t) \to 0$ as $t \to \infty$

However, for choices of large p there is asymptotic convergence to a point mass but no selfsimilar rates

For $p_0 < p$ then $p \in (p_0, \infty)$ so that $\mu'(p) > 0$ then

$$\lim_{t\to\infty} e^{d\,\mu_* t} f(|v|e^{-\mu_* t}, t) = \delta_0$$

Study of the spectral function $\mu(p)$ associated to the linearized collision operator

Theorem: *(Bobylev, Cercignani, I.M.G,06)* The self-similar asymptotic function $F_{\mu(p)}(|v|)$ does NOT have finite moments of all orders if the energy dissipates, i.e. $\mu(1) < 0$.

For any initial state $\varphi(x) = 1 - x^p + x^{(p+\varepsilon)}$, $p \leq 1$.

Decay rates in Fourier space: $(p+\varepsilon)[\,\mu(p) - \mu(p+\varepsilon)\,]$

both for or finite *(p=1)* or infinite *(p<1)* initial energy.

For $p_0>1$ and $0<p<(p+\varepsilon)<p_0$

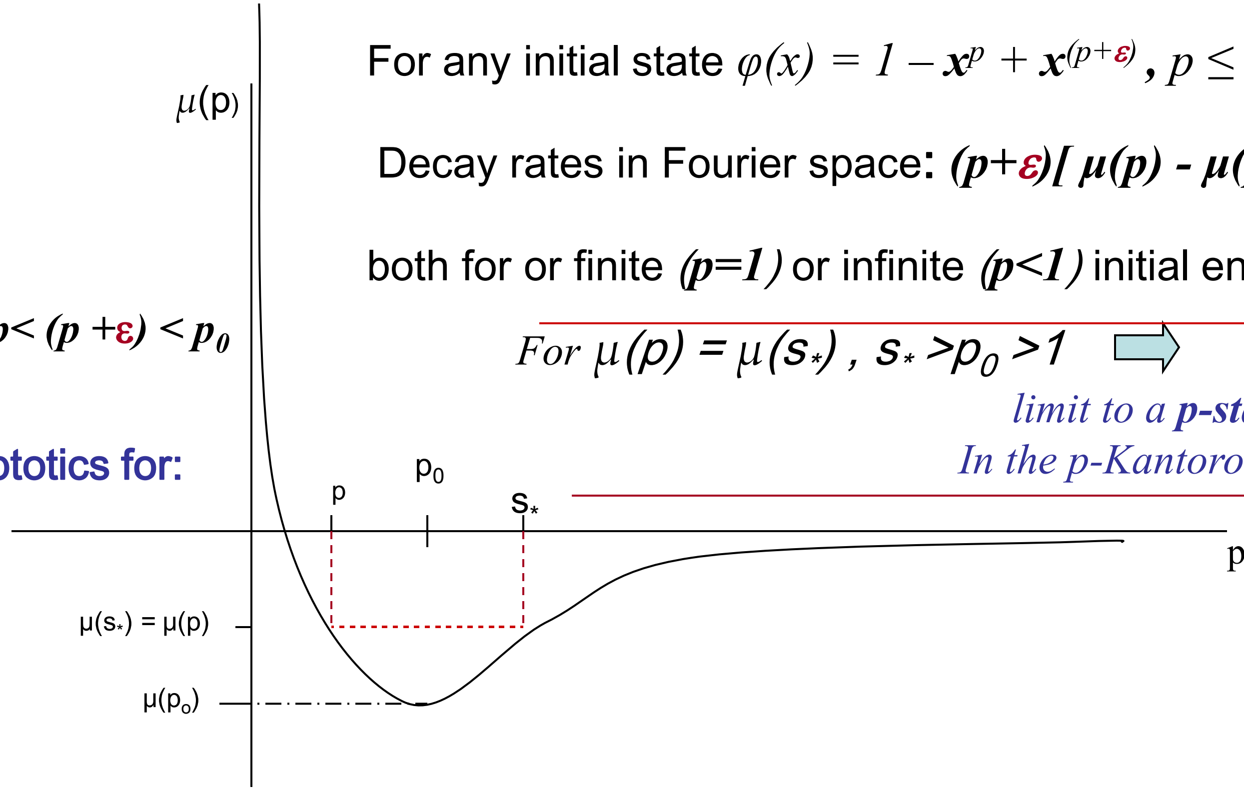
⇓

Self similar asymptotics for:

For $\mu(p) = \mu(s_*)$, $s_* > p_0 > 1$ ⇒ Power tails

*limit to a **p-stable law***

In the p-Kantorovich distance



⇒ If $p = 1$ (finite initial energy) then, $m_q \leq \infty$ only for $0 \leq q \leq p_*$, where $p_* \geq 1$ is the unique maximal root of the equation $\mu(p_*) = \mu(1)$.

Finite (p=1) or infinite (p<1) initial energy ⇒ If $0 \leq p \leq 1$ then, $m_q = \int_{\mathbb{R}^3} F_{\mu(p)}(|v|)|v|^q dv \leq \infty; \quad 0 \leq q \leq p$

For $p_0 < 1$ *and* $p=1$ ⇒ **No self-similar asymptotics with finite energy**

In general we can see that

***1**. For more general systems multiplicatively interactive stochastic processes the lack of entropy functional **does not impairs** the understanding and realization of global existence by extension of the Wild summation method (in the sense of positive Borel measures), long time behavior from spectral analysis and self-similar asymptotics.*

***2.** "power tail formation for high energy tails" of self similar states is due to lack of total energy conservation, **independent** of the process being micro-reversible (elastic) or micro-irreversible (inelastic).*

*It is also possible to see **Self-similar solutions may be singular at zero**.*

***3.** The long time asymptotic dynamics and decay rates are fully described by the **continuum spectrum associated to the linearization about singular measures**.*

***4.** Recent probabilistic interpretation of our workw as given by **F. Bassetti and L. Ladelli:** **connects to evolution of expecations, m-ary convolution trees (Mc Kean approach of Wild sums), filtrations and stable laws with power law decay rates,** (Annals in Probability'11)*

***5.** Study of the convergence properties of the corresponding cumulative function (Kolmogorov distances) is not cover by the analysis in BCG, CMP'10 and it is work in progress with R. Srinivasan .*

***6.** Study of self-similarity for systems in work under progress with F. Bolley and Srinivasan (in progress)*

Further applications to agent interactions

- ***information percolation models***

  (Duffie, Giraux, Malamud and Manso, 08-09)

  - **Percolation information** (Duffie, Giraux & Manso, 08) (already discussed)
  - **Information percolation in segmented markets** (Duffie, Malamud &Manso, 2010)
    systems of Maxwell type interaction
  - **Information percolation with equilibrium search dynamics** (Duffie,Malamud &Manso'09) beyond Maxwell type → moment summability methods techniques?

- ***M-game multi agent model*** (Bobylev Cercignani, Gamba, CMP'09)

Information aggregation model with equilibrium search dynamics *(Duffie, Malamud & Manso 08)*

For any search-effort policy function C(n), the cross-sectional distribution f_t of precisions and posterior means of the i-agents is almost surely given by

$$f_t(n; x; w) = \mu^C(n,t) \;\; p_n(x \mid Y(w))$$

where $\mu_t(n)$ is the fraction of agents with ***information precision n at time t,*** *which is the unique solution of the differential equation below (of generalized Maxwell type)*
and $p_n(x \mid Y(w))$ is the Y-conditional Gaussian density of $E(Y \mid X_1; \ldots; X_n)$, for any n signals $X_1; \ldots; X_n$.

This density has conditional mean

$$\frac{n\rho^2 Y}{1+\rho^2(n-1)}$$

and

conditional variance

$$\sigma_n^2 = \frac{n\rho^2(1-\rho^2)}{(1+\rho^2(n-1))^2}.$$

$\mu_t(n)$ satisfies the dynamic equation

$$\frac{d}{dt}\mu_t = \eta(\pi - \mu_t) + \mu_t^C * \mu_t^C - \mu_t^C \, \mu_t^C(\mathbb{N}),$$

with ***π(n)*** *a given distribution independent of any pair of agents*

Where $\mu_t^C(n) = C(n) \; \mu(n,t)$ is the effort-weighted measure such that: **C(n) is the search-effort policy function**

Example from information search (percolation) model **not of Maxwell type!!**

For $\mu_t(n)$ for the fraction of agents with precision n (related to the cross-sectional distribution μ_t of information precision at time t in a given set*) its the evolution equation is given by*

$$\frac{d}{dt}\mu_t = \eta(\pi - \mu_t) + \mu_t^C * \mu_t^C - \mu_t^C \, \mu_t^C(\mathbb{N}),$$

Where $\mu_t^C(n) = C(n) \ \mu(n,t)$ is the effort-weighted measure such that: **C(n) is the search-effort policy function**

Linear term: represents the replacement of agents with newly entering agents.

Gain Operator: The convolution of the two measures $\mu_t^C * \mu_t^C$ represents the gross rate at which new agents of a given precision are created through matching and information sharing.

$$(\mu_t^C * \mu_t^C)(n) = \sum_{k=1}^{n-1} \mu_t(k)C(k)C(n-k)\mu_t(n-k).$$

Loss operator: The last term of captures the rate $\mu_t^C \ \mu_t^C(\boldsymbol{N})$ of replacement of agents with prior precision n with those of some new posterior precision that is obtained through matching and information sharing, where

$$\mu_t^C(\mathbb{N}) = \sum_{n=1}^{\infty} C_n \, \mu_t(n)$$

is the cross-sectional average search effort

This is an aggregation model of "non-Maxwell" type where Pego-Menon does not apply, but variable potential interactions (Bobylev, Panferov, Villani, I.M.G or Laurencot, Mishler,Escobedo may be adjusted)

Conclusions- future directions

- Systems of different agent types, p-stable law dynamics (with Bolley and Srinivasan)

- Local interaction frequency → moment methods?

Control of interaction frequency-- mean field games formulation.

- Networks -spatial dependence

- Friction –unisotropic states in multi dimensional agent/type space

Thank you very much for your attention!

Preprints: *http://rene.ma.utexas.edu/users/gamba/publications-web.htm*
And references therein

Revision of the Boltzmann transport equation and connections to continuum models

Self similar solutions – Moments equations of the limiting (p-stable law) state

where $\Psi_{\mu_*}(x)$ satisfies:

$$1 \geq \Psi_{\mu_*}(x) \geq e^{-x} \ , \qquad \lim_{x\to\infty} \Psi_{\mu_*}(x) = 0 \ ,$$

and there exists a generalized non-negative function $R_{\mu_*}(\tau)$, $\tau \geq 0$, s.t.

$$\Psi_{\mu_*}(x) = \int_0^\infty d\tau \, R_{\mu_*}(\tau) \, e^{-\tau x} \ , \qquad \int_0^\infty d\tau \, R_{\mu_*}(\tau) = \int_0^\infty d\tau \, R_{\mu_*}(\tau) \, \tau = 1 \ .$$

In addition: for $p_0 > 1$ and $p = 1$: the $R(\tau)$ satisfies (using the Laplace transform)

$$-\mu(1)\frac{\partial}{\partial\tau}\tau R(\tau) + R(\tau) = Z(R) = \mathcal{L}^{-1}[\Gamma(w)]$$ $\Longleftrightarrow$ fractional moment equations

for $Z(R) = \sum_{n=1}^{M} \alpha_n Z_n(R) \ , \quad \sum_{n=1}^{M} \alpha_n = 1 \ , \quad Z_n(R) = \int_{\mathbb{R}^n_+} da_1, \dots, da_n \frac{A_n(a_1,\dots,a_n)}{a_1 a_2 \dots a_n} \prod_{k=1}^{n}{}^{*} R_k\left(\frac{\tau}{a_k}\right) \ ,$

$$\prod_{k=1}^{n}{}^{*} R_k(\tau) = R_1 * R_2 * \dots * R_n \ , \qquad R_1 * R_2 = \int_0^\tau d\tau' \, R_1(\tau') R_2(\tau - \tau') \ .$$

p-Stable laws (we show here p=1 case, it generalizes to p<1)

2 - Properties for moments equations: $-\mu(1)\frac{\partial}{\partial\tau}\tau R(\tau)+R(\tau)=\mathcal{L}^{-1}[\Gamma(w)]$

$m_s > 0$ for all $s>1$.

Set $m_s = \int_0^\infty d\tau\, R(\tau)\tau^s \ , \ s>0$; with $m_0 = m_1 = 1$.

Then multiply by τ^s and integrate to obtain

$s\,[\mu(1)-\mu(s)]m_s = \sum_{n=2}^{N} \alpha_n I_n(s)$ for $s>1$, with $\mu(1)=$ energy dissipation rate

Now, one can show that $0 \le \sum_{n=2}^{N} \alpha_n I_n(s) \le C_N m_{s-1}$, then

while $\mu(1)-\mu(s)>0$ then $0 < m_s \le \dfrac{C_N}{s\,[\mu(1)-\mu(s)]}\, m_{s-1}$ is finite,

otherwise, if $\mu(1)-\mu(s)<0$ then m_s must be unbounded.

$\Rightarrow$ the following **Theorem** holds:

[i] If the equation $\mu(s)=\mu(1)$ has the only solution $s=1$, then $m_s<\infty$ for any $s>0$.

[ii] If $\mu(s)=\mu(1)$ has two solutions $s=1$ and $s=s_*>1$, then $m_s<\infty$ for $s<s_*$ and $m_s=\infty$ for $s>s_*$.

[iii] $m_{s_*}<\infty$ only if $I_n(s_*)=0$ in the above equation, for all $n=2\ldots N$.

In addition, the corresponding Fourier Transform of the self-similar pdf admits an integral representation by distributions $M_p(|v|)$ with kernels $R_p(\tau)$, for $p = \mu^{-1}(\mu_*)$.

They are given by:

$$F_p(|v|) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^N} dv\, \Psi_{\mu(p)}\left(\frac{k^2}{2}\right) e^{ik\cdot v} = \int_0^\infty d\tau\, R_p(\tau)\, \tau^{-\frac{d}{2p}} M_p\left(|v|\tau^{-\frac{1}{2p}}\right)$$

This property generalizes infinite divisibility (as in aggregation models)

where

$$M_p(|v|) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^N} dk\, e^{-|k|^{2p} + ik\cdot v} \quad ,$$

↕

i.e. with $\lambda(1)=2$

(Bertoin,Menon,Pego,…)

Similarly, by means of Laplace transform inversion, for $v \geq 0$ and $0 < p \leq 1$

$$\Phi_p(v) = \int_0^\infty R_p(\tau)\tau^{-\frac{1}{p}} N_p(v\tau^{-\frac{1}{p}})\, d\tau \quad \textbf{with} \quad N_p(v) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{-x^p + xv}\, dx \,,$$

These representations explain the connection of self-similar solutions with stable distributions

Two very important properties of the self-similar solutions:

$$F_p(|v|) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^d} \Psi_p(\frac{k^2}{2}) e^{ik\cdot v} dv$$

1 - Long time asymptotics:

case 1.1: Convergence to non-trivial stationary states:

For $1 \le p_0$ with $\mu(p_0) = \min_{p>0} \mu(p)$ and $p \in (0, p_0)$ then $\mu'(p) < 0$

$\Rightarrow$ Self-similar asymptotics $\Rightarrow$ Dynamically stable law (CLT) to NESS:

For initial states $\hat{f}_0(\frac{|v|^2}{2}) = 1 - x^p + O(x^{p+\varepsilon})$, such that $0 < p < p + \varepsilon < p_0$

Theorem: dynamically scaled stable laws *(Kintchine type of CLT)*

for $\mu_* = \mu(p)$, then $(p+\varepsilon)(\mu_* - \mu(p+\varepsilon)) = (p+\varepsilon)(\mu(p) - \mu(p+\varepsilon)) > 0$

so that

$$e^{d\mu(p)t} f(|v|e^{-\mu(p)t}, t) \to F_p(|v|) \quad \text{as } t \to \infty$$

⇒ The boundedness properties of the moments m_s of R_p implies the boundedness of moments for the self-similar solutions constructed by Fourier or Laplace transform methods: with $v \geq 0$, $0 < p \leq 1$:

$$F_p(|v|) = \int_0^\infty d\tau \, R_p(\tau)\tau^{-\frac{d}{2p}} M_p(|v|\tau^{-\frac{1}{2p}}) \ , \qquad \text{then}$$

$$m_{2s}(F_p) = m_{2s}(M_p) m_{s/p}(R_p) \quad \text{(for Fourier Transform)},$$

and

$$\Phi_p(v) = \int_0^\infty d\tau \, R_p(\tau)\tau^{-\frac{1}{p}} N_p(v\tau^{-\frac{1}{p}}) \ , \ \text{then}$$

$$m_s(\Phi_p) = m_s(N_p) m_{s/p}(R_p) \quad \text{(for Laplace Transform)} \ .$$

⇒ the following **Theorem**:

1- If $0 < p < 1$, then $m_{2s}(F_p)$ and $m_s(\Phi_p)$ are finite if and only if $0 < s < p$.

2- For $p = 1$ the result holds for $m_s = m_{2s}(F_1)$ and for $m_s = m_s(\Phi_1)$.

⇒ $F(|v|)$ can not have all (even) moments bounded ≡ power tails.

Interactions with equilibrium dynamics

Example: Description of the Weakly Coupled Binary Mixture Problem *(Bobylev, I.M.G. JSP '06)*

Construction of **explicit solutions** to:

$$\frac{\partial f(v,t)}{\partial t} = \int_{w\in\mathbb{R}^3}\int_{\sigma\in S^2} B(|u|,\mu)[f('v,t)f('w,t) - f(v,t)f(w,t)]d\sigma dw$$
$$+ \quad \theta_b \int_{w\in\mathbb{R}^3}\int_{\sigma\in S^2} B(|u|,\mu)[f('v,t)M_T('w) - f(v,t)M_T(w)]d\sigma dw$$

with $M_T(v) = \frac{e^{\frac{-|v|^2}{(2T)}}}{(2\pi T)^{3/2}}$, $B(|u|,\mu) = C_\lambda = \frac{1}{4\pi}$, $\beta = 1.0$, θ_b - depending on the asymptotics and T being the background temperature.

- A system of two different particles with the same mass is considered. One set of particles is assumed to be at equilibrium i.e., with a Maxwellian distribution with temperature $T(t)$.
- Second set of particles is assumed to collide with themselves (first integral) and the background particles(Linear Boltzmann Collision Integral).

The collisions are assumed to be ***locally* elastic** i.e., $|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$ but the above form leads to ***global*** energy dissipation i.e., $\int_{\mathbb{R}^3} |v|^2 f(v,t)dv \neq 0$.

Explicit solutions an elastic model in the presence of a cold thermostat for d ≥ 2

Mixtures of colored particles (same mass ***β=1***): (Bobylev & I.M.G., JSP'06)

In the space of characteristic functions:

$$\frac{\partial \varphi}{\partial t} = \int_{-}^{1} ds\, G(s)\, \{\varphi(sx)\, \varphi[(1-s)x] + \theta \varphi[(1-\beta\, s)x] - \varphi(x)\, [1+\theta]\} =$$

$$= \int_0^1 ds\, \varphi[(1-s)x][\varphi(sx)+\theta] - (1+\theta)\varphi(x) \;, \quad \textit{with} \;\; \varphi(0)=1 .$$

Set β=1

and set $\varphi(xe^{-\mu t}) = \psi(\eta) \simeq 1 - c(p)\, \eta^p \;,\qquad \text{for } \eta \to 0 \;,\; p > 0$

1. *Laplace transform of ψ:* $w(z) = \mathcal{L}(\psi)(z)$ ***Transforms*** → ***The eq. into*** $\mu(zw)'' + (1+\theta)w' + w\left(w + \frac{\theta}{z}\right) = 0$

2- *set group transformations*

$$u(z) = zw(z) = \int_0^\infty dx\, e^{-x}\psi\left(\frac{x}{z}\right) \;,$$ *and* $y(z) = z^{-2}\, u(z^q) + B$, *B constant*

By the choice of parameters,

→ $\alpha = q(5\mu q + 1 + \theta - \mu)$ *and* $\beta = 2B - 1 + 4\mu q^2 + 2q(1+\theta-\mu)$

3- *and* $\alpha=\beta=0 = B(B-1)$ → $\mu q^2 y'' + y^2 = 0$ ***with*** $\theta=\mu - 1 - 5\mu q$ *and* $6\mu q^2 = \pm 1$

Painleve type eq.

Theorem: *the equation for the slowdown process in Fourier space, has* ***exact self-similar solutions*** *satisfying the condition*

$$\psi(x) \simeq 1 - \frac{x^p}{\Gamma(p+1)}, \qquad x \to 0, \quad p > 0 \qquad \textit{for} \qquad \psi_i(x) = \mathcal{L}^{-1}\left[\frac{u_i(z)}{z}\right], \qquad i = 1, 2,$$

for the following values of the parameters θ(p) and μ(p):

Case 1: $\mu(p) = -\frac{1}{6p^2}, \quad \theta(p) = \frac{(3p-1)(1-2p)}{6p^2};$

Case 2: $\mu(p) = \frac{2}{3p^2}, \quad \theta(p) = \frac{(3p+1)(2-p)}{3p^2}$

where the solutions are given by equalities

Case 1:

$$u(z) = \left(1 + \frac{1}{2}z^{-p}\right)^{-2}$$

Infinity energy SS solutions

Case 2:

$$u(z) = 1 - (1 + z^{p/2})^{-2},$$

Finite energy SS solutions

For p = 1/3 and p=1/2 then θ=0 → the Fourier transf. Boltzmann eq. for one-component gas → These exact solutions were already obtained by Bobylev and Cercignani, JSP'03

$$\mu = \frac{2}{3}, \quad \theta = \frac{4}{3}, \qquad f(|v|, t) = e^t F(|v|e^{t/3}),$$

$$F(|v|) = \frac{4}{\pi}\int_0^\infty ds \frac{\exp(-|v|^2/2s^2)}{(2\pi s^2)^{3/2}(1+s^2)^2}.$$

after transforming Fourier back in phase space

- *For self similar asymptotics we study $t \to \infty$ so $\hat{T} \to T$ in $f_T^{ss}(v,t)$ (i.e. the particle distribution temperature approaches the background temperature as expected due to the linear coll. op.)*, ***both, for infinite and finite energy cases***

Qualitative results for Case 2 with finite energy:

- Interesting NESS behavior can be observed if $T \to 0$: Set $\hat{T} = s^2 e^{\frac{-2t}{3}}$ so $f_0^{ss}(|v|)$ is explicit.

- Then $f(|v|e^{-t/3}, t) \to_{t\to\infty} e^t f_0^{ss}(|v|)$ where $f_0^{ss}(|v|) = \frac{4}{\pi}\int_0^\infty \frac{e^{-|v|^2/(2s^2)}}{(2\pi s^2)(1+s^2)^2} ds$

- $f_0^{ss}(|v|) = O(\frac{1}{|v|^6})$ as $|v| \to \infty$, and
  $f_0^{ss}(|v|) = O(\frac{1}{|v|^2})$ as $|v| \to 0$

Also, rescaling back w.r.t. to $M(k)$ **and Fourier transform back** $f_0^{ss}(|v|) = M_T(v)$ **and the similarity asymptotics holds as well.**

Computations: spectral Lagrangian methods in collaboration with Harsha Tharkabhushaman JCP'09 and JCM'09

Weak Formulation & fundamental properties of the collisional integral and the equation: Conservation of moments & entropy inequality

$$(\frac{\partial}{\partial t} + \nabla_x) \int_{I\!R^d} f(t,x,v)\, \varphi(v)\, dv = \int_{I\!R^d} Q(f,f)(t,x,v)\, \varphi(v)\, dv \; =$$

$$\frac{\kappa(t)}{2} \int_{\mathbb{R}^{2d}} \int_{S_+^{d-1}} f f_* \,(\varphi' + \varphi'_* - \varphi - \varphi_*) |u|^\gamma \, \tilde{b}(\sigma)\, d\sigma\, dv_*\, dv$$

x-space homogeneous (*or periodic boundary condition*) problem: Due to symmetries of the collisional integral one can obtain (after interchanging the variables of integration):

Invariant quantities (or observables) - These are statistical moments of the 'pdf'

conservation of mass ρ and momentum J: set $\varphi(v) = 1$ and $\varphi(v) = v$

Using local conservation of momentum on the test function: $'v + 'v_* = v + v_*$

$$\frac{\partial}{\partial t} \int_{I\!R^d} f\, \{1, v_i\} dv = \kappa(t) \int_{I\!R^d} Q(f,f)(v)\{1, v_i\}\, dv = 0, \quad i = 1,2,3.$$

holds, both for the **Elastic** and **Inelastic** cases

Next, set $\varphi(v) = |v|^2 \Rightarrow$ It **conserves energy for $e = 1$ – ELASTIC**:

Using local conservation of energy on the test function: $|'v|^2 + |'v_*|^2 = |v|^2 + |v_*|^2$

➔ $\frac{\partial}{\partial t}\Theta(t) = \kappa(t) \int_{I\!R^d} Q(f,f)(v)\, |v|^2\, dv = 0$ *Conservation of energy*

Recall Boltzmann H-Theorem for **ELASTIC interactions:**

$$\frac{\partial}{\partial t}\int f \log f \, dv = \kappa(t)\int_{I\!R^d} Q(f,f)\log f \, dv =$$

$$\frac{\kappa(t)}{2}\int_{I\!R^d}\int_{I\!R^d}\int_{S_+^{d-1}} (ff_* - f'f'_*)\log\frac{f'f'_*}{ff_*}|u|^{\gamma}b(\sigma)d\sigma\, dv\, dv_* \leq 0$$

Time irreversibility is expressed in this inequality $\Longrightarrow$ **stability**

In addition:

The Boltzmann Theorem: *there are only N+2 collision invariants* $\Longleftrightarrow$

$$\int_{I\!R^N} Q(f,f)\log f\, dv = 0 \iff \log f(\cdot,v) = A + B\cdot v + C|v|^2 \iff$$

$f(\cdot,v) = M_{A,B,C}(v)$ Maxwellian (Gaussian in v-space) parameterized by A, B, C

related the first $N+2$ moments of the initial probability state of $f(0,v) = f_0(v)$

Elastic (conservative) Interactions

Time Irreversibility and relation to Thermodynamics

- **Stability** $\lim_{t\to\infty} \| f(t,v) - M_{A,B,C} \|_{L^1_2} \to 0$ where $\{A,B,C\} \longleftrightarrow$ $\{\rho, u, w\}$, $\rho = \int f_0\, dv$, $\rho u = \int v f_0\, dv$ and $\rho w = \int |v|^2 f_0\, dv$

- **Macroscopic balance equations:** For the space inhomogeneous problem:

Under the ansatz of a Maxwellian state in v-space

$$\boxed{f(t,x,v) = M_{a,b,\mathbf{u}} = a e^{-(b|v-\mathbf{u}|^2)}}$$

where the dependance of (t,x) is only though the parameters $(a,b,\mathbf{u})$:

$\mathbf{u} = \dfrac{\mathbf{J}}{\rho}$ mean velocity and $\Theta = \rho \mathrm{w} = \dfrac{1}{2}\rho \mathbf{u} + \rho\,\mathrm{e}$ kinetic energy, $\mathrm{e} =$ internal energy

choosing $a = \dfrac{3^{3/2}\rho}{(4\pi\, e)^{3/2}}$; $b = \dfrac{3}{4e}$

plus equilibrium constitutive relations : $P = \dfrac{2}{3}\rho\, e$ **pressure.**

→ yields the compressible Euler equations ➔

Elastic (conservative) Interactions: Connections to

Hydrodynamic limits: evolution models of a 'few' statistical moments

(mass, momentum and energy)

One obtains the Euler equations:

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\rho\, \mathrm{u_i}) = 0\,,$$

$$\frac{\partial}{\partial t}(\rho\, \mathrm{u_i}) + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\rho\, \mathrm{u_i u_j} + p) = 0\,, \quad (j = 1,2,3)$$

$$\frac{\partial}{\partial t}(\rho(\frac{1}{2}|\mathrm{u}|^2 + e)) = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\rho\, \mathrm{u_i}(\frac{1}{2}|\mathbf{u}|^2 + \mathrm{e} + \frac{\mathrm{p}}{\rho})) = 0\,.$$

- **Hydrodynamic limits:** for ϵ-perturbations of Maxwellians plus constitutive relations $\Rightarrow \{A, B, C\}(t, x)$ the corresponding macroscopic system satisfy compressible Euler

or ϵ-Navier-Stokes equations with higher order partial derivatives terms proportional to an $O(\epsilon)$ deviations from Gaussian (Maxwellian) distributions.

Reviewing Inelastic (dissipative) properties: loss of classical hydrodynamics

Set $\varphi(v) = |v|^2$ and using local energy dissipation:

$$|v|^2 + |v_*|^2 - |'v|^2 - |'v_*|^2 = -\frac{1-e^2}{4}(1-\nu\cdot\sigma)|v-v_*|^2$$

INELASTIC Boltzmann collision term:

It dissipates total energy for ***e=e(z)*** < 1 (by Jensen's inequality):

$$\frac{\partial}{\partial t}\Theta(t) = -c_d\frac{(1-e^2)}{4}\kappa(t)\int_{I\!R^{2d}} f f_* |v-v_*|^{2+\gamma}\, dv_*\, dv \le -c_d\frac{(1-e^2)}{4}\kappa(t)\Theta(t)^{\frac{\gamma+2}{2}}$$

and there is no classical H-Theorem if ***e*** = constant < 1

$$\int_{I\!R^d} Q(f,f)\log f\, dv = \frac{1}{2}\int_{I\!R^{2d}\times S^{d-1}} f f_* \left(\log\frac{f' f'_*}{f f_*} - \frac{f' f'_*}{f f_*} + 1\right)|u|^\gamma b(\sigma)\, d\sigma\, dv\, dv_* + \frac{1-e^2}{2e^2}\int_{I\!R^{2d}} f f_* |u|^\gamma\, dv\, dv_*.$$

→ **Inelasticity brings loss of micro reversibility**

→but keeps **time irreversibility !!**: That is, there are stationary states and, in some particular cases we can show stability to stationary and self-similar states → However: Existence of **NESS**: Non Equilibrium Statistical States (**stable stationary states are non-Gaussian** pdf's)

→ ***f(v,t)*** $\to \delta_0$ as $t \to \infty$ **to a singular concentrated measure** (unless there is 'source')

→*(Multi-linear Maxwell molecule equations of collisional type and variable hard potentials for collisions with a background thermostat)*