Minimal surfaces and entire solutions of the Allen-Cahn equation

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The Allen-Cahn Equation

(AC) $$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^n$$

Euler-Lagrange equation for the *energy functional*

$$J(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{4}\int (1-u^2)^2$$

$u = +1$ and $u = -1$ are *global minimizers* of the energy representing, in the gradient theory of phase transitions, two distinct phases of a material.

Of interest are solutions of (AC) that connect these two values. They represent states in which the two phases coexist.

Solutions that "connect" the values -1 and $+1$ along some direction, say x_N:

$$\lim_{x_N \to -\infty} u(x', x_N) = -1, \quad \lim_{x_N \to +\infty} u(x', x_N) = +1, \quad \text{for all} \quad x' \in \mathbb{R}^{N-1}$$

The case $N = 1$. The function

$$w(t) := \tanh\left(\frac{t}{\sqrt{2}}\right)$$

connects monotonically -1 and $+1$ and solves

$$\boxed{w'' + w - w^3 = 0, \quad w(\pm\infty) = \pm 1, \quad w' > 0.}$$

Canonical examples

For any $p, \nu \in \mathbb{R}^N$, $|\nu| = 1$, $\nu_N > 0$ the functions

$$u(x) := w(\,(x - p) \cdot \nu\,)$$

solve equation (AC) and connect -1 and $+1$ along x_N.

De Giorgi's conjecture (1978): *Let u be a bounded solution of equation*

$$\text{(AC)} \qquad \Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N,$$

which is monotone in one direction, say $\partial_{x_N} u > 0$. Then, **at least** *when $N \le 8$, there exist p, ν such that*

$$u(x) = w(\,(x - p) \cdot \nu).$$

This statement is equivalent to:

At least when $N \leq 8$, all level sets of u, $[u = \lambda]$ must be hyperplanes.

Parallel to **Bernstein-Fleming's conjecture** for minimal surfaces which are entire graphs.

$$H_\Gamma := \nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^{N-1}. \qquad (MS)$$

Entire minimal graph in $\mathbb{R}^N$: For F as above,

$$\Gamma = \{(x', F(x')) \in \mathbb{R}^{N-1} \times \mathbb{R} \ / \ x' \in \mathbb{R}^{N-1}\}.$$

Bernstein-Fleming conjecture: *All entire minimal graphs are hyperplanes, namely any entire solution of (MS) must be a linear affine function*:

True for $N \leq 8$: Bernstein (1910), De Giorgi (1965), Fleming (1962), Almgren (1966), Simons (1968). **False** for $N \geq 9$: Bombieri-De Giorgi-Giusti found a counterexample (1969).

De Giorgi's Conjecture: *u bounded solution of (AC), $\partial_{x_N} u > 0$ then level sets $[u = \lambda]$ are hyperplanes.*

• True for $N = 2$. Ghoussoub and Gui (1998).

• True for $N = 3$. Ambrosio and Cabré (1999).

• True for $4 \leq N \leq 8$ (Savin (2009), thesis (2003)) if in addition u *connects* -1 *and* $+1$ *along* x_N, namely

$$\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1 \quad \text{for all} \quad x' \in \mathbb{R}^{N-1}.$$

The Bombieri-De Giorgi-Giusti minimal graph (1969) :

Explicit construction by super and sub-solutions. $N = 9$:

$$H(F) := \nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^8.$$

$$F : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}, \quad (\mathbf{u}, \mathbf{v}) \mapsto F(|\mathbf{u}|, |\mathbf{v}|).$$

In addition, $F(|\mathbf{u}|, |\mathbf{v}|) > 0$ for $|\mathbf{v}| > |\mathbf{u}|$ and

$$F(|\mathbf{u}|, |\mathbf{v}|) = -F(|\mathbf{v}|, |\mathbf{u}|).$$

Introduce polar coordinates:

$$|\mathbf{u}| = r\cos\theta, \; |\mathbf{v}| = r\sin\theta, \quad \theta \in (0, \frac{\pi}{2})$$

Fact: (del Pino, Kowalczyk, Wei) There is a function $g(\theta)$, $g > 0$ in $(\frac{\pi}{4}, \frac{\pi}{2})$ such that for some $\sigma \in (0,1)$ and all large r

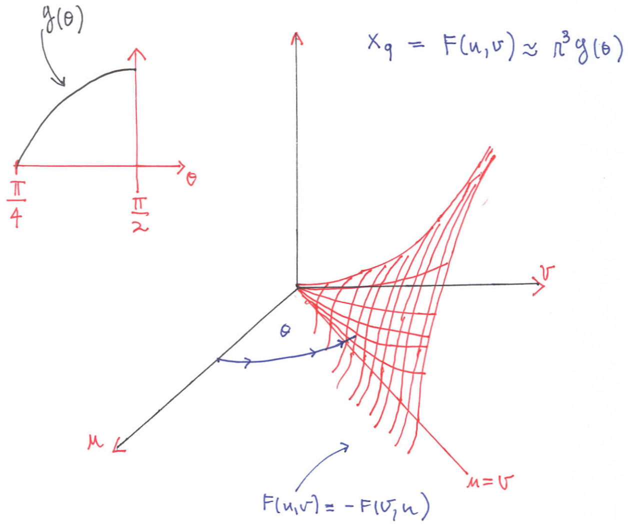
$$\boxed{r^3 g(\theta) \ \leq \ F(r,\theta) \leq r^3 g(\theta) + A r^{-\sigma} \quad \text{as } r \to +\infty.}$$

$$\frac{21 g \sin^3 2\theta}{\sqrt{9g^2 + g'^2}} + \left(\frac{g' \sin^3 2\theta}{\sqrt{9g^2 + g'^2}} \right)' = 0 \quad \text{in } \left(\frac{\pi}{4}, \frac{\pi}{2}\right),$$

$$g\left(\frac{\pi}{4}\right) = 0 = g'\left(\frac{\pi}{2}\right).$$

This problem has a solution g positive in $(\frac{\pi}{4}, \frac{\pi}{2})$.

The BDG surface:



Theorem (del Pino, Kowalczyk, Wei)

Let Γ *be a BDG minimal graph in* $\mathbb{R}^9$ *and* $\Gamma_\varepsilon := \varepsilon^{-1}\Gamma$. *Then for all small* $\varepsilon > 0$, *there exists a bounded solution* u_ε *of* (AC), *monotone in the* x_9*-direction, with*

$$u_\varepsilon(x) = w(\zeta) + O(\varepsilon), \quad x = y + \zeta\nu(\varepsilon y), \quad y \in \Gamma_\varepsilon, \ |\zeta| < \frac{\delta}{\varepsilon},$$

$$\lim_{x_9 \to \pm\infty} u(x', x_9) = \pm 1 \quad \textit{for all} \quad x' \in \mathbb{R}^8.$$

u_ε is a "counterexample" to De Giorgi's conjecture in dimension 9 (hence in any dimension higher).

Sketch of the proof

Let Γ be a fixed BDG graph and let ν designate a choice of its unit normal. Local coordinates near Γ:

$$x = y + z\nu(y), \quad y \in \Gamma, \ |z| < \delta$$

Laplacian in these coordinates:

$$\boxed{\Delta_x = \partial_{zz} + \Delta_{\Gamma^z} - H_{\Gamma_z}(y)\,\partial_z}$$

$$\Gamma^z := \{y + z\nu(y) \ / \ y \in \Gamma\}.$$

Δ_{Γ^z} is the Laplace-Beltrami operator on Γ^z acting on functions of y, and $H_{\Gamma^z}(y)$ its mean curvature at the point $y + z\nu(y)$.

Let $k_1, \ldots, k_N$ denote the principal curvatures of Γ. Then

$$\boxed{H_{\Gamma^z} = \sum_{i=1}^{8} \frac{k_i}{1 - zk_i}}$$

For later reference, we expand

$$H_{\Gamma^z}(y) = H_\Gamma(y) + z\,|A_\Gamma(y)|^2 + z^2 \sum_{i=1}^{N} k_i^3 + \cdots$$

where

$$\underbrace{H_\Gamma = \sum_{i=1}^{8} k_i = 0,}_{mean\ \ curvature} \qquad \underbrace{|A_\Gamma|^2 = \sum_{i=1}^{8} k_i^2}_{norm\ \ second\ \ fundamental\ \ form} \quad .$$

Letting $f(u) = u - u^3$ the equation

$$\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^9$$

becomes, for

$$u(y,\zeta) := u(x), \quad x = y + \zeta\nu(\varepsilon y), \quad y \in \Gamma_\varepsilon,\ |\zeta| < \delta/\varepsilon,$$

ν unit normal to Γ with $\nu_N > 0$,

$$S(u) := \Delta u + f(u) \ =$$

$$\Delta_{\Gamma_\varepsilon^\zeta} u - \varepsilon H_{\Gamma^{\varepsilon\zeta}}(\varepsilon y)\,\partial_\zeta u \ + \partial_\zeta^2 u + f(u) \ = \ 0.$$

- We look for a solution of the form (near Γ_ε)

  $$u(x) = w(\zeta - \varepsilon h(\varepsilon y)) + \phi(y, \zeta - \varepsilon h(\varepsilon y)), \quad x = y + \zeta\nu(\varepsilon y)$$

  for a function h defined on Γ, left as a parameter to be adjusted and ϕ small.
- Let $r(y', y_9) = 1 + |y'|$. We assume a priori on h that

  $$\|(1+r^3)D_\Gamma^2 h\|_{L^\infty(\Gamma)} + \|(1+r^2)D_\Gamma h\|_{L^\infty(\Gamma)} + \|(1+r)h\|_{L^\infty(\Gamma)} \ \leq \ M$$

  for some large, fixed number M.

$$u(x) = w(t) + \phi(y,t), \quad x = y + (t + \varepsilon h(\varepsilon y))\nu(\varepsilon y)$$

Equation in terms of $\phi = \phi(t,y)$

$$\partial_{tt}\phi + \Delta_{\Gamma_\varepsilon}\phi + B\phi + f'(w(t))\phi + N(\phi) + E \;=\; 0.$$

where B is a small linear second order operator, and

$$E = S(w(t)), \quad N(\phi) = f(w+\phi) - f(w) - f'(w)\phi \approx f''(w)\phi^2.$$

The error of approximation.

$$E := S(w(t)) \ =$$

$$\varepsilon^4 |\nabla_{\Gamma_{\varepsilon\zeta}} h(\varepsilon y)|^2 w''(t) - [\varepsilon^3 \Delta_{\Gamma_{\varepsilon\zeta}} h(\varepsilon y) \ + \varepsilon H_{\Gamma_{\varepsilon\zeta}}(\varepsilon y)]\, w'(t),$$

and

$$\varepsilon H_{\Gamma_{\varepsilon\zeta}}(\varepsilon y) = \varepsilon^2 (t+\varepsilon h(\varepsilon y))|A_\Gamma(\varepsilon y)|^2 + \varepsilon^3 (t+\varepsilon h(\varepsilon y))^2 \sum_{i=1}^{8} k_i^3(\varepsilon y) + \cdots$$

A crucial fact: (L. Simon (1989)) $k_i = O(r^{-1})$ as $r \to +\infty$. In particular

$$|E(y,t)| \ \le \ C\varepsilon^2 r(\varepsilon y)^{-2}.$$

Equation

$$\partial_{tt}\phi + \Delta_{\Gamma_\varepsilon}\phi + B\phi + f'(w(t))\phi + N(\phi) + E \; = \; 0.$$

makes sense only for $|t| < \delta\varepsilon^{-1}$.

A **gluing procedure** reduces the full problem to

$$\boxed{\partial_{tt}\phi + \Delta_{\Gamma_\varepsilon}\phi + B\phi + f'(w)\phi + N(\phi) + E = 0 \quad \text{in } \mathbb{R}\times\Gamma_\varepsilon,}$$

where E and B are the same as before, but cut-off far away. N is modified by the addition of a small nonlocal operator of ϕ.

We find a small solution to this problem in **two steps**.

Infinite dimensional Lyapunov-Schmidt reduction:

Step 1: Given the parameter function h, find a a solution $\phi = \Phi(h)$ to the problem

$$\partial_{tt}\phi + \Delta_{\Gamma_\varepsilon}\phi + B\phi + f'(w(t))\phi + N(\phi) + E \ = \ c(y,\phi)w'(t) \quad \text{in } \mathbb{R}\times\Gamma_\varepsilon,$$

$$\int_{\mathbb{R}} \phi(t,y)w'(t)\,dt \ = \ 0 \quad \text{for all} \quad y\in\Gamma_\varepsilon.$$

$$c(y,\phi) := \frac{1}{\int_{\mathbb{R}} w'^2\,dt}\int_{\mathbb{R}}(E + B\phi + N(\phi)\,w'\,dt = 0.$$

Step 2: Find a function h such that for all $y\in\Gamma_\varepsilon$,

$$c(y,\Phi(h)) = 0.$$

For **Step 1** we solve first the linear problem

$$\partial_{tt}\phi + \Delta_{\Gamma_\varepsilon}\phi + f'(w(t))\phi = g(t,y) - c(y)w'(t) \quad \text{in } \mathbb{R}\times\Gamma_\varepsilon$$

$$\int_{\mathbb{R}} \phi(y,t)w'(t)\,dt = 0 \quad \text{in } \Gamma_\varepsilon,\ \ c(y) := \frac{\int_{\mathbb{R}} g(y,t)w'(t)\,dt}{\int_{\mathbb{R}} w'^2\,dt}.$$

There is a unique bounded solution $\phi := A(g)$ *if* g *is bounded. Moreover, for any* $\nu > 0$ *we have*

$$\|(1+r(\varepsilon y)^\nu)\phi\|_\infty \ \le\ C\,\|(1+r(\varepsilon y))^\nu g\|_\infty.$$

We write the problem of **Step 1**,

$$\partial_{tt}\phi + \Delta_{\Gamma_\varepsilon}\phi + B\phi + f'(w(t))\phi + N(\phi) + E \;=\; c(y)w'(t) \quad \text{in } \mathbb{R}\times\Gamma_\varepsilon,$$

$$\int_{\mathbb{R}} \phi(t,y)w'(t)\,dt \;=\; 0 \quad \text{for all} \quad y\in\Gamma_\varepsilon,$$

in fixed point form

$$\phi \;=\; A(B\phi + N(\phi) + E).$$

Contraction mapping principle implies the existence of a unique solution $\phi := \Phi(h)$ with

$$\|r^2(\varepsilon y)\phi\|_\infty \;=\; O(\varepsilon^2).$$

Finally, we carry out **Step 2**. We need to find h such that

$$\int_{\mathbb{R}} [E + B\Phi(h) + N(\Phi(h))]\,(\varepsilon^{-1}y, t)\, w'(t)\, dt = 0 \ \forall\, y \in \Gamma.$$

Since

$$-E(\varepsilon^{-1}y, t) = \varepsilon^2 t w'(t)\, |A_\Gamma(y)|^2 + \varepsilon^3 [\Delta_\Gamma h(y) + |A_\Gamma(y)|^2 h(y)\,]\, w'(t)$$

$$+\ \varepsilon^3 t^2 w'(t) \sum_{j=1}^{8} k_j(y)^3 + \ \textit{smaller terms}$$

the problem becomes

$$\mathcal{J}_\Gamma(h) := \Delta_\Gamma h + |A_\Gamma|^2 h \ = \ c \sum_{i=1}^{8} k_i^3 + \mathcal{N}(h) \quad \text{in } \Gamma,$$

where $\mathcal{N}(h)$ is a small operator. We solve this problem with the aid of barriers for the linear operator and a fixed point argument.

Loosely speaking: The method described above applies to find an entire solution u_ε to $\Delta u + u - u^3 = 0$ with transition set near $\Gamma_\varepsilon = \varepsilon^{-1}\Gamma$ whenever Γ is a minimal hypersurface in $\mathbb{R}^N$, that splits the space into two components, and for which enough control at infinity is present to invert globally its Jacobi operator.

An important example for $N = 3$: finite Morse index solutions.

Theorem (del Pino, Kowalczyk, Wei)

Let Γ be a complete, embedded minimal surface in $\mathbb{R}^3$ with finite total curvature: $\int_\Gamma |K| < \infty$, K Gauss curvature.

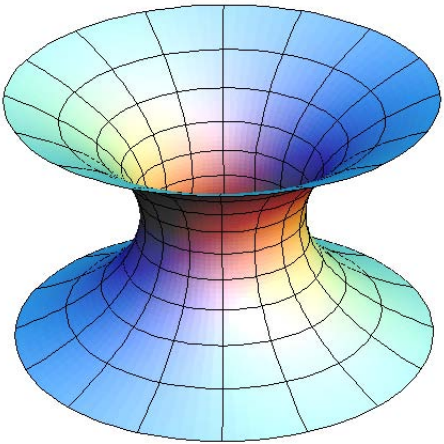
If Γ is non-degenerate, namely its bounded Jacobi fields originate only from rigid motions, then for small $\varepsilon > 0$ there is a solution u_ε to (AC) with

$$u_\varepsilon(x) \approx w(t), \quad x = y + t\nu_\varepsilon(y).$$

In addition $i(u_\varepsilon) = i(\Gamma)$ where i denotes Morse index.

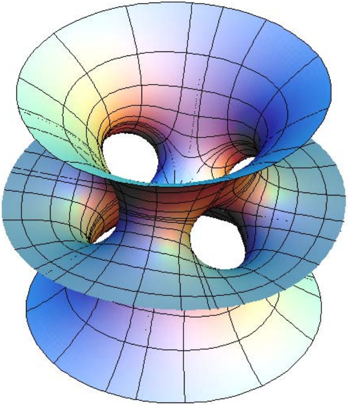
Examples: nondegeneracy and Morse index are known for the catenoid and Costa-Hoffmann-Meeks surfaces (Nayatani (1990), Morabito, (2008)).

$\Gamma =$ a catenoid: $\quad \exists\ u_\varepsilon(x) = w(\zeta) + O(\varepsilon),\ x = y + t\nu_\varepsilon(y).$



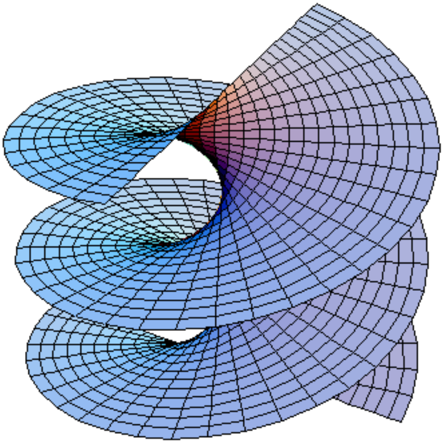
u_ε axially symmetric: $u_\varepsilon(x) = u_\varepsilon(\sqrt{x_1^2 + x_2^2}, x_3)$, x_3 rotation axis coordinate. $i(u_\varepsilon) = 1$

Γ = CHM surface genus $\ell \geq 1$:



$\exists\ u_\varepsilon(x) = w(\zeta) + O(\varepsilon)$, $x = y + \zeta\nu_\varepsilon(y)$. $i(u_\varepsilon) = 2\ell + 3$.

An example with infinite total curvature $\int_\Gamma |K| = \infty$: The **helicoid**



$$H_\lambda = \{(r\cos\theta, r\sin\theta, z) \in \mathbb{R}^3 \ / \ z = \frac{\lambda}{\pi}\theta\}$$

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Theorem (del Pino, Musso, Pacard)

• *If* $\lambda > \pi$ *There exists a solution to the Allen Cahn equation in* $\mathbb{R}^3$ *whose zero level set is exactly* H_λ

• *If* $\lambda \leq \pi$ *then any solution which vanishes on* H_λ *must be identically zero.*

The zero-level set of u: the helicoid $\boxed{z = \frac{\lambda}{\pi}\theta.}$

As $r \to +\infty$, $v(r,s) \approx w(s)$ where w is the unique solution of

$$\boxed{w'' + f(w) = 0, \quad w(\lambda) = 0 = w(0)}$$

$w \neq 0$ exists and it is unique up to translations if and only if $\lambda > \pi$.

Screw-motion invariant solutions. If $\lambda > \pi$, there exists a solution $u(r, \theta, z)$, whose zero set corresponds exactly to the helicoid $z = \frac{\lambda}{\pi}\theta$, invariant under *screw motion*:

$$u(r, \theta, z) = u(r, \theta - \alpha, z - \frac{\lambda}{\pi}\alpha) = u(r, 0, z - \frac{\lambda}{\pi}\theta) \quad \text{for all} \quad \alpha.$$

Look for

$$\boxed{u(r, \theta, z) \equiv v(r, z - \frac{\lambda}{\pi}\theta),}$$

$$v_{ss} + v_{rr} + \frac{v_r}{r} + \frac{\lambda^2}{r^2\pi^2} v_{ss} + f(v) = 0, \quad v(r, 0) = 0 = v(r, \lambda)$$

Solutions in $\mathbb{R}^3$ with nodal set with *multiple components*

Theorem (Agudelo, del Pino, Wei)

1. There exists an axially symmetric solution with nodal sets Γ_i, Γ_o *made up of two components diverging logarithmically from a largely dilated catenoid,* $\varepsilon^{-1}\Gamma_0$, *one inside, the other outside. graphs for* $r > \frac{1}{\varepsilon}$ *of functions*

$$\varphi_i(r) \sim 4\varepsilon^{-1}\log(r\varepsilon) + 2\log r, \quad \varphi_o(r) \sim 4\varepsilon^{-1}\log(r\varepsilon) - 2\log r$$

This solution has **infinite Morse index**.

2. There exists an axially symmetric solution with nodal set made up of two components $\Gamma_\pm$ *which are graphs of two functions*

$$\varphi_\pm(r) \sim \pm 2\log(1+\varepsilon r) \pm \log\frac{1}{\varepsilon} \text{as } r \to +\infty.$$

This solution has **Morse index 2.**

Another application of the BDG minimal graph: Overdetermined semilinear equation

Ω smooth domain, f Lipschitz

$$\Delta u + f(u) = 0, \;\; u > 0 \quad \text{in } \Omega, \;\; u \in L^\infty(\Omega) \qquad (S)$$

$$u = 0, \quad \partial_\nu u = constant \quad \text{on } \partial\Omega$$

Let us assume that (S) is solvable. What can we say about the geometry of Ω?

Serrin (1971) proved that if Ω is **bounded** and there is a solution to (S) then Ω must be a ball.

We consider the case of an entire *epigraph*

$$\Omega = \{(x', x_N) \; / \; x' \in \mathbb{R}^{N-1}, \; x_N > \varphi(x')\}, \quad \Gamma = \partial\Omega.$$

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- Berestycki, Caffarelli and Nirenberg (1997) proved that if φ is Lispchitz and *asymptotically flat* then it must be linear and u depends on only one variable. They asked whether this should be true for an arbitrary smooth function φ.
- Farina and Valdinoci (2009) lifted asymptotic flatness for $N = 2, 3$ and for $N = 4, 5$ and $f(u) = u - u^3$.

Theorem (del Pino, Pacard, Wei)

In Dimension $N \geq 9$ there exists a solution to Problem (S) with $f(u) = u - u^3$, in an entire epigraph Ω which is not a half-space.

The proof consists of finding the region Ω for which

$$\partial\Omega = \{y + \varepsilon h(\varepsilon y)\nu(\varepsilon y) \ / \ y \in \Gamma_\varepsilon\}.$$

for h a small decaying function on Γ, with Γ a BDG graph.
The construction carries over for more general surfaces Γ

Let us set

$$u_0(x) = w(t), \quad x = y + (t + \varepsilon h(\varepsilon y))\nu(\varepsilon y) \quad \Omega = \{t > 0\}.$$

Again for $x = y + \varepsilon(t + \varepsilon h(\varepsilon y))$, we look for a solution for $t > 0$ with $u(t, y) = w(t) + \phi(t, x)$. Then at main order ϕ should satisfy

$$\partial_{tt}\phi + \Delta_{\Gamma_\varepsilon}\phi + f'(w(t))\phi \approx E$$

$$\phi(0, y) = 0, \phi_t(0, y) \approx \alpha \quad \forall y \in \Gamma_\varepsilon$$

$$E = \Delta u_0 + f(u_0) \; =$$

$$\varepsilon^4 |\nabla_{\Gamma^{\varepsilon\zeta}} h(\varepsilon y)|^2 w''(t) - [\varepsilon^3 \Delta_{\Gamma^{\varepsilon\zeta}} h(\varepsilon y) \; + \varepsilon H_{\Gamma^{\varepsilon\zeta}}(\varepsilon y)] \, w'(t),$$

$$E = \varepsilon H_\Gamma(\varepsilon y) \, w'(t) + O(\varepsilon^2)$$

Integrating the equation for ϕ we wind

$$-w'(0)\phi_t(0,y) \approx \int_0^\infty E(y,t) w'(t) dt = -\varepsilon H_\Gamma(\varepsilon y) \int_0^\infty w'(t)^2 dt + O(\varepsilon^2)$$

We need

$$H_\Gamma \equiv H = constant$$

Namely Γ should be a constant mean curvature surface. Then we solve imposing $\alpha = \varepsilon (H/w'(0)) \int_0^\infty w'(t)^2 dt$.

Let us assume that that Γ is a smooth surface such that

$$H_\Gamma \equiv H = constant$$

The approximation can be improved as follows:

For $x = y + \varepsilon(t + \varepsilon h(\varepsilon y))$, we look now for a solution for $t > 0$ with

$$u(t, y) = w(t) + \phi(t, y), \quad \phi(0, y) = 0.$$

Imposing $\alpha = (H/w'(0)) \int_0^\infty w'(t)^2 dt$. we can solve

$$\psi'' + f'(w(t))\psi = Hw'(t), \quad t > 0, \quad \psi(0) = 0, \psi'(0) = \alpha$$

which is solvable for ψ bounded. Then the approximation $u_1(x) = w(t) + \varepsilon\psi(t)$ produces a new error of order ε^2. And the equation for $\phi = \varepsilon\psi(t) + \phi_1$ now becomes

$$\partial_{tt}\phi_1 + \Delta_{\Gamma_\varepsilon}\phi_1 + f'(w(t))\phi_1 = E_1 = O(\varepsilon^2)$$

$$\phi_1(0, y) = 0, \phi_{1,t}(0, y) = 0$$

The construction follows a scheme similar to that for the entire solution, but it is more subtle in both theories needed in Steps 1 and 2.