The non local obstacle problem: Regularity of solutions and free boundaries

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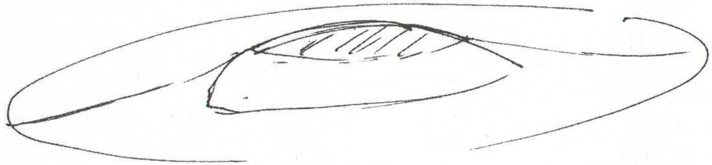
Classical Obstacle Problem

Some heuristics: Variational approach

Obstacle φ in domain $\Omega = u_0$ minimizes energy: $E = \int (\nabla u)^2$

among $u = 0$ or $\partial\Omega$, $u \geq \varphi$ in Ω



One side perturbations always allowed

$w \in C_0^\infty$, $w \geq 0$, u admissible $\quad \Longrightarrow \quad$ $u + \varepsilon w$ admissible

Therefore: - u_0 is superharmonic in Ω

- u_0 is harmonic in the "non-coincidence" set:

$$D = \{u_0 > \varphi\}.$$

In particular $\Delta u_0 = 0$ on D and $\Delta u_0 = \Delta\varphi$ at the interior, Λ^0, of the contact set Λ.

What about at detachment, $\partial\Lambda$??

Since Δu_0 is a negative measure, and at $x_0 \in \partial\Lambda$, u_0 is controlled below byφ, *no* singular part

$$\Delta u_0 = (\Delta\varphi)\chi_{u=\varphi} < 0$$

and *bounded* $\implies$ $u_0 \in W^{2,p}\ \forall\ p.$

Remark

u_0, least supersolution above φ.

Proof.

Maximum principle: If u super, $u \geq \varphi$, $u - u_0$ cannot have a negative minimum, neither in D, nor in Λ. $\square$

Conclusions

- Existence
- Considerable regularity
- Uniqueness
- Rather general operator

$$D_i a_{ij} D_j u \to \text{div } F_j(\nabla u)$$

(Regularity depending on a_{ij})

- Mainly comparison principle
- Divergence $\implies$ RHS is a distribution

Suggest non-variational approach: (Perron-“Viscosity”):

(game theory)

- Equation with comparison principle, notion of supersolution, find least supersolution above φ.
- If the operator is translation invariant

Quasiconvexity: If u_0 is the solution and φ is $C^{1,1}$, u_0 is semiconvex ($D^2u \geq -CI$).

Heuristic: From translation invariance, for any direction e, h small

$$\frac{1}{2}\Big[u(x+he)+u(x-he)\Big]$$

is a supersolution (of whatever the linear operator is). But $u \geq \varphi$, so

$$\frac{1}{2}\Big[u(x+he)+u(x-he)\Big] \geq \frac{1}{2}\Big[\varphi(x+he)+\varphi(x-he)\Big]$$

since φ is C^2,

$$\frac{1}{2}\Big[\varphi(x+he)+\varphi(x-he)-2\varphi(x)\Big] \geq -Ch^2$$

Therefore

$$\frac{1}{2}\Big[u(x+he)+u(x-he)\Big]+C|h|^2 \geq \varphi(x)$$

Since $u(x)$ is the "least supersolution" above $\varphi(x)$, and $\frac{1}{2}[u(x+h)+u(x-h)]+C|h|^2$ is a supersolution above φ (some care along $\partial\Omega$ is necessary), we have

$$\frac{1}{2}\left[u(x+h)+u(x-h)\right]+C\,|h|^2>u(x)$$

u is semiconvex.

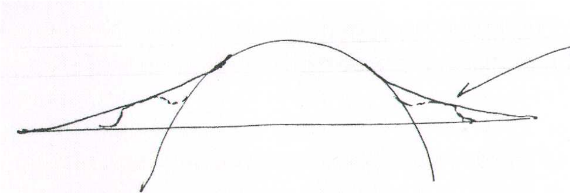
Corollary for Δ

Since $\Delta u \leq 0$ *and* $D^2u \geq -C$

$$u \ \ is \ \ C^{1,1}$$

Free boundary regularity

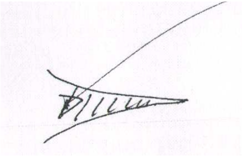
a) Need for non degeneracy:



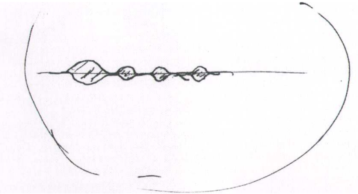
“fake free boundary”
any compact domain

Need $\Delta\varphi < 0$, or transversal transition from $\Delta\varphi > 0$ to $\Delta\varphi < 0$.

b) Still counterexamples in analytic case: Contact set (Shaefer)



C^∞ right hand side:



any compact set
on the line of
empty interior

Need smoothness of operators $\begin{cases} \text{divergence} \\ + \\ \text{non-divergence.} \end{cases}$

Obstacle problems for non-local operators

- "Surface diffusion" - (variational)
- Game theory (non-variational)
- Geometric (non-local minimal surface)

Surface diffusion

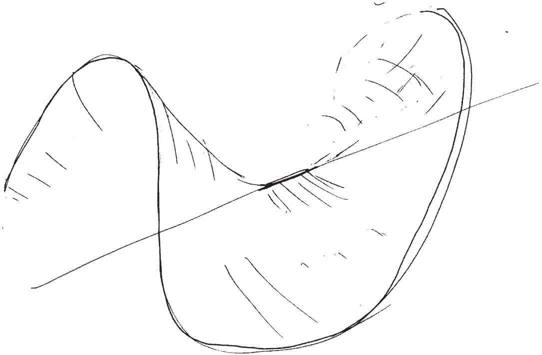
- Thin obstacle
- Semipermeable membranes
- Optimal insulation
- Signorini problem

Topics: the half Laplacian

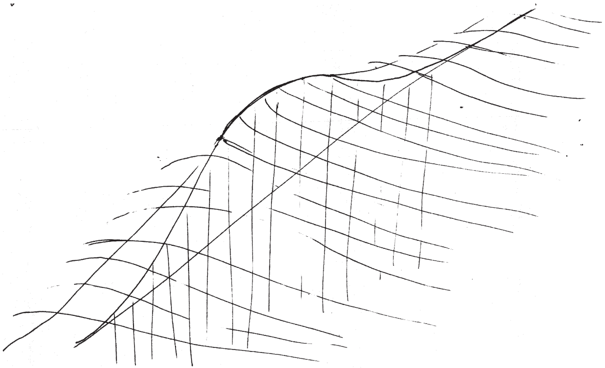
- **Some basic facts**
    - Some extra regularity for the half Laplacian
    - What should the optimal regularity be?
    - A monotonicity formula for weighted energy
    - The Almgreen frequency formula and global (blow-up) profiles
    - Cones, monotonicity and free boundary regularity
    - Fractional Laplacians for all values between zero and one
    - An extension property
    - Basic changes for optimal and free boundary regularity

Thin obstacle: Two versions

"Potato chip geometry"



Global geometry



From global geometry we see:

This is a non-local diffusion problem:

We minimize in $(\mathbb{R}^{n+1})^+$ (we denote points by (x', y)):

$$D(u) = \int\!\!\int (\nabla u)^2 \, dx \, dy$$

among all u's such that $u(x, 0) \geq \varphi(x)$ (or in $\mathbb{R}^{n+1}$ for u symmetric).

But $u(x,y)$ is just the harmonic extension of $u^*(x) = u(x,0)$, so we can write:

$$Du = \iint_{(\mathbb{R}^{n+1})^+} \big(\nabla u(x,y)\big)^2 \, dx\, dy = -\int_{\mathbb{R}^n} u(x',0) u_y(x',0)\, dx'$$

To write $u_y(x',0)$ in terms of $u(x',0)$, we write

$$u(x',h) = C \int u(x'+z',0) \underbrace{\frac{h}{\left((z')^2+h^2\right)^{\frac{n+1}{2}}}}_{\text{Poisson kernel, with } \int = 1}$$

So

$$\frac{u(x',h) - u(x',0)}{h} = \frac{C}{\cancel{h}} \int \left[u(x'+z,0) - u(x',0)\right] \frac{\cancel{h}}{\left((z')^2 + h^2\right)^{\frac{n+1}{2}}}$$

$\Longrightarrow$

$$u_y(x',0) = C \int \underbrace{\begin{array}{ccc} u(x'+z,0) & - & u(x',0) \\ \| & & \| \\ u^*(x'+z) & - & u^*(x') \end{array}}_{\Delta^{1/2} u^*} \cdot \frac{1}{|x|^{n+1}}$$

The Dirichlet energy becomes:

$$\begin{aligned} D(u) &= \int dx' \int u^*(x') \left[u^*(x'+z) - u^*(x')\right] |z|^{-n+1} dz \\ \text{or} \quad &= \int dz'\, dz \iint \left[u^*(x'+z) - u^*(x')\right]^2 |z|^{-n+1} \quad (H^{1/2} \text{ norm}) \end{aligned}$$

The Dirichlet energy of the extension $u(x,y)$

equals

The $H^{1/2}$ norm (square) of the trace $u^*(x) = u(x,0) = u(x,0)$

and

The Euler Lagrange equation for the obstacle problem: $u^* \geq \varphi$,

$$\Delta u(x,y) \leq 0 \quad , \quad \Delta u(x,y) = 0$$

when $u > \varphi$ (and outside $y = 0$)

Becomes $u^* \geq \varphi$,

$$u_\nu(x,0) = \Delta^{1/2} u^*(x) \leq 0 \ ,$$

and

$$u_\nu(x,0) = 0 \quad \text{when} \quad u^* > \varphi$$

That is an obstacle problem involving the "half Laplacian", a non-local operator operator.

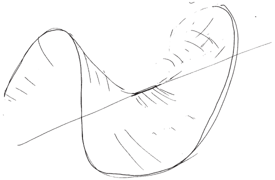
In the case of the "potato chip" configuration, the half Laplacian is not exactly u_y, but the operators have the same singularity. They basically differ from their behavior at infinity.

We will concentrate on the “potato chip” configuration.

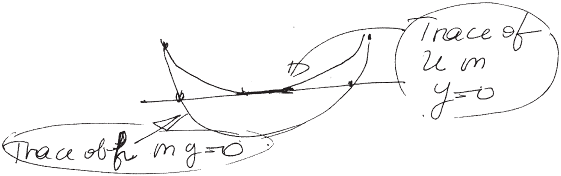
Basic properties:

- Localization of the contact set
- Semiconvexity
- u^* is Lipschitz and u_y is bounded

Recall the Potato Chip Configuration:



Localization: Simply compare with the harmonic function h of same boundary data



Quasiconvexity in x (for *all* (x, y) in Ω, not just $y = 0$.

- $u(x, y)$ has a C^2 extension near $\partial\Omega$.
- Therefore $\frac{1}{2}\left[u(x+he, y) + u(x-he, y)\right] + C|h|^2$

is an *admissible* supersolution.

It follows that

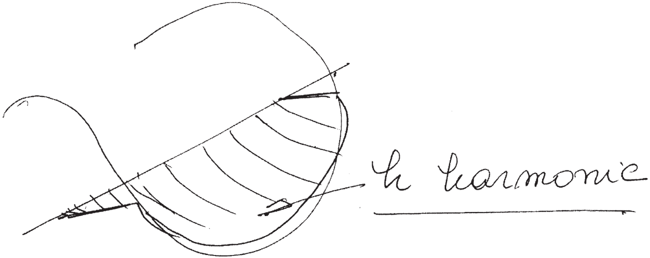
$$\frac{1}{2}\frac{[u(x+he) + u(x-he) - 2u(x)]}{h^2} \geq -C$$

Corollary

Also, $u_{yy} \leq -C$.

u is also Lipschitz in x, but that does not imply boundness of u_y.

Boundness of u_y follows from a lower barrier.



Also using a “thick obstacle”.

Is the the optimal regularity we may expect?

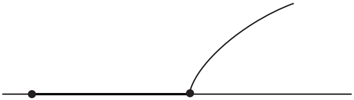
On one hand, we may be tempted to say:

As for the second order case $\Delta^{1/2}u = \Delta^{1/2}\chi_{u=\varphi}$

(in the global configuration).

Thus: $\Delta^{1/2}u$ is discontinuous.

On the other hand, a closer inspection indicates that, being semiconvex, if u is not C^1 at detachment: it would form a corner:



i.e., $$u^*(x) \geq Cx^+ - O(x^+)^2$$

Let us assume that u^* is *actually* convex.

We can go for (Hölder) continuity of u_x or u_y.

(Hölder because the problem rescales, and any gain of oscillation will iterate.)

But u_y vanishes on the *exterior* of a convex set.

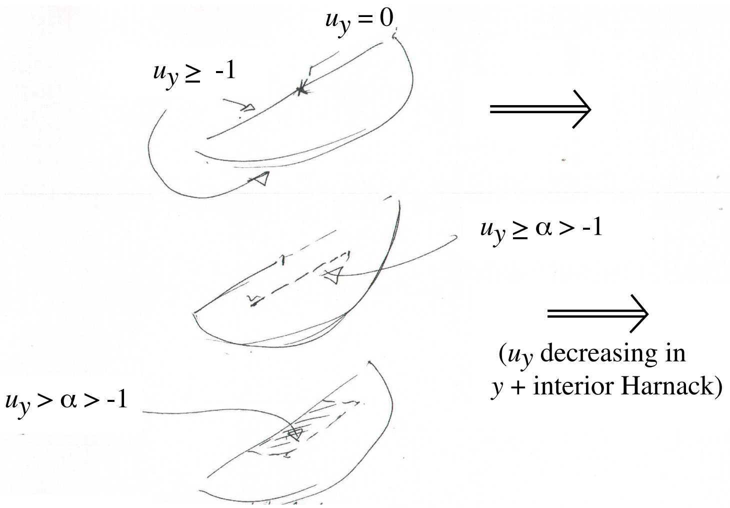
While u_x in the complement of a *concave set*:

So u_y is more promising.

Let (O, o) be a free boundary point, and assume that

$$-1 \leq u_y \ \ (\leq 0) \ \text{ in } \ B_1^+$$

Then we have the following configuration:

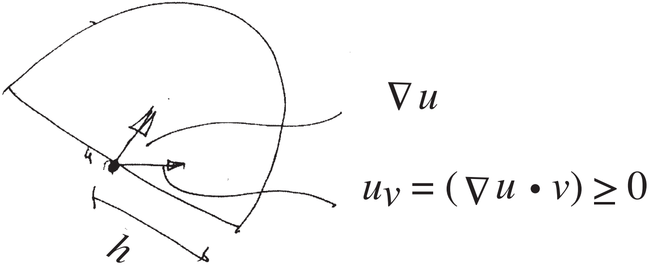


Iteration $\Longrightarrow$ Holder

For the non-convex (semiconvex) case, we need two observations:

- Half ball condition: u semiconvex, $u(x_0) = h^2$,

  $\exists$ a half ball of radius $\sim h$, where $u \gtrsim ch^2$



On the other hand, u is quasiconcave in y: $(u_y - y)$ is monotone decreasing in y. (Note: renormalize)

Lemma

If $(0,0)$ is now a free boundary point, and $\tilde{h}$ is the (sub) harmonic function $-x^2 + Cy^2$, then given a cylinder, $B_r \times [0,r] = \Gamma_r(0,0)$.

The function $g(x,y) = u(x,y) + \tilde{h}(x,y)$ attains a positive maximum on the "lateral boundary": ∂B_r or $y = r$.

With this lemma at hand, we have two cases:

Case a): The maximum is attained at $y = 0$.

Then, from the half ball condition $u^*(x) > 0$ in half a ball of radius $\sim r$ centered at distance r^2 from zero.

In such a half ball, $u_y = 0$.

Case b): The half ball $\bar{B}_r$ (in x) is centered at a point (x_0, y_0) with $0 < y_0 < r^2$ and there, $u(x, y_0) \geq -Cr^2$. If we now go from (x, y_0) to $(x, 0)$, we have

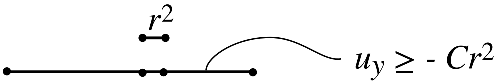
b$_1$) $\quad u(x, y_0) > 0$, and $u_y^*(x) = 0$

or

b$_2$) $\quad u(x, 0) = 0, \quad -Cr^2 \leq u(x, y_0) -$

$$-u(x, 0) = \int_0^r u_y \leq u_y(x, 0) + Cr^2.$$

In any case in B_r we have the picture



Since we are shooting for $u_y^* \geq -Cr^\alpha$ in the cylinder Γ_r of size r:

$B_r \times [0, r]$.

The term $-Cr^2$ should be easily absorbed:

We make a first dilation that makes u "almost convex": $u_{xx}^* \geq -\varepsilon_0$,

$u_{yy} \leq \varepsilon_0$, $u_y^* \geq -\varepsilon_0 r^2$ on the half ball centered at (x_0, y_0).

If then $u_y\big|_{\Gamma_k} \geq -r_k^\alpha \quad (\alpha \text{ small})$

$$\Gamma_k = B_{r_k} \times [0, r_k]\,, \qquad r_k = 2^{-k}\,,$$

we renormalize the picture to Γ_1, u to $w_k = \dfrac{1}{(r_k)^\alpha} u(r_k x, r_k y) + \varepsilon_0 r_k^{1-\alpha}$

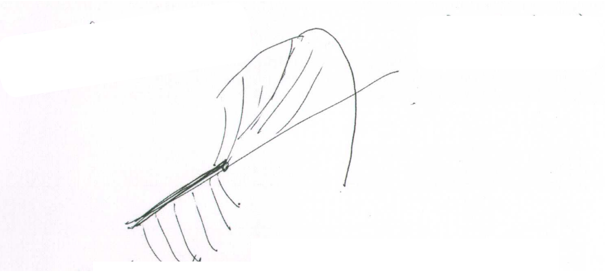
and obtain

$$u_y\big|_{\Gamma_{k+1}} \geq -(r_{k+1})^\alpha$$

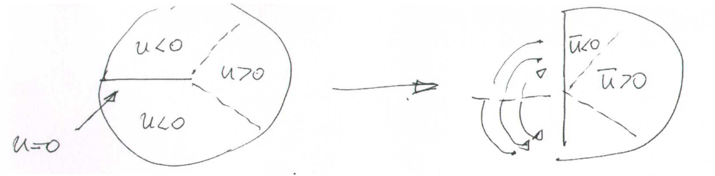
Optimal regularity

What should be the optimal regularity of the solution u^*?

In 2-d, assume we have the local configuration for $u(x, y)$.



If we make the change of variables $z = w^2$, we get



$$\bar{u}(\bar{x},\bar{u}) = y^k + P_k + \bar{v}\,, \qquad \bar{v} = O(r^{k+1})$$

That is $$u(x,y) = \begin{cases} r^{3/2}\cos\frac{3}{2}\theta \\ r^2\cos 2\theta \\ \text{higher order} \end{cases}$$

Optimal regularity $C^{1,\frac{1}{2}}$

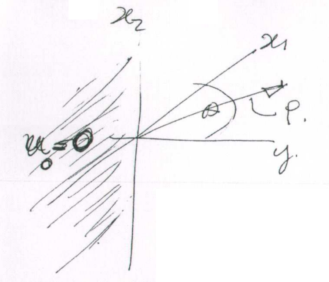
(In 2-d: Hans Levy-Richardson)

Theorem 1

$u(x,y)$ is $C^{1,\frac{1}{2}}$ (in R^{n+1})

If we make a blow up (a sequence of dilations) with homogeneity below quadratic (in particular of order $3/2$) semiconvexity becomes convexity.

We assume $u(x,y)$ convex and (in x), bounded in B_1. We expect that the critical profile for regularity, in some system of coordinates should be $r^{3/2}\cos\frac{3}{2}\theta$,



and we would like to have a measure of that "extremality" through a monotonicity formula that estimates the deviation of u from that profile.

Again, from u, u_x and u_y, the natural choice is $u_y := w$. It has a sign (so we can expect $w_0 = (u_0)_y$ to be an extremal).

And for a convex function $u(x,y)$, the support of $u_y = w$ is convex, so we may expect w to decrease from its extremal limit w_0, when we expand away from the origin. A natural candidate then becomes the weighted energy,

$$J(R) = \frac{1}{R} \iint\limits_{B_R} \frac{(\nabla w)^2}{r^{n-2}}\, dx, y$$

Indeed, $J(r)$ is constant in w_0, and as the (convex) support of w shrinks away from the half space, we can expect the "w-energy" to decrease (Lemma: $J(r) \nearrow r$).

Lets compute

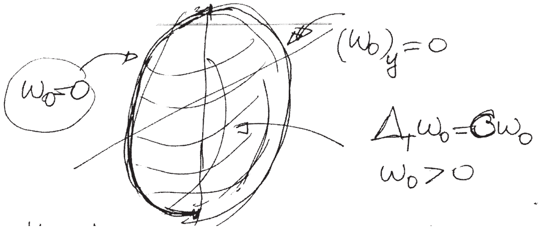
$$J'(r)\big|_{r=1} = -\int_{B_1} \frac{|\nabla w|^2}{r^{n-2}} + \int_{\partial B_1} |\nabla w|^2 = \text{ Bad + Good}$$

At this point we note that $w_0 = r^{1/2} \cos \theta/2$ is extremal for the following problem:

Minimize the Raleigh quotient

$$\int_{S_1^+} (\nabla_T w)^2 \, dA \Big/ \int_{S_1^+} w^2 \, dA$$

among all those functions in the half unit sphere that vanish in at least half of the sphere.



w_0 is the first eigenfunction for this problem.

The next computation is *exact* for w_0: $\quad (\Delta \frac{w^2}{2} = (\nabla w)^2$

$$\iint\limits_{B_1^+} \frac{(\nabla w)^2}{r^{n-2}}\, dx, y = \int\limits_{\partial B_1^+} w w_\nu + \frac{(n-2)}{2} w^2$$

$$\leq \int_\partial \frac{w^2}{4} + w_\nu^2 + \frac{(n-2)}{w^2} \qquad \text{(exact for } w_0\text{)}$$

While the good term: $\int -\partial B_1 (\nabla w)^2$ is simply $\int_{\partial B_1} (w_\nu)^2 + (\nabla_t w)^2$

For the Good to control the Bad, we need

$$\int_{\partial B_1} (\nabla_T w)^2 \text{ to be } \geq \text{ than } \left(\int w^2 \right) \cdot \frac{2n-3}{4}$$

But the computation is exact for w_0, so for any other admissible w, the inequality holds. (The semiconvex case needs an approximation argument and a correction.)

Corollary

a) $w(x) \leq d^{1/2}(x, \partial\Lambda)$

b) w *is* $C^{1/2}$ – *(a+ apriori estimates)*

Proof.

If $0 \in \partial\Lambda$, we have

1) $J(r) \leq J(1) \leq C_0$, thus
2) $\fint_{B_R} (\nabla w)^2 \leq \frac{1}{R^2} \int_{B_R} \frac{(\nabla w)^2}{r^{n-2}} \leq \frac{1}{R} C_0$
3) From Poincare (w vanishes in at least half a ball for $y = 0$),
$$\fint_{B_R} w^2 \leq R$$
4) w^2 is subharmonic, so $w^2\big|_{B_{R/2}} \leq CR$.

Frequency formula and global profiles:

We start this section with a formula due to Almgreen, that quantifies how much a harmonic function deviates from begin homogeneous when we expand away from the origin.

Theorem 2

Let $u(x,y)$ be a harmonic function (or a solution to our problem with $0 \in \lambda$). Then,

a) $H(R) = R \int_{B_R} (\nabla u)^2 \Big/ \int_{\partial B_R} u^2 \, dA$ *is monotone in R.*

b) $H' = 0$ *if and only if u is homogeneous of degree t. In that case* $H \equiv t$.

Some preliminaries:

1) In our case we may substitute B_R by B_{R+}, also $uu_y = 0$ on $\partial\Lambda$

2) $\displaystyle\int_{B_R} (\nabla u)^2 = \int_{\partial B_r} uu_\nu$ since $\displaystyle\Delta \frac{u^2}{2} = u\Delta u + (\nabla u)^2$

We compute:

$$\log H^1(R)\Big|_{R=1} = \frac{1 + \displaystyle\int_{\partial B_r} (\nabla u)^2}{\displaystyle\int_{B_1} (\nabla u)^2} + \frac{\displaystyle\int_\partial uu_r + (n-1)\int_\partial u^2}{\displaystyle\int_\partial u^2}$$

$$= [-n+2] + \frac{\displaystyle\int (u_\nu)^2 + (\nabla_T u)^2}{\displaystyle\int uu_\nu} + \frac{\displaystyle\int uu_\nu}{\displaystyle\int u^2}$$

We must get rid of $(\nabla_T u)^2$.

We use the vector field $= \vec{v} = (\nabla u)^2 x - 2\langle x, \nabla u\rangle \cdot \nabla u$

$$\int_{B_1} \operatorname{div} \vec{v} = \int_{\partial B_1} \vec{v} \cdot \nu = \int_{\partial B_1} (\nabla u)^2 - 2(u_\nu)^2$$

or

$$\int_{\partial B_1} (u_\nu)^2 + (\nabla_T u)^2 = 2(u_\nu)^2 + \int_{B_1} \operatorname{div} \vec{v}$$

But

$$\mathrm{div}\, v = \mathrm{div}\, |\nabla u|^2\, x - \langle x, \nabla u\rangle\, \nabla u =$$

$$n|\nabla u|^2 + \cancel{2u_i\, u_{ij}\, x_j} - 2\, \cancel{\langle x, \nabla u\rangle} - |\nabla u|^2$$

$$- \cancel{2x_i\, u_{ij}\, u_j} = (n\text{-}2)\, |\nabla u|^2$$

So

$$\int_{\partial B_1} (u_\nu)^2 + (\nabla_T u)^2 = 2 \int (u_\nu)^2 + (n-2)|\nabla u|^2$$

We substitute, recalling that $\int_B (\nabla u)^2 = \int_{\partial B_1} u u_\nu$

$$\log H^2(R)\Big|_{R=1} = \frac{2\int (u_\nu)^2}{\int u u_\nu} - \frac{2\int u u_\nu}{\int u^2} \geq 0\ ?$$

Just Hölder

$$\left(\int u_\nu^2\right)\left(\int u^2\right) \geq \left[\int (u u_\nu)\right]^2$$

with equality if $u_\nu = \lambda u$.

If we review the calculation for our (free boundary) configuration, we realize that the terms that come from the differentiation of J, divide in those "area terms" that come from the radial variation:

like $\int_{\partial B_1} (u_\nu)^2 + (\nabla_T u)^2$, that only lives on ∂B_1^+ in our case (does not involve the "flat part" $y = 0$) and $\int_{\partial B_1^+} u u_\nu$ as part $D_r \int u^2$, and the rest of the terms involve $u u_\nu$ that has a uniform order of vanishing, $y^{1/2}$ when y goes to zero.

Free Boundary Regularity

We are now ready to consider the issue of free boundary regularity.

As in the second case, there are stable free boundary points and unstable ones, as the one dimensional case shows:

The $r^{1/2} \cos \frac{1}{2}\theta$ configuration, being the lowest one, seems to be stable, while all other ones would change a lot under small perturbations.

Let us try to make this statement more precise.

Some observations

- The frequency formula is “semicontinuous” on the point:

If at x_0, $\lim_{r \to 0} H(r) = D(x_0)$, that means that for some r_0,

$H(r_0) \leq D(x_0) + \varepsilon$, but $H(r_0)$ is continuous on the point x_0.

Therefore, for $x \to x_0$, $\overline{\lim}\, D(x) \leq D(x_0)$.

- The case of harmonic functions

$D(0)$ is an integer (the degree of the first non zero polynomials), so $D(x) \leq D(x_0) = n$ for x in a full neighborhood of x_0, and if $D(x_0) = 1$, $D(x) = 1$ nearby (i.e. linear behavior is stable for harmonics).

In our case, then, we would like to prove that

- The possible limits $D(x_0)$ consist of $D^0 = 3/2$, corresponding to the basic profile u_0.

$D^1 = 2$, for quadratic polynomials and all other $D > 2$.

This will produce a gap, that will make D^0 points stable.

At every free boundary point, there is a limiting blow up profile

- Let $D(x_0) = \lim\limits^{\searrow} H(r)$, the homogeneity of u at the origin, and

$$I(r) = r^{-2D_0} \fint_{\partial B_r} u^2 \ ,$$

then

$$F'(r) = r^{-2D_0} \left(2 \fint_{\partial B_r} u u_r - \frac{2D_0}{r} \fint_{\partial B_r} u^2 \right) \geq 0$$

(from Frequency)

Thus, $F(r)$ has a limit as $r \to 0$.

Further

$$F'(r) = \frac{F(r)}{r}\Big(H(r) - H(0)\Big) \Longrightarrow$$
$$(\log F)' = \frac{1}{r}\Big(H(r) - H(0)\Big) \leq \frac{\varphi_0}{r} \quad \text{for } r \text{ small}$$

Then, for $0 < h < s < \varepsilon_0$, we have

$$1 \leq \frac{F(s)}{F(h)} = \left(\frac{h}{s}\right)^{2D(x_0)} \frac{\displaystyle\fint_{B_s} u^2}{\displaystyle\fint_{B_h} u^2} \leq \left(\frac{s}{h}\right)^{\varepsilon_0}$$

It follows that:

If we take a sequence of dilations of u, around a free boundary point

$$u_k = \mu_k u(\lambda_k x)$$

renormalized so that $\int_{\partial B_1} (u_k)^2 = 1$ there is a subsequence, that converges uniformly (locally) to a global solution, homogeneous of degree $D(x_0)$.

Classification of global cones for $0 < D_0 \leq 2$

It follows, from optimal regularity that $D_0 = D(x_0) \geq 3/2$.

If $D_0 < 2$, since

$$\fint_{\partial B_s} u^2 \Big/ \fint_{\partial B_h} u^2 \leq \left[\left(\frac{s}{h}\right)^2\right]^{2-\varepsilon}$$

the sequence of normalizations has homogeneity less than 2, i.e.,

$$\mu_k < \lambda_k^{-2}$$

It follows that the **limiting global profile** is **convex in x**

Lemma

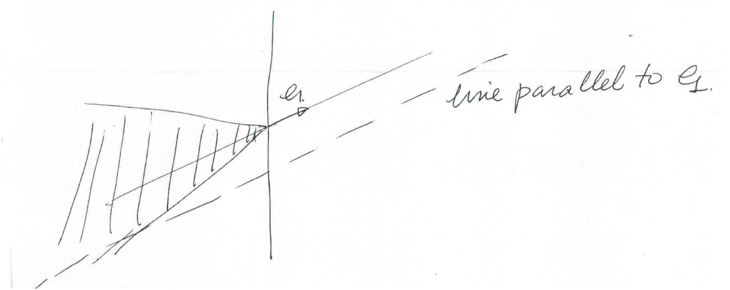
In some coordinates the limiting profile is

$$u_0 = r^{3/2} \cos \frac{3}{2}\theta$$

in polar coordinates in x_1 and y.

Proof: In $R^n = (x, 0)$, u_0 must vanish in a cone with non empty interior. If not, it would be harmonic across.

To fix ideas, say that the line in the direction of e_1 becomes interior to the cone



Along any line parallel to e_1, u is convex, and becomes negative at te_1 for t negative enough.

It follows that $D_{x_1} u$ is

- non negative harmonic and homogeneous in the complement of the cone.

But the homogeneity of a non negative (homogeneous) harmonic function is monotone on the cone (the larger the *complement* the higher the homogeneity).

Therefore, the homogeneity of u would go below 3/2, unless the cone is a half plane and $u = u_0$.

Case $D_0 = 2$

u should be harmonic across and a quadratic polynomial.

If not, as before, u will be zero in a nontrivial cone of R^n but directional derivatives $D_{x_1} u$ must change sign.

In order to have linear growth the positivity and negativity cones of $D_{x_1} u$ must be spherical cups and $(D_{x_1} u)^{+/-}$ linear functions, i.e., Cy^+, Cy^- — a contradiction.

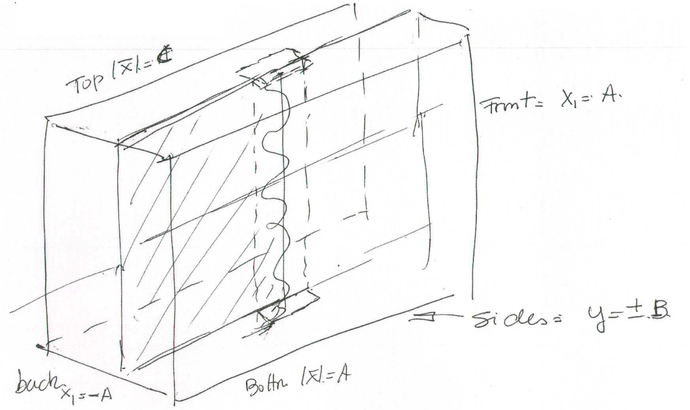
We are now ready to prove that the free boundary is a $C^{1,\alpha}$ surface in a neighborhood of a point with homogeneity $3/2$.

The proof divides in two parts.

- If we know that in a neighborhood of the free boundary point $(0,0)$, there is a cone of non negative directional derivatives $D_\tau u$, with $\langle \tau, e\rangle \geq 1-\delta$ (i.e., u is monotone in any line with direction τ), the F.B. is Lipschitz and an application of the boundary Harnack inequality to the quotient of directional derivatives implies the regularity.
- The monotonicity of u on those lines is an application of a "barrier" argument.

Let's do first the barrier argument.

From the blow up theorem, at a 3/2 point after a blow up (large dilation) our solution gets very close to a u_0 profile



Consider a directional derivative $D_e u$, with e close to e_1.

Since u is very close to the front $u_0 = r^{3/2} \cos 3/2\ \theta$, $D_e u$ is very close to $D_e u_0 = D_{e_1} u_0 = r^{1/2} \cos 1/2\ \theta$.

In fact $D_{e_1} u$ can become negative only in a small tubular neighborhood of the front.

Assume that $D_e u(x^*, y^*) < 0$, with $y^* \leq \delta$, $x_1^* \leq \delta$ and $|x^*| < C/2$.

We use the strict positivity of $D_e u$ on the sides: $y = \pm B$, to get a contradiction. There, $D_e u \sim y^{1/2} \sim 1$.

Let

$$g(x, y) = D_e u(x, y) + \theta\left[(x - x^*)^2 - ny^2\right]$$

with $\delta \ll \theta \ll 1$.

$g(x^*, y^*) \leq 0$, subharmonic outside Λ, and positive on Λ.

It must be negative on the boundary of the box.

But on the front and back, $(y = \pm B)$, u beats $-\theta y^2$, and on the sides $\theta(x - x^*)^2$ beats $D_e u$ in the small tubular regions.

(We use $D_e u$ is $C^{1/2}$.)

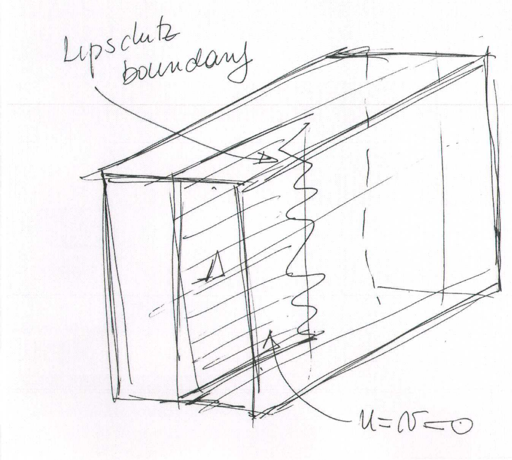
The Boundary Harnack inequality

Let $u, v > 0$ and

harmonic in the box Λ.

$u = v$ on Λ. Then $\dfrac{u}{v}$ is a

Hölder function *up to* Λ.



We apply the Harnack boundary principle to $\dfrac{D_e}{D_{e_1}}$ for e in the cone $\langle e, e_1 \rangle \geq 1 - \sigma$.

Then $\dfrac{D_e}{D_{e_1}}$ is C^α, but given e_2, we can consider $e = \sigma e_2 + e_1$.

Then $\dfrac{\sigma D_{e_2} + D_{e_1}}{D_{e_1}}$ is $C^\alpha \Longrightarrow \dfrac{D_{e_2}}{D_{e_1}}$ is C^α

All level surfaces of u are $C^{1,\alpha}$ uniformly up to Λ

Some facts about $\Delta^s u$: $\quad 0 < s < 1$

a) $\Delta^s u(x) = (1-s)\int \left[u(x+z) - u(x)\right]\, |z|^{-(n+2s)}$

b) $\widehat{(\Delta^s u)}(\zeta) = -|\zeta|^{2s}\widehat{u}(\zeta)$

c) Euler-Lagrange of $\iint \left[u(z) - u(x)\right]^2 |x-z|^{-(n+2s)}$

the H^s-norm of u ("s-derivatives in L^2")

$$= \int |\zeta|^{2s} |\widehat{u}(\zeta)|^2 \, d\zeta$$

d) ($\Delta^s u$ is a stable process).

Note that we realized

$\Delta^{1/2}u^*$ as the normal derivative of the harmonic extension of u^* to the upper half space.

Ansatz For any $0 < s < 1$

$\Delta^s u^*$ should be realized in some way as the normal derivative for an upper half space extension of u^* by an operator of the form

$$\frac{1}{y^a}\operatorname*{div}_{x,y} y^a \nabla_{x,y} u = 0$$

Heuristic reason:

The equation $y^{-a}\operatorname{div} y^a \nabla u = \Delta_{x,y} u + \frac{a}{y} D_y u$ is invariant under dilations in the x, y space (and translation invariant).

If, as before, we represent $u(0,h) - u(0,0)$ by

$$u(0,h) - u(0,0) = \int \left[u(z,0) - u(0,0)\right] P(h,z)$$

$P(h,z)$ will have a scaling of the form

$$h^\gamma P(1,\, z/h) \;\text{ for some }\; \gamma(a)$$

Therefore

$$\frac{u(0,h)-u(0,0)}{h} = h^{\gamma-1} \int \big[u(z,0)-u(0,0)\big] P(1,\, z/h)\, dz$$

We also expect $P(1,z^*)$ to have a power decay at ∞:

$(P(1,z^*) = |z^*|^{-\beta})$

Therefore $\quad P(1,\, z/h) \to h^{\beta}|z|^{-\beta}$, $(\beta = n+2s)$.

In other words, for some appropriate power α,

$$\frac{u(0,h)-u(0,0)}{h^{\alpha}} \to \Delta^s u^* \ \text{ as } \ h \to 0$$

How to calculate the relation $a \leftrightarrow s$, and what is good about this?

If $a = 1$, the equation

$$\Delta_x u + \frac{1}{y} u_y = \Delta_x u + \text{``}u_{rr} + \frac{1}{r} u_r\text{''} \qquad (y = r)$$

corresponds to harmonic functions in R^{n+2} radially symmetric in the "last two coordinates y_1, y_2" that then becomes "r".
In particular, u is, say, a bounded function "R^{n+2}", harmonic except in the "R" subspace ("$r = 0$"), it is "harmonic across".

Therefore, we cannot expect to prescribe data for $a = 1$ (or $a > 1$).

For $a = -1$, the equation

$$\Delta_x u + \left(u_{rr} - \frac{1}{r} u_r \right)$$

correspond to the "stream" function, "harmonic conjugate" of

$a = -1$, so our range of a must be limited to $-1 < a < 1$.

On the other hand, for this range the weight "y^a" is an Δ_p weight and work of Jerison, Fabes, Kenig, Serapioni shows regularity of solutions up to $y = 0$ for the Dirichlet problem, interior and boundary Harnack, etc.

Important heuristic interpretation:

Following the integer case, we can formally think on the extension $L_q = \frac{1}{y^a} \operatorname{div} y^a \nabla u$, as a “harmonic extension” of $u*$ into a fractional dimension: $n + (1 + a)$.

If we do that, we can guess many different formulas; for instance

- The fundamental solution at the origin has the form $(X = (x, y))$:

$$V = C|X|^{-(n+(1+a)-2)}$$

- The Poisson kernel:

$$P(x,y) = \frac{C\, y^{1-a}}{(|x|^2 + y^2)^{n+(1-a)/2}}$$

In particular

$$u(0,y) - u(0,0) = C \int \frac{[u(x,0) - u(0,0)]y^{1-a}}{(|x|^2 + y^2)^{n+(1-a)/2}}\, dx$$

and

$$\lim_{y\to 0} \left[\frac{u(0,y) - u(0,0)}{y}\right] y^a = \Delta^{1-a/2} u^*$$

- The Dirichlet integral

$$\iint\limits_{B_r(0)} (\nabla_{x,y} u)^2$$

becomes

$$\iint\limits_{B_r(0)} (\nabla_{x,y} u)^2 y^a \, dx \, dy$$

and

$$\iint\limits_{\partial B_r(0)} u^2 \, dA(x,y) \ , \qquad \iint\limits_{\partial B_r(0)} u^2 y^a \, dx \, dy$$