Reaction-diffusion equations and minimal surfaces

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- The Simons cone
- Saddle-shaped solutions

$\Longrightarrow$ Weights from axial symmetry: $s^{m-1} t^{k-1} ds\, dt$

2 **Antisymmetry for problems with weights**

3 **The explosion problem in axially symmetric domains**

4 **Isoperimetric inequalities with homogeneous weights**

Minimal surfaces

Thm [Simons] $E \subset \mathbb{R}^n$ of minimal perimeter.
If $n \leq 7$ then $\partial E =$ hyperplane.

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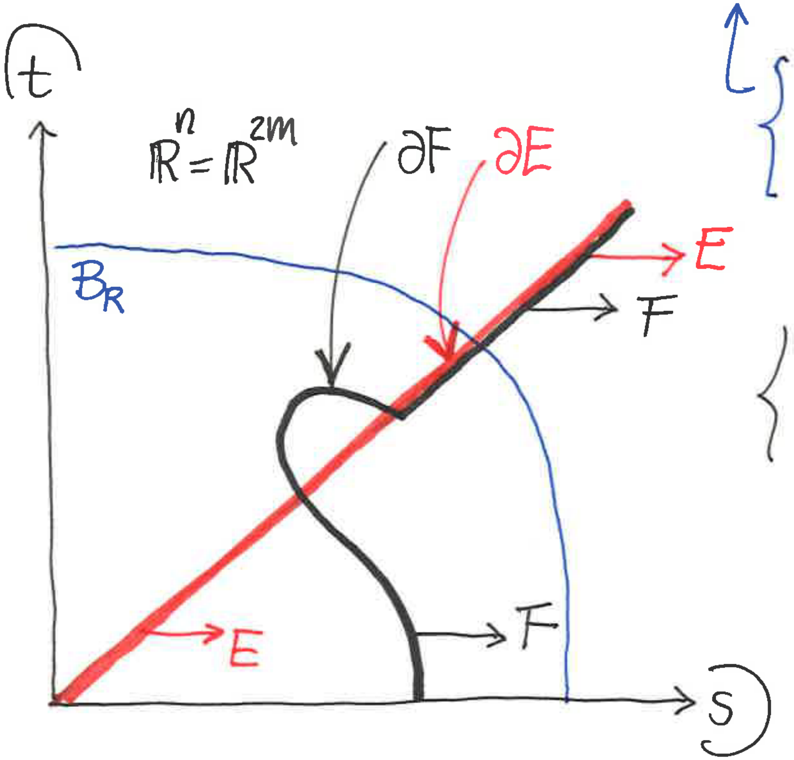

$\uparrow$ $\left\{ \begin{array}{l} \text{Area}(B_R \cap \partial E) \\ \leq \text{Area}(B_R \cap \partial F) \end{array} \right.$

$\left\{ \begin{array}{l} s = \sqrt{x_1^2 + \cdots + x_m^2} \\ t = \sqrt{x_{m+1}^2 + \cdots + x_{2m}^2} \end{array} \right.$

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$(E = \{ s > t \})$

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$\forall n = 2m$ stationary (mean curv $= 0$)

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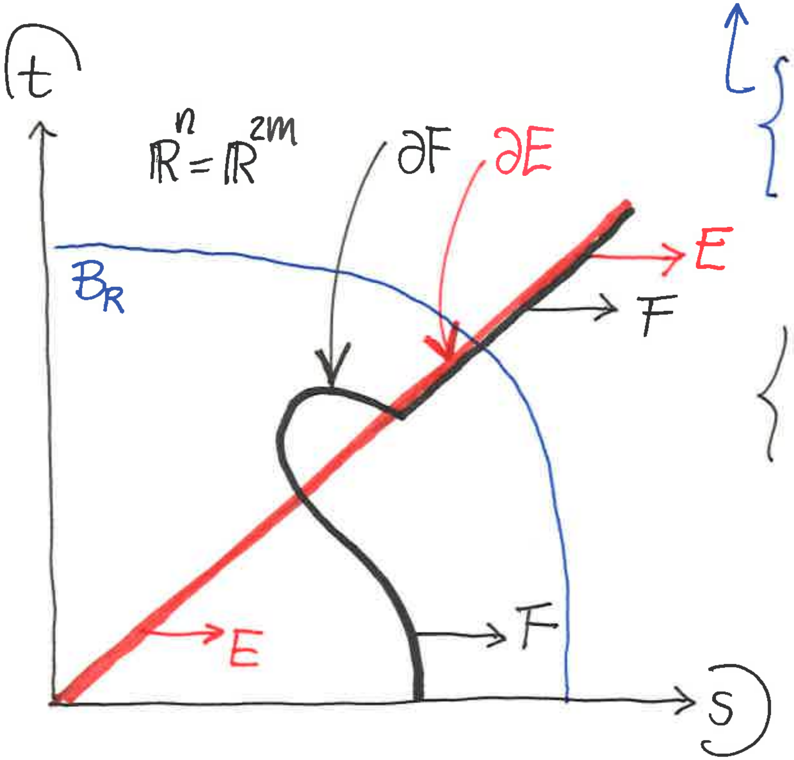

$$\mathrm{Area}(B_R \cap \partial E) \leq \mathrm{Area}(B_R \cap \partial F)$$

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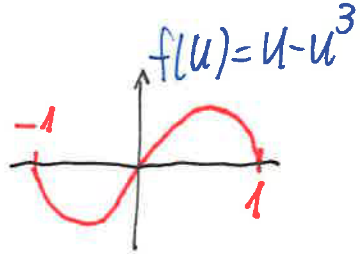
$\forall n = 2m$ stationary
(mean curv $= 0$)

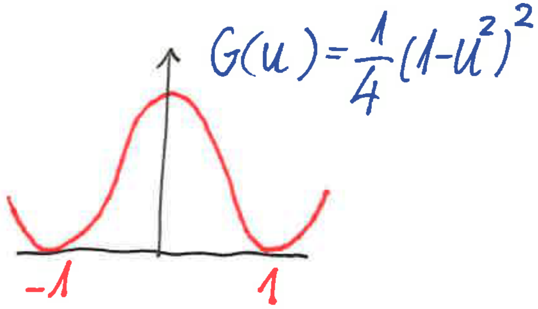
Thm [Bombieri-DeGiorgi-Giusti]
Simons cone $\mathcal{C} \subset \mathbb{R}^{2m}$ minimal $\iff$ **$2m \geq 8$**.

Allen-Cahn equation. A conjecture of De Giorgi

$$\text{(AC)} \qquad -\Delta u = \underline{u - u^3} \quad \text{in } \mathbb{R}^n$$

$$\hookrightarrow \quad E_{B_R}(u) = \int_{B_R} \frac{1}{2}|\nabla u|^2 + \underline{G(u)}$$

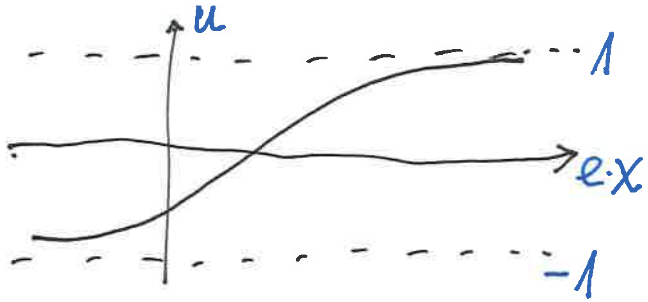




Thm [Savin '03]

u global minimizer of (AC) in $\mathbb{R}^n$. If $\mathbf{n \leq 7}$, then u is **1-D**, i.e., $\{u = \lambda\}$ = hyperplanes.

$$\boxed{u(x) = \tanh\left(\frac{e\cdot x + c}{\sqrt{2}}\right)} \quad , x\in\mathbb{R}^n;\ e\in\mathbb{R}^n,\ |e|=1,\ c\in\mathbb{R}$$



$\hookrightarrow$ is a 1D solution of AC: $-\Delta u = u - u^3$ in $\mathbb{R}^n$

& $\partial_e u = \nabla u \cdot e > 0$

In fact, it is a global minimizer, SINCE:

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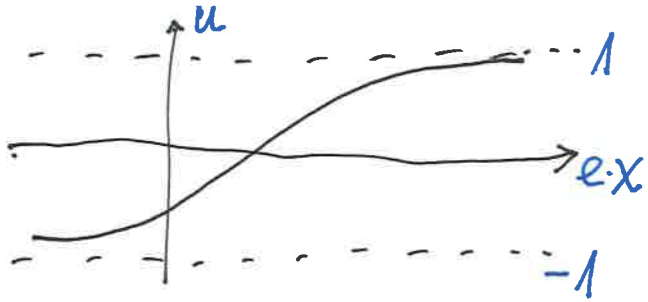

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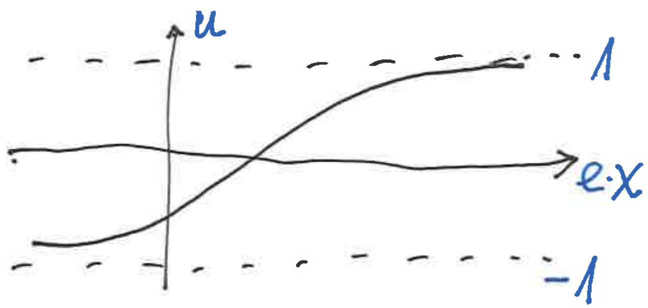
Thm [Alberti-Ambrosio-Cabré '01]

$$\left.\begin{array}{l} |u|<1 \ \&\ -\Delta u = u - u^3 \text{ in } \mathbb{R}^n, \\ \partial_{x_n} u > 0 \text{ in } \mathbb{R}^n \ \&\ \lim\limits_{x_n\to\pm\infty} u(x', x_n) = \pm 1 \quad \forall x'\in\mathbb{R}^{n-1} \end{array}\right\} \Rightarrow u \text{ is a global minimizer.}$$

$$\boxed{u(x) = \tanh\left(\frac{e\cdot x + c}{\sqrt{2}}\right)}\quad , x\in\mathbb{R}^n;\ e\in\mathbb{R}^n,\ |e|=1,\ c\in\mathbb{R}$$



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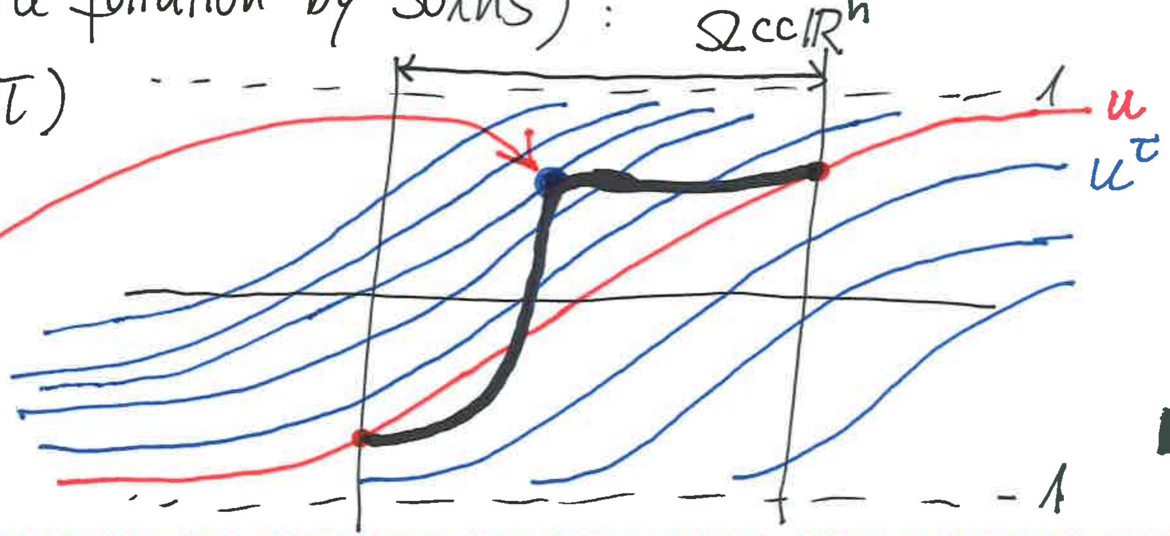
$|u|<1$ & $-\Delta u = u - u^3$ in $\mathbb{R}^n$,

$\partial_{x_n} u > 0$ in $\mathbb{R}^n$ & $\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1 \quad \forall x' \in \mathbb{R}^{n-1}$ $\Big\}$ $\Rightarrow$ u is a global minimizer.

Proof (sliding to produce a foliation by sol'ns):

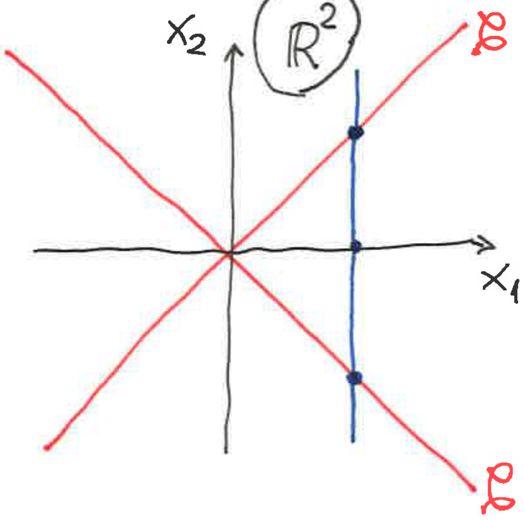
$\tau \in \mathbb{R}$, $u^\tau(x) = u(x', x_n + \tau)$

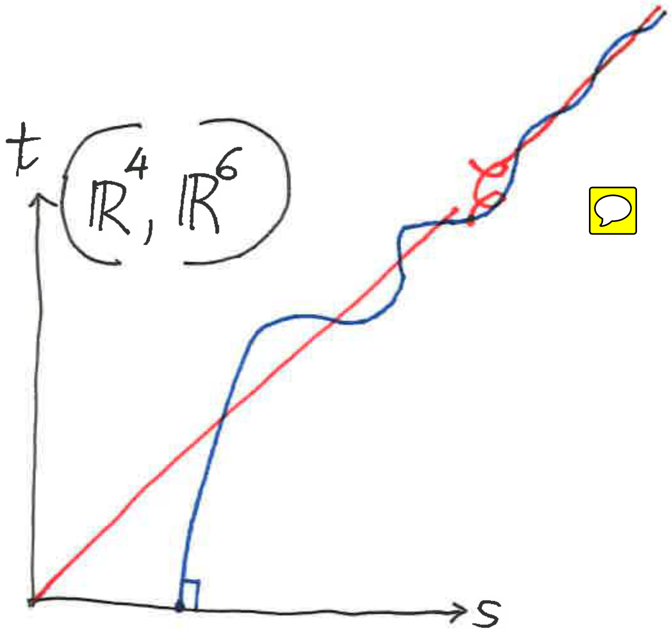
all sol'ns of AC



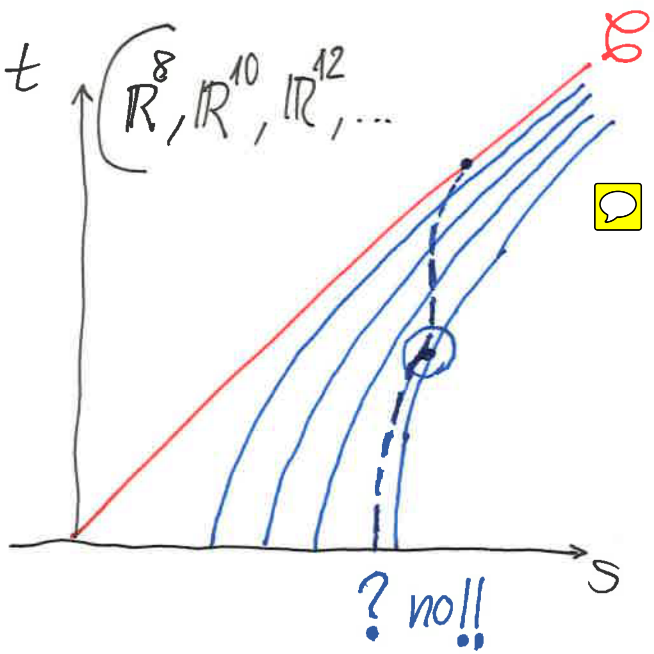
∎

Simons cone. Foliations.





Minimizers $\left\langle \begin{array}{c} \text{are not} \\ \text{are} \end{array} \right\rangle$



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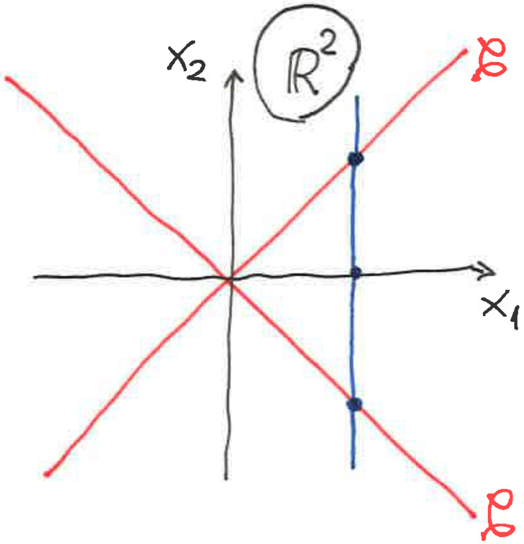
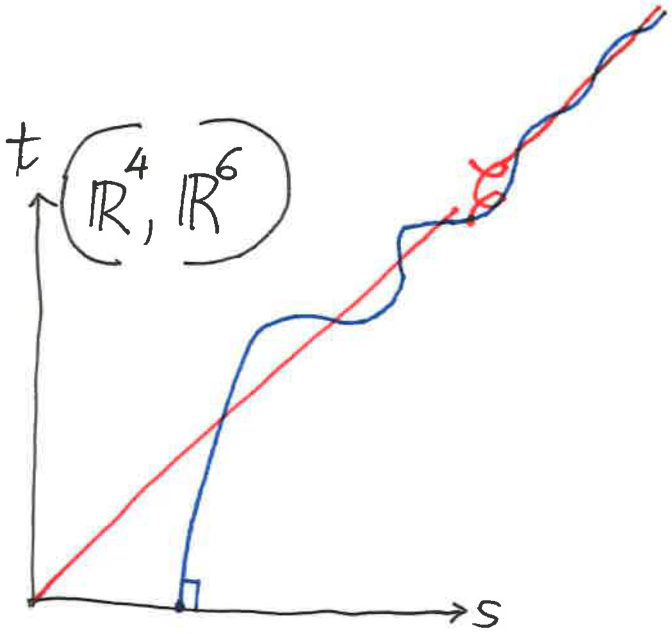
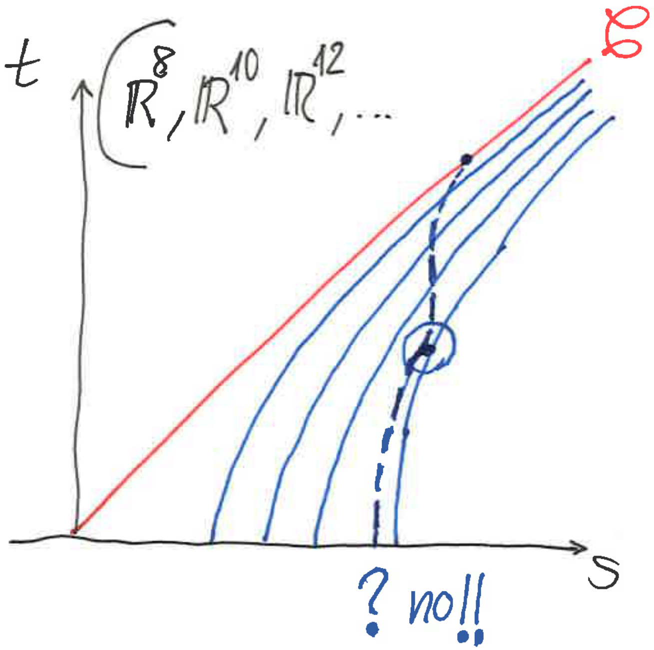




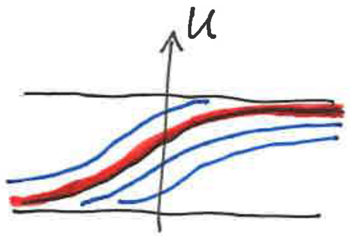


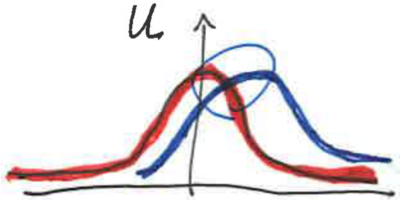
Foliation by stationary

$\Downarrow$

Minimizers

[Weierstrass - Caratheodory]



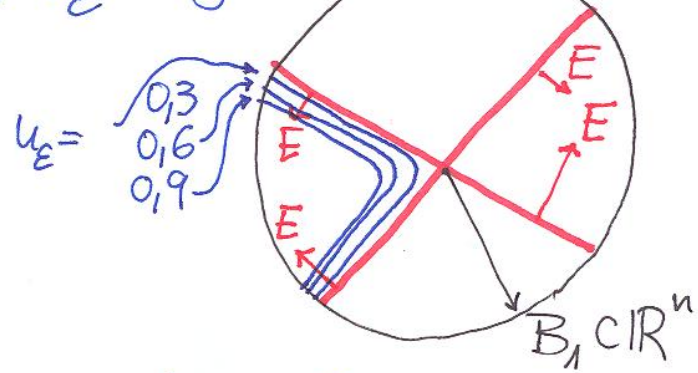


Modica-Mortola thm

$-\Delta u = u - u^3 = f(u)$ in $\mathbb{R}^n$

$u_\varepsilon(x) = u(x/\varepsilon) = u(Rx)$, $x \in B_1 \subset \mathbb{R}^n$. $R = 1/\varepsilon$ large

$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2} f(u_\varepsilon)$



Thm [MM]

Minimizers u_ε in $B_1 \subset \mathbb{R}^n$.

$u_\varepsilon \xrightarrow[\varepsilon \downarrow 0]{} \begin{cases} 1 & \text{in } E \\ -1 & \text{in } B_1 \setminus E \end{cases}$ & E is of minimal perimeter in B_1.

"Pf"

$$E_{B_1}(u_\varepsilon) = \int_{B_1} \varepsilon \frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{1}{2} 2G(u_\varepsilon) \frac{1}{\varepsilon}$$

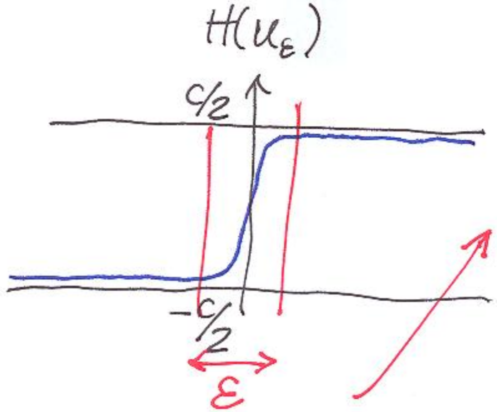
$$\geq$$

$$\int_{B_1} \sqrt{\varepsilon} |\nabla u_\varepsilon| \cdot \sqrt{2G(u_\varepsilon)} \frac{1}{\sqrt{\varepsilon}}$$

(=) if parallel level sets

$$=$$

$$\int_{B_1} |\nabla H(u_\varepsilon)|. \xrightarrow[\varepsilon \downarrow 0]{} \int_{B_1} c \cdot |\nabla \mathbb{1}_E| = c \cdot \text{perimeter}_{B_1}(E)$$



$\int |\partial_x H(u_\varepsilon)| =$

$= \int \partial_x H(u_\varepsilon) = \text{jump}$

$= C$

Allen-Cahn equation. A conjecture of De Giorgi

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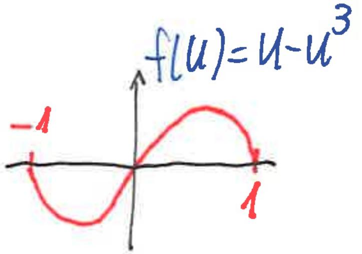
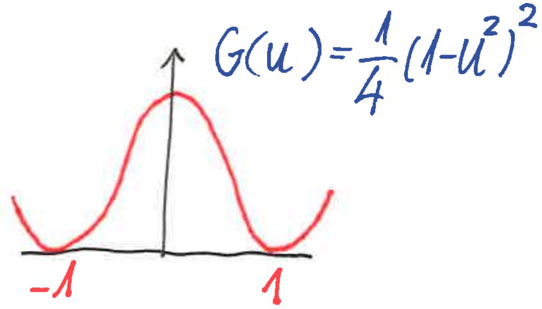
$\hookrightarrow \quad E_{B_R}(u) = \int_{B_R} \frac{1}{2}|\nabla u|^2 + G(u)$





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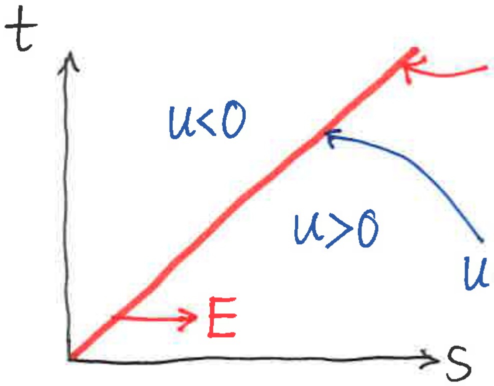
Thm [delPino-Kowalczyk-Wei '08]

$\exists$ u global minimizer of (AC) in $\mathbb{R}^9$, u not 1D, with $u_{x_9} > 0$.

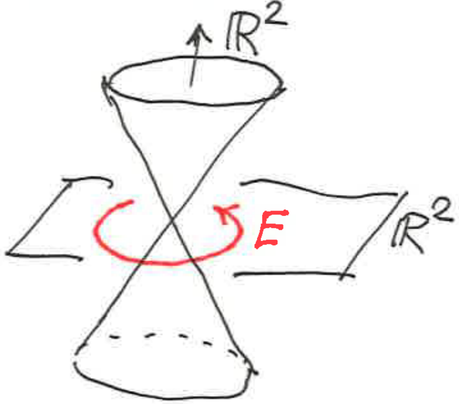
Open pb $\quad n=8$? $\longrightarrow$ Saddle-shaped solutions of (AC):

Saddle-shaped sol'ns of (AC)



$\mathcal{C} = \partial E = \{s = t\}$: Simons cone

$E = \{s > t\}$



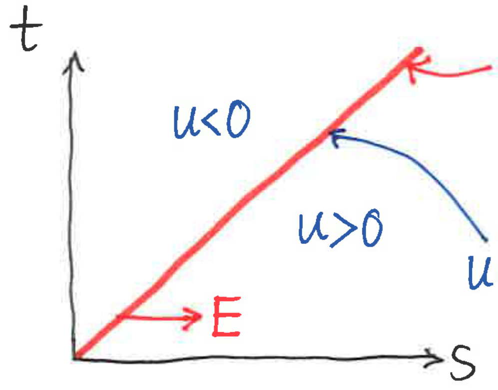
$u = u(s,t)$ $\quad \begin{cases} |u| < 1 \\ u(t,s) = -u(s,t) \end{cases}$

$$-\Delta u = u - u^3 \text{ in } \mathbb{R}^{2m} \iff$$

$$u_{ss} + u_{tt} + (m-1)\left\{ \frac{u_s}{s} + \frac{u_t}{t} \right\} + u - u^3 = 0$$

for $s>0, t>0$

Saddle-shaped sol'ns of (AC)



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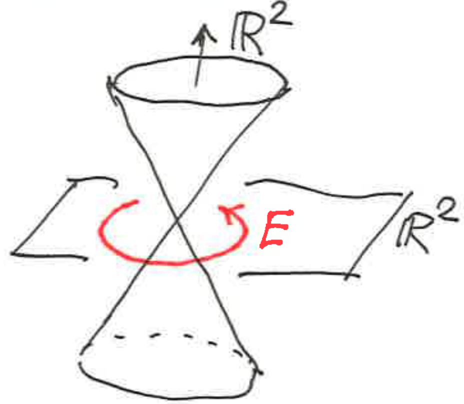

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Thm [C.-Terra][’09 ’10]

$\exists$ saddle sol'n in $\mathbb{R}^{2m}$ $\forall m \geq 1$. It is **unstable** if $2m = \mathbf{2, 4, 6}$.

Its Morse index $= 1$ in $\mathbb{R}^2$ $\leftarrow$ [Schatzman]

$= \infty$ in $\mathbb{R}^4, \mathbb{R}^6$.

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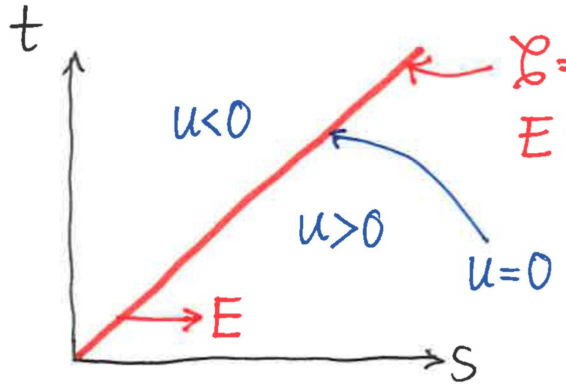

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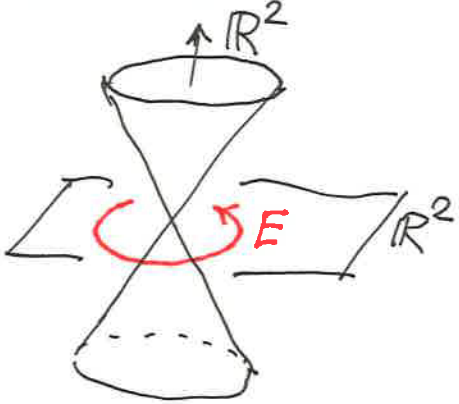

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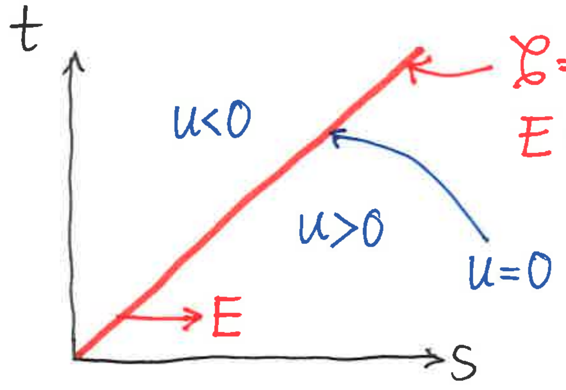
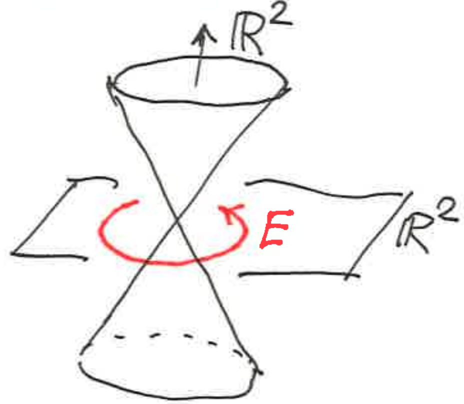
Thm [C. ’10] In $\mathbb{R}^{14}$ saddle sol'n *is* stable.

Thm [C.’10] In $\mathbb{R}^{2m}$, the saddle sol'n is unique.

Saddle-shaped sol'ns of (AC)





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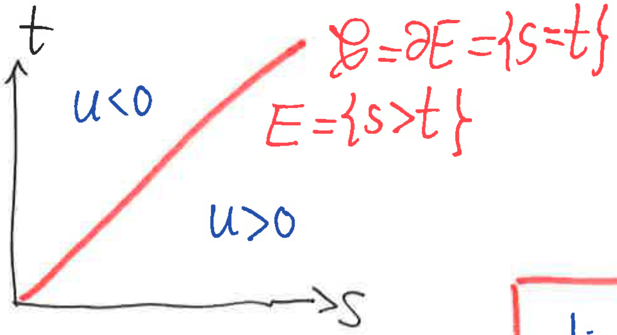
Thm [C. '10] In $\mathbb{R}^{14}$ saddle sol'n is stable.

Thm [Pacard-Wei '11] In $\mathbb{R}^8$ $\exists$ stable sol'n not 1D.

Thm [C'10] In $\mathbb{R}^{2m}$, the saddle sol'n is unique.

Saddle-shaped soln's to (AC)



$u = u(s,t)$

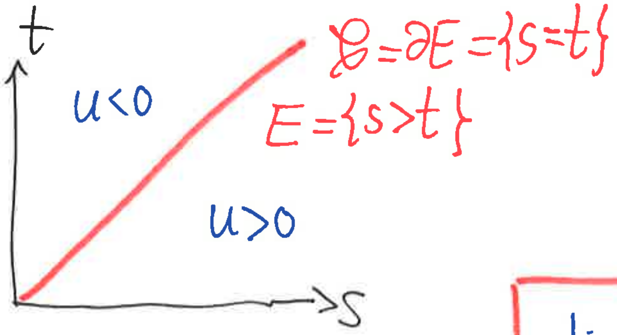
$$u_{ss} + u_{tt} + (m-1)\left\{ \frac{u_s}{s} + \frac{u_t}{t} \right\} + u - u^3 = 0$$

$$\boxed{\operatorname{dist}_{\mathbb{R}^{2m}}(x, \mathscr{C}) = \frac{s-t}{\sqrt{2}}}$$

Asymptotic behaviour at ∞:

Let $\boxed{U(x) := u_0\left(\frac{s-t}{\sqrt{2}}\right) = \tanh\left(\frac{s-t}{2}\right).}$

Saddle-shaped soln's to (AC)



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$$\boxed{\operatorname{dist}_{\mathbb{R}^{2m}}(x, \Gamma) = \frac{s-t}{\sqrt{2}}}$$

$+$

LIOUVILLE THMS in $\mathbb{R}^{2m}$ & $\mathbb{R}^{2m}_+$ for (AC)

Asymptotic behaviour at ∞:

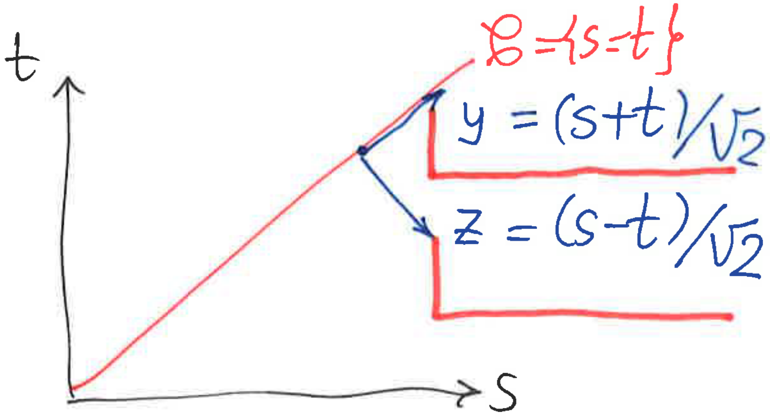
Thm [C-Terra '09] Let $\boxed{U(x) := u_0\left(\frac{s-t}{\sqrt{2}}\right) = \tanh\left(\frac{s-t}{2}\right).}$

u saddle sol'n in $\mathbb{R}^{2m}$, $\forall m \Longrightarrow$

$$\| \, |u - U| + |\nabla u - \nabla U| \, \|_{L^\infty(\mathbb{R}^{2m} \setminus B_R(0))} \longrightarrow 0 \text{ as } R \to \infty.$$

Instability in $\mathbb{R}^4$ & $\mathbb{R}^6$: (C-Terra '09)

(in $\mathbb{R}^2$: [Dang-Fife-Peletier '92] [Schatzman '95])



(AC):

$$u_{yy} + u_{zz} + \frac{2(m-1)}{y^2 - z^2}(y u_y - z u_z) + f(u) = 0$$

$$\Downarrow$$

$$0 = \{\Delta_{2m} + f'(u)\} u_z - \frac{2(m-1)}{y^2 - z^2} u_z + \frac{4(m-1)z}{(y^2 - z^2)^2}(y u_y - z u_z).$$

Instability in $\mathbb{R}^4$ & $\mathbb{R}^6$: (C-Terra '09)

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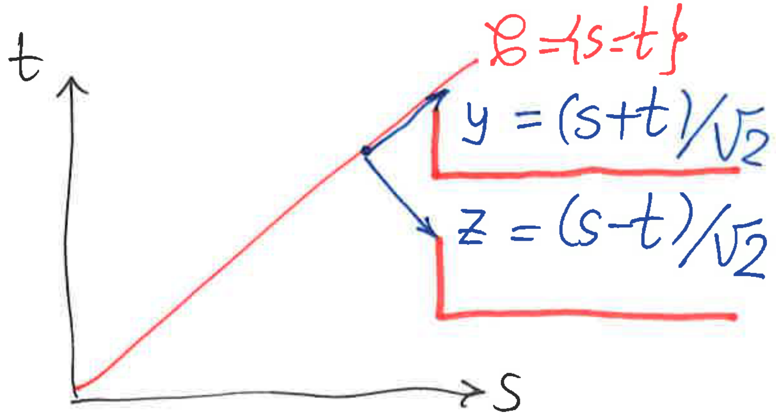

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$$\boxed{D^2 E(u)(\xi,\xi) = \int_{\mathbb{R}^{2m}} |\nabla \xi|^2 - f'(u)\,\xi^2 < 0} \quad \text{for}$$

$\xi(y,z) = \eta\left(\frac{y}{a}\right) u_z(y,z)$

& let $a \to +\infty$: HARDY ineq.

Towards uniqueness in $\mathbb{R}^{2m}$ & stability in $\mathbb{R}^{14}$

Prop'n [C'10] u saddle sol'n in $\mathbb{R}^{2m}$ $\Longrightarrow$
$L_u := \Delta + f'(u(x))$ satisfies the maximum principle in $\mathcal{O} = \{s>t\}$.

(i.e., $L_u v \geq 0$ in $\mathcal{O}$, $v \leq 0$ on $\partial\mathcal{O}$ & $\limsup_{x\in\mathcal{O}, |x|\to\infty} v(x) \leq 0$

$\Longrightarrow v \leq 0$ in $\mathcal{O}$)

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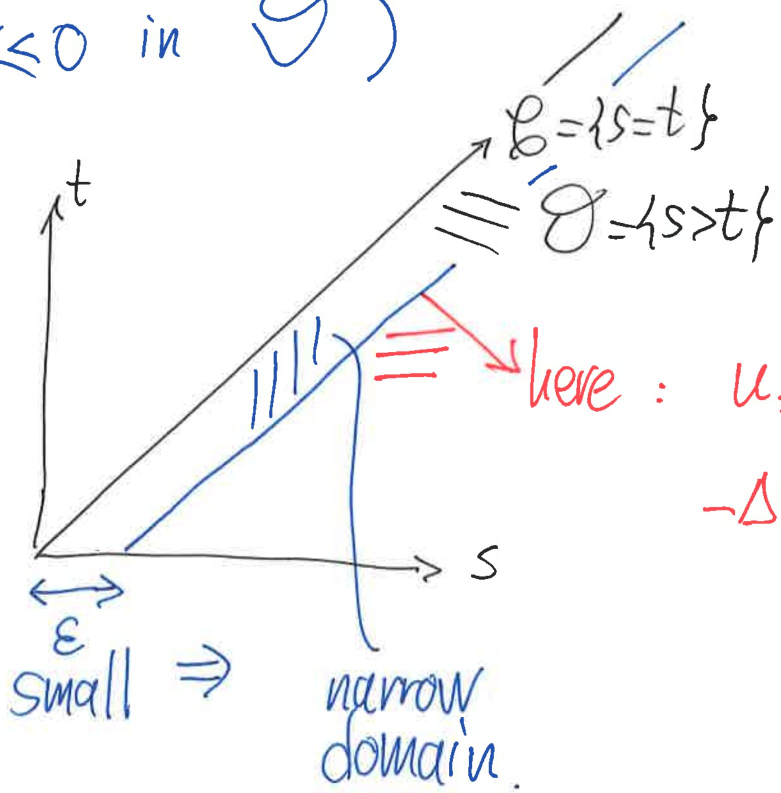
$\Rightarrow v \leq 0$ in $\mathcal{O}$)

Proof. uses



here: $u \geq \delta > 0$ &

$-\Delta u = f(u) \geq f'(u)u$

$-L_u u \geq 0$

(supersol'n, $\geq \delta > 0$) $\square$

Maximum principle in $\mathcal{O}$ for L_u

$\oplus$

Asymptotics of saddle sol'ns at ∞

$\oplus$

$\exists$ of smallest saddle in $\mathcal{O}$

$\Longrightarrow$

Thm [C'10] (**Uniqueness**) The saddle sol'n in $\mathbb{R}^{2m}$ is unique, $\forall 2m \geq 2$.

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Pf $\quad \underline{u} \leq u$ in $\mathcal{O}$

$\uparrow$ smallest saddle in $\mathcal{O}$

$\searrow \quad -\Delta(u-\underline{u}) = f(u) - f(\underline{u}) \leq f'(u)(u-\underline{u})$ in $\mathcal{O}$

$\Downarrow$ Asymptotics + Max. Pr.

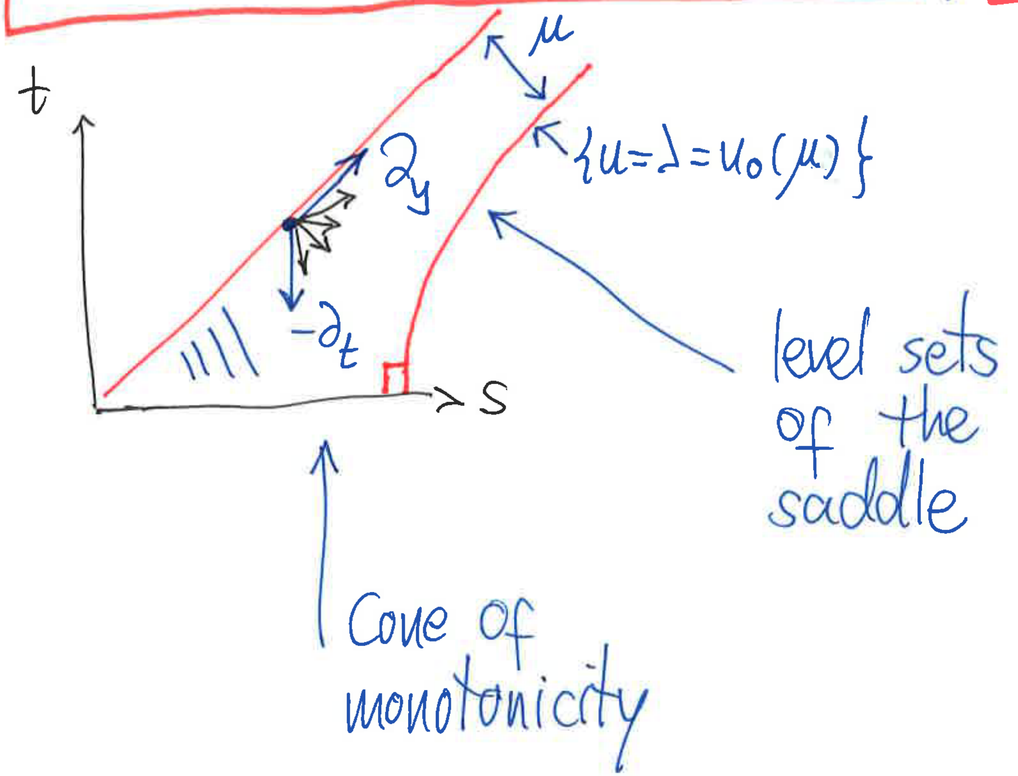
$u-\underline{u} \leq 0$ in $\mathcal{O}$. $\square$

Maximum principle in $\mathcal{O}$ $\oplus$ Asymptotics at ∞ $\Downarrow$

$\hookrightarrow$ Monotonicity & convexity properties of saddles.

Thm [C'10] u saddle sol'n in $\mathbb{R}^{2m}$, $2m \geq 2$. Then:

in $\mathcal{O} \setminus \{t=0\} = \{s>t>0\}$: $u_y > 0$, $-u_t > 0$, $u_{st} > 0$.



Maximum principle in $\mathcal{O}$ $\oplus$ Asymptotics at ∞ $\Downarrow$

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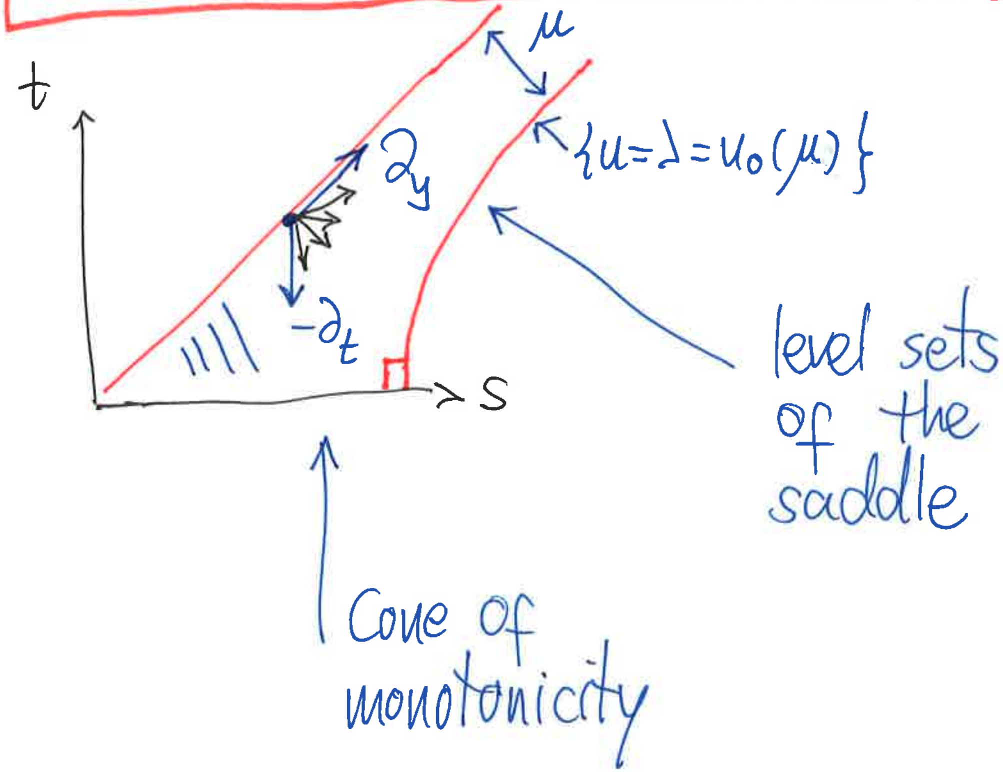

Pf: MPrinciple $\oplus$ asympt. ∞

$\oplus$

$$\{\Delta + f'(u)\} u_y = \frac{m-1}{s^2} u_y + \frac{(m-1)(s^2-t^2)}{\sqrt{2}\, s^2 t^2} u_t$$

$$\{\Delta + f'(u)\} u_t - \frac{m-1}{t^2} u_t = 0$$

$$\{\Delta + f'(u)\} u_{st} - (m-1)\left(\frac{1}{s^2} + \frac{1}{t^2}\right) u_{st} \leq 0$$

$\square$

Thm [C'10] (stability in $\mathbb{R}^{14}$, $\mathbb{R}^{2m}$ for $2m \geq 14$)

$2m \geq 14 \iff \exists b \in \mathbb{R}$ s.t. $b(b-m+2)+m-1 \leq 0$. Then:

$\varphi = \varphi(s,t) := t^{-b} u_s - s^{-b} u_t$ $(b>0)$

satisfies

$$\left.\begin{array}{l} \varphi > 0 \\ \{\Delta + f'(u)\}\varphi \leq 0 \end{array}\right\} \text{ in } \mathbb{R}^{2m} \setminus \{st=0\}.$$

$\Longrightarrow$ Stability of the saddle in $\mathbb{R}^{2m}$, $2m \geq 14$.

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$\Longrightarrow$ Stability of the saddle in $\mathbb{R}^{2m}$, $2m \geq 14$.

Pf: φ is even w.r.t. $\mathscr{C}$ $\Longrightarrow$ $\varphi > 0$ in $\mathcal{O}$ ($\Rightarrow$ in $\mathbb{R}^{2m} \setminus \{st=0\}$)

$$\Delta u_s + f'(u) u_s - \frac{m-1}{s^2} u_s = 0 \quad ; \quad \Delta u_t + f'(u) u_t - \frac{m-1}{t^2} u_t = 0$$

$$\Delta t^{-b} = b(b-m+2)\, t^{-b-2} \quad ; \quad \Delta s^{-b} = b(b-m+2)\, s^{-b-2}$$

$$\boxed{\varphi = t^{-b} u_s - s^{-b} u_t}$$

$$\longrightarrow \quad \underline{\Delta\varphi + f'(u)\varphi} = u_s\, t^{-b}\left\{(m-1)s^{-2} + b(b-m+2)t^{-2}\right\}$$

$$+ (-u_t)\, s^{-b}\left\{(m-1)t^{-2} + b(b-m+2)s^{-2}\right\}$$

$$+ u_{st}\, 2b\left\{s^{-b-1} - t^{-b-1}\right\}$$

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$u_{st} > 0$ in $\mathcal{O}$

$$\le u_s\, t^{-b}\{(m-1)s^{-2} + b(b-m+2)t^{-2}\} + (-u_t)\, s^{-b}\{(m-1)t^{-2} + b(b-m+2)s^{-2}\} \quad \text{in } \mathcal{O}$$

$$\downarrow$$

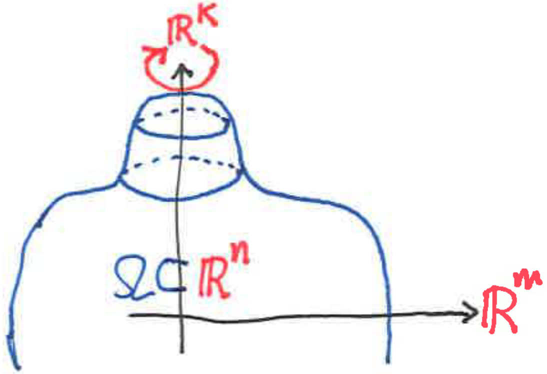
$$
\begin{aligned}
\{\Delta + f'(u)\}\varphi &\leq t^{-b}(u_s + u_t)\{(m-1)s^{-2} + b(b-m+2)t^{-2}\} \\
&\quad -s^{-b}u_t\{(m-1)t^{-2} + b(b-m+2)s^{-2}\} \\
&\quad -t^{-b}u_t\{(m-1)s^{-2} + b(b-m+2)t^{-2}\} \\
&= u_y\sqrt{2}t^{-b}\{(m-1)s^{-2} + b(b-m+2)t^{-2}\} \\
&\quad +(-u_t)\ (m-1)(s^{-b}t^{-2} + t^{-b}s^{-2}) \\
&\quad +(-u_t)\ b(b-m+2)(s^{-2-b} + t^{-2-b}) \\
&\leq u_y\sqrt{2}t^{-b}(m-1)\{s^{-2} - t^{-2}\} \\
&\quad +(-u_t)\ (m-1)(s^{-b}t^{-2} + t^{-b}s^{-2} - s^{-2-b} - t^{-2-b}) \\
&\leq (-u_t)\ (m-1)(s^{-b}t^{-2} + t^{-b}s^{-2} - s^{-2-b} - t^{-2-b})
\end{aligned}
$$

2 Antisymmetry —or oddness— for problems with weights

$$\min_{u(\pm L)=\pm m} \int_{-L}^{L} \left\{ \frac{1}{2}\dot{u}^2 + \frac{1}{4}(1-u^2)^2 \right\} a(x)dx$$

- Weights from axial symmetry

$\Omega \subset \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k$ radially symmetric w.r.t. to $(x_1, \dots, x_m)$ and $(x_{m+1}, \dots, x_{m+k} = x_n)$



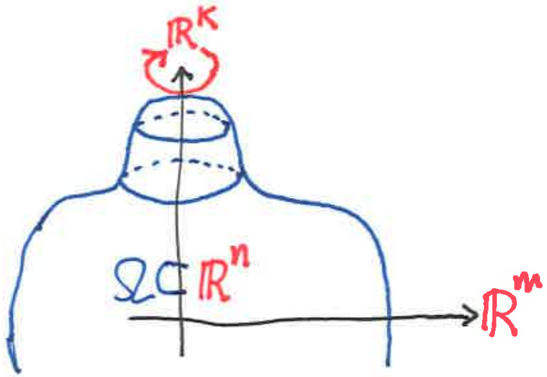
$$\begin{cases} s = \sqrt{x_1^2 + \dots + x_m^2} \geq 0 \\ t = \sqrt{x_{m+1}^2 + \dots + x_n^2} \geq 0 \end{cases}$$

$\Delta u + f(u) = 0$ in $\Omega \subset \mathbb{R}^n$ $\rightsquigarrow$ $u_{ss} + u_{tt} + \frac{m-1}{s} u_s + \frac{k-1}{t} u_t = 0$

in $\tilde{\Omega} \subset \mathbb{R}_+ \times \mathbb{R}_+ \subset \mathbb{R}^2$

- Weights from axial symmetry

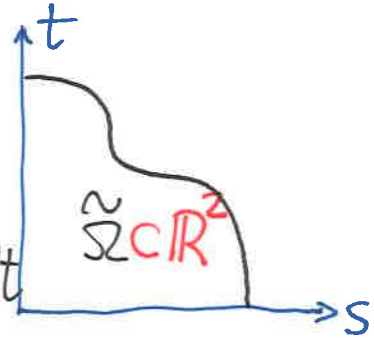
$\Omega \subset \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^K$ radially symmetric w.r.t. to $(x_1, \dots, x_m)$ and $(x_{m+1}, \dots, x_{m+K} = x_n)$

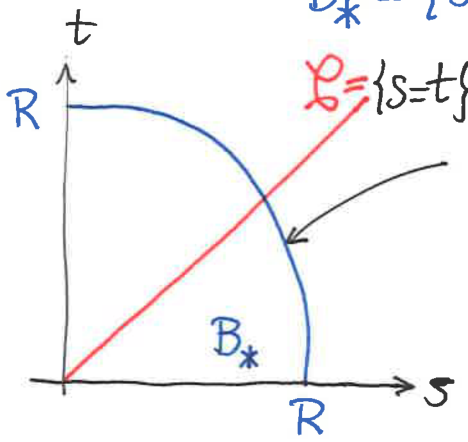


$$\begin{cases} s = \sqrt{x_1^2 + \dots + x_m^2} \geq 0 \\ t = \sqrt{x_{m+1}^2 + \dots + x_n^2} \geq 0 \end{cases}$$

$$\begin{cases} \Delta u + f(u) = 0 \text{ in } \Omega \subset \mathbb{R}^n \rightsquigarrow u_{ss} + u_{tt} + \frac{m-1}{s} u_s + \frac{K-1}{t} u_t = 0 \\ \qquad\qquad \text{in } \tilde{\Omega} \subset \mathbb{R}_+ \times \mathbb{R}_+ \subset \mathbb{R}^2 \\ E(u) = \int_\Omega \left\{ \frac{1}{2} |\nabla u|^2 + G(u) \right\} dx \\ \qquad (G' = -f) \end{cases}$$

$$E(u) = c \cdot \int_{\tilde{\Omega}} s^{m-1} t^{K-1} \left\{ \frac{1}{2} |\nabla u|^2 + G(u) \right\} ds\,dt$$





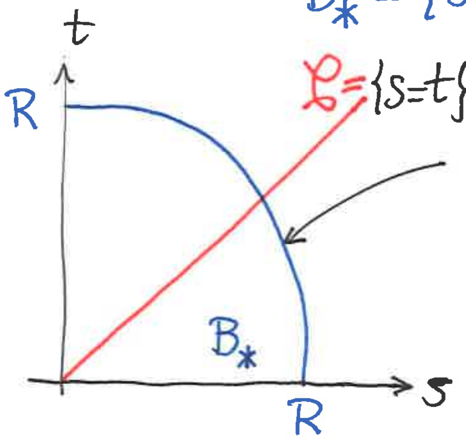
$B_* = \{s>0, t>0, s^2+t^2<R\}$

$\mathscr{C} = \{s=t\}$ = Simons cone $\subset \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m}$

$u = g : \partial B_* \to [-1,1]$ antisymmetric (or odd) w.r.t. $\mathscr{C}$

$$\min_{u|_{\partial B_*} \equiv g} \int_{B_*} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1-u^2)^2 \right\} s^{m-1} t^{m-1}\, ds\, dt$$

- Question : Is a minimizer odd w.r.t. $\mathscr{C}$ (i.e., $u(s,t) = -u(t,s)$) ?



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- **Question**: Is a minimizer odd w.r.t. $\mathscr{C}$ (i.e., $u(s,t) = -u(t,s)$) ?
- **Fact**: If $2m = 2, 4, 6$, for R large, any minimizer is not odd.
- Minimizers are expected to be odd for $2m \geq 8$ & R large. ? (Open pb)

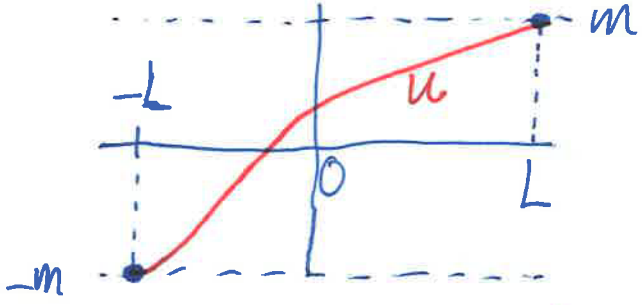
If $2m \leq 6$ odd solutions are unstable

(In addition, minimizers are 1-d if $2m \leq 6$)

} Conjecture of De Giorgi

• Oddness of minimizers & solns in 1D with weights

$$\min_{u(\pm L)=\pm m} \int_{-L}^{L} \left\{ \frac{1}{2} \dot{u}^2 + G(u) \right\} a(x)\, dx \quad ; \quad a \text{ even in } (-L,L)$$



(EL) $\quad -(a u')' - a f(u) = 0$

$$-u'' = (\log a)' u' + f(u)$$

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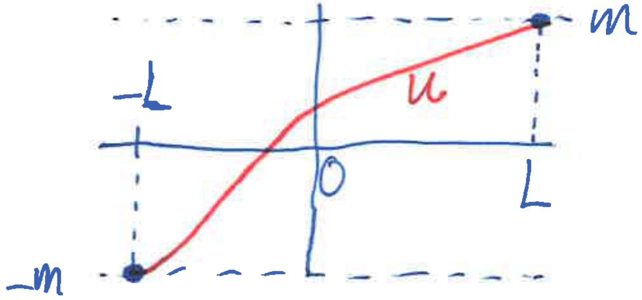

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Thm [Berestycki-Nirenberg '88] $(\log a)'' \geq 0 \Rightarrow \left\{ \begin{array}{l} \text{any solution} \\ u \text{ is odd } \& \nearrow \end{array} \right.$

& $G \in C^{1,1}$ $(f \in Lip)$

• sliding method

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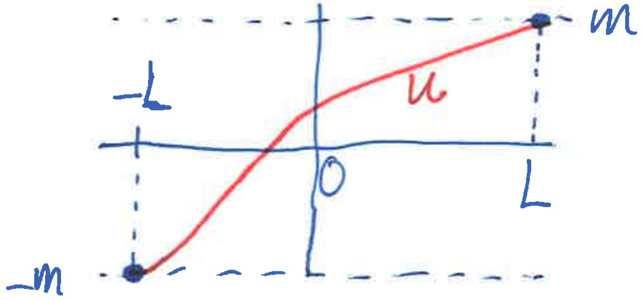

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& $G \in C^{1,1}$ ($f \in Lip$)

Thm [Cabré–Lucia–Sanchón–Villegas '12]

Any soln u is odd and $\nearrow$ if

(a) $(\log a)'' \geq 0$ & $G \in C^0$

or

(b) $a' \geq 0$ in $(0,L)$ & $G' \leq 0$ in $(0,m)$

• sliding method

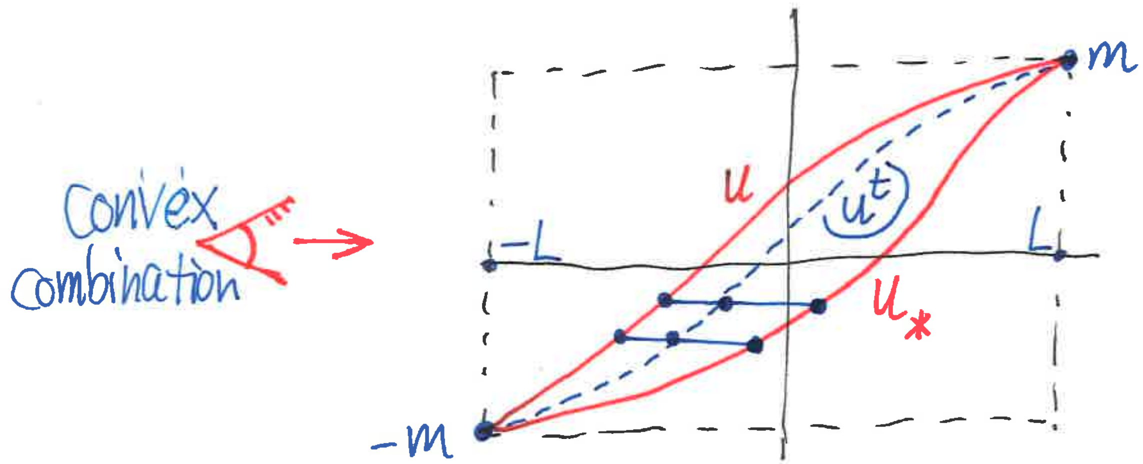
• continuous odd rearrangement

• calibration with Hamiltonian

- **Continuous odd rearrangement**:

$$\int \left\{ \frac{1}{2} \dot{u}^2 + G(u) \right\} \alpha(x)\, dx \xrightarrow[\alpha(x)dx = dy]{} \int \left\{ \frac{1}{2} \dot{v}^2 \tilde{\alpha}(y) + G(v) \right\} dy$$

$$\left. \begin{array}{l} u_*(x) = -u(-x) \\ \hookrightarrow \text{flipped of } u \end{array} \right\}$$



$\rho = u^{-1} \rightsquigarrow \rho^t(\lambda) := t\rho(\lambda) + (1-t)\rho_*(\lambda)$

$\begin{cases} u(x) = \lambda \\ \rho(\lambda) = x \end{cases}$

$u^t = (\rho^t)^{-1}$: the continuous odd rearrang.

3 The explosion problem in axially symmetric domains

- Mean curvature of the level sets and

  a geometric Sobolev inequality

- Regularity up to $n \leq 7$ in axially symmetric domains

• L^∞ estimate for stable solutions

$$(*)\begin{cases} -\Delta u = f(u) & \text{in } \Omega\subset\mathbb{R}^n \\ u=0 & \text{on } \partial\Omega \end{cases}$$

$$\begin{cases} f>0,\ f(0)>0,\ f\nearrow \ \&\ \text{superlinear at } \infty & (f(u)=e^u, (1+u)^p, p>1, \text{etc.}) \\ u \text{ stable solution of } (*) \end{cases}$$

• **Open pb**: $n\leq 9 \Rightarrow u\in L^\infty(\Omega)$

True if • $\Omega = B_R$ & $n\leq 9$

or • $n\leq 3$

or • $n=4$ & Ω convex.

[for instance $u=u^*$: the extremal solution

when $f(u) \rightsquigarrow \lambda f(u)$]

- **L^∞ estimate for stable solutions**

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or
- $n = 4$ & Ω convex.

[for instance $u = u^*$: the extremal solution when $f(u) \rightsquigarrow \lambda f(u)$]

Thm [Cabré-Sanchón '12] $n \geq 5 \Rightarrow u \in L^{\frac{2n}{n-4}}(\Omega)$

Thm [" " "] (Geometric Sobolev inequality)

$\left.\begin{array}{l} p \geq 1,\ r \geq 1 \\ p(1+r) < n \end{array}\right\} \Rightarrow \forall v \in C_c^\infty(\mathbb{R}^n)$ $\quad \|v\|_{L^{p_r^*}} \leq C_{n,p,r} \| \, |H_v|^r |\nabla v| \, \|_{L^p}$, where $\rightarrow$

weight = mean curvature of level sets

(p subcritical)

$p \geq 1,\ r \geq 1,\ p(1+r) < n \Rightarrow \forall v \in C_c^\infty(\mathbb{R}^n)$

$$\|v\|_{L^{p_r^*}} \leq C_{n,p,r} \left(\int_{\{|\nabla v| > 0\}} |H_v|^{pr} |\nabla v|^p \right)^{1/p}$$

(Also $\|\cdot\|_\infty$ & Trudinger ineq.)

p supercritical

p critical

where $\displaystyle \frac{1}{p_r^*} = \frac{1}{p} - \frac{1+r}{n}$

$H_v = \mathrm{div}\left(\frac{\nabla v}{|\nabla v|}\right)$: mean curvature of the level sets of v

(p subcritical)

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$H_v = \operatorname{div}\left(\frac{\nabla v}{|\nabla v|}\right)$: mean curvature of the level sets of v

(Also $\|\cdot\|_\infty$ & Trudinger ineq.)

p supercritical

p critical

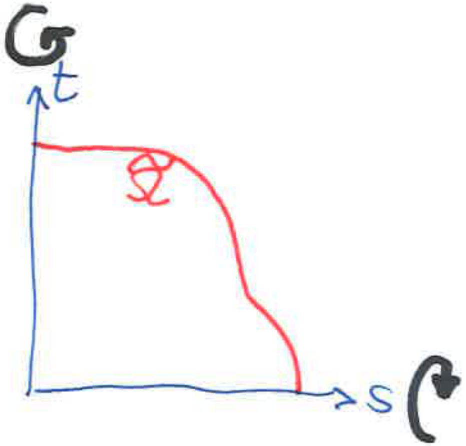
where $\frac{1}{p_r^*} = \frac{1}{p} - \frac{1+r}{n}$

- RHS controlled by stability of the sol'n : test fcn $= |\nabla u| \cdot \eta(x)$
- Proof uses classical isoperimetric inequality in $\mathbb{R}^n$

  ⊕ Michael-Simon & Allard Sobolev inequality $\mapsto$

$$S = S_{n-1} \subset \mathbb{R}^n \Rightarrow |S|^{\frac{n-2}{n-1}} \leq C_n \cdot \int_S |H| \, d\sigma$$

• L^∞ estimate in "axially symmetric domains"



$\Omega \subset \mathbb{R}^m \times \mathbb{R}^k = \mathbb{R}^n$, $m \geq 2$, $k \geq 2$;

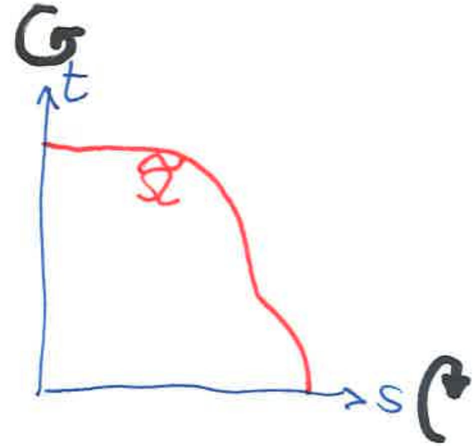
$\llcorner$ of double revolution

$$\int_{\tilde{\Omega}} \left\{ \frac{1}{2} |\nabla u|^2 - F(u) \right\} \underline{s^{m-1} t^{k-1}} \, ds \, dt$$

<u>Thm</u> [<u>Cabré - Ros</u> '12] Ω convex & u stable sol'n $\Longrightarrow$

$$\boxed{\begin{array}{l} \bullet\ \underline{n \leq 7} \Rightarrow u \in L^\infty(\Omega). \\ \bullet\ n \geq 8 \Rightarrow u \in L^p(\Omega) \end{array}} \quad \forall p < 2 + \frac{4}{\frac{m}{2+\sqrt{m-1}} + \frac{k}{2+\sqrt{k-1}} - 2}.$$

• L^∞ estimate in "axially symmetric domains"



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↳ of double revolution

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__Thm__ [__Cabré–Ros__ '12] Ω convex & u stable sol'n $\Longrightarrow$

- $\underline{n \leq 7}$ $\Rightarrow$ $u \in L^\infty(\Omega)$.
- $n \geq 8$ $\Rightarrow$ $u \in L^p(\Omega)$ $\forall p < 2 + \dfrac{4}{\dfrac{m}{2+\sqrt{m-1}} + \dfrac{k}{2+\sqrt{k-1}} - 2}$.

__Proof__ uses a __new__ Sobolev inequality:

$$\forall u \in C_c^1(\mathbb{R}^2) \left(\int_{(\mathbb{R}_+)^2} \sigma^a \tau^b |u|^{q^*}\, d\sigma\, d\tau \right)^{1/q^*} \leq C_{a,b,q} \left(\int_{(\mathbb{R}_+)^2} \sigma^a \tau^b |\nabla u|^q\, d\sigma\, d\tau \right)^{1/q}$$

$$\left(a > -1,\ b > -1,\ D = 2 + a + b,\ 1 \leq q < D,\ q^* = \frac{Dq}{D-q} \right)$$

4 Isoperimetric inequalities with homogeneous weights

- Monomial weights
- Homogeneous weights in cones
- Homogeneous weights in exterior domains

BASIC TOOL:

Cabré's proof of the classical isoperimetric inequality; see:

Butl. Soc. Catalana Mat. 15 (2000) in CATALAN

Discrete Contin. Dyn. Syst. 20 (2008).

- Isoperimetric inequality with monomial weights

$x \in \mathbb{R}^n$, $\boxed{x^A = |x_1|^{A_1} \cdots |x_n|^{A_n}}$, $A_i > 0$

$m(\Omega) := \int_\Omega x^A \, dx$, $m(\partial\Omega) := \int_{\partial\Omega} x^A \, d\sigma$ for $\Omega \subset \mathbb{R}^n$ bdd Lipschitz

Let $\mathbb{R}^n_* = (0, +\infty)^n$

- Isoperimetric inequality with monomial weights

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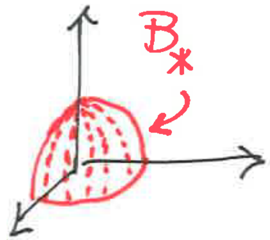
Let $\mathbb{R}^n_* = (0, +\infty)^n$

Thm [Cabré–Ros '12] $\forall \Omega \subset \mathbb{R}^n$

$$\frac{m(\partial\Omega)}{m(\Omega)^{\frac{D-1}{D}}} \geq \frac{m(\partial B_*)}{m(B_*)^{\frac{D-1}{D}}}$$

where $B_* = B_1(0) \cap \mathbb{R}^n_*$

& $D = n + |A| := n + A_1 + \cdots + A_n$



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$m(\Omega) := \int_\Omega x^A\,dx$, $m(\partial\Omega) := \int_{\partial\Omega} x^A\,d\sigma$ for $\Omega \subset \mathbb{R}^n$ bdd Lipschitz

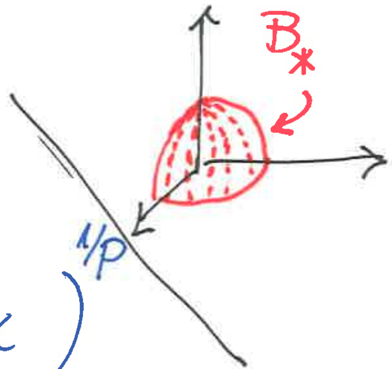
Let $\mathbb{R}^n_* = (0,+\infty)^n$

Thm [Cabré–Ros '12] $\forall \Omega \subset \mathbb{R}^n$

$$\frac{m(\partial\Omega)}{m(\Omega)^{\frac{D-1}{D}}} \geq \frac{m(\partial B_*)}{m(B_*)^{\frac{D-1}{D}}}$$

where $B_* = B_1(0) \cap \mathbb{R}^n_*$

& $D = n + |A| := n + A_1 + \cdots + A_n$



Thm [Cabré–Ros '12] $1 \leq p < D \Rightarrow \forall u \in C^1_c(\mathbb{R}^n)$

$$\left(\int_{\mathbb{R}^n_*} x^A |u|^{p^*} dx \right)^{1/p^*} \leq C_p \left(\int_{\mathbb{R}^n_*} x^A |\nabla u|^p dx \right)^{1/p}$$

$p^* = \frac{pD}{D-p}$.

best ctt is explicit (Γ function) &

extremals $= (a + b|x|^{\frac{p}{p-1}})^{1-\frac{D}{p}}$ $(p>1)$

- Also a C^α, $\alpha = 1 - \frac{D}{p}$, Morrey inequality for $p > D$.
- <u>Note</u> $\exists A_i \geq 1 \Rightarrow |x|^A$ not Muckenhoupt.
- Proof of Sobolev: direct from <u>isoperimetric</u>

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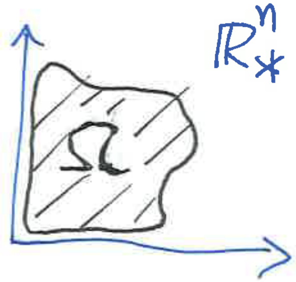
- Proof of the isoperimetric ineq:

$$\begin{cases} \operatorname{div}(x^A \nabla u) = c x^A & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 1 & \text{on } \partial\Omega. \end{cases} \longleftarrow c = \frac{m(\partial\Omega)}{m(\Omega)}$$

May assume $\overline{\Omega} \subset \mathbb{R}^n_*$



(Note: $\Omega = B_* \longrightarrow$ sol'n $= c \cdot |x|^2 = c(x_1^2 + \dots + x_n^2)$)

Γ_u: lower contact set of u; $\Gamma_u^* = \Gamma_u \cap (\nabla u)^{-1}(B_*)$

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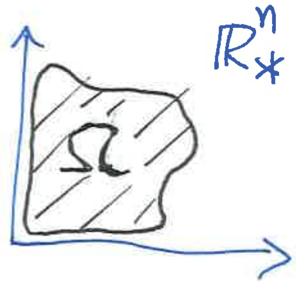

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Γ_u : lower contact set of u ; $\Gamma_u^* = \Gamma_u \cap (\nabla u)^{-1}(B_*)$

$$B_* = \nabla u(\Gamma_u^*) \Longrightarrow m(B_*) = \int_{\nabla u(\Gamma_u^*)} p^A \, dp \leq \int_{\Gamma_u^*} (\nabla u)^A \det D^2 u \, dx =$$

$$= \int_{\Gamma_u^*} \underbrace{\left(\frac{u_1}{x_1}\right)^{A_1} \cdots \left(\frac{u_n}{x_n}\right)^{A_n} \det D^2 u}_{} \cdot \underline{x^A dx}$$

$$w_1^{\lambda_1}\cdots w_k^{\lambda_k} \leq \left(\frac{\lambda_1 w_1 + \dots + \lambda_k w_k}{\lambda_1 + \dots + \lambda_k}\right)^{\lambda_1+\dots+\lambda_k}$$

$$\left(\frac{u_1}{x_1}\right)^{A_1}\cdots\left(\frac{u_n}{x_n}\right)^{A_n}\det D^2u \leq \left(\frac{A_1\frac{u_1}{x_1} + \dots + A_n\frac{u_n}{x_n} + \Delta u}{A_1+\dots+A_n+n}\right)^{A_1+\dots+A_n+n}$$

$$A_1\frac{u_1}{x_1} + \dots + A_n\frac{u_n}{x_n} + \Delta u = x^{-A}\operatorname{div}(x^A\nabla u) \equiv c \quad !!$$

$\blacksquare$

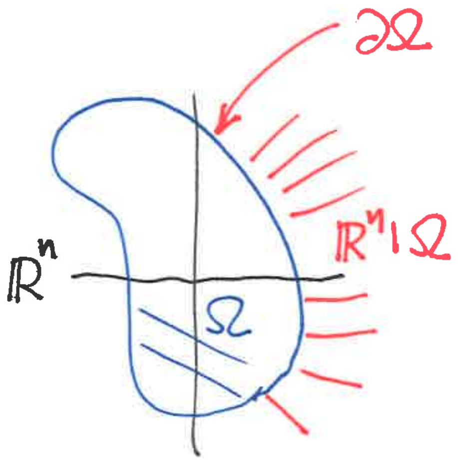
• An example of new isoperimetric inequality with best constant & with nonradial homogeneous weight.

• Thm [Cabré–Ros-Oton–Serra '12]

$|||\cdot|||$ a norm in $\mathbb{R}^n$
$a > n$
$0 \in \Omega \subset \mathbb{R}^n$
↳ bdd & regular

$$\frac{\displaystyle\int_{\partial\Omega} |||x|||^{-a}\, d\sigma(x)}{\left(\displaystyle\int_{\mathbb{R}^n\setminus\Omega} |||x|||^{-a}\, dx\right)^{\frac{n-a-1}{n-a}}}$$

is minimized by balls $B_R(0)$.



For instance:

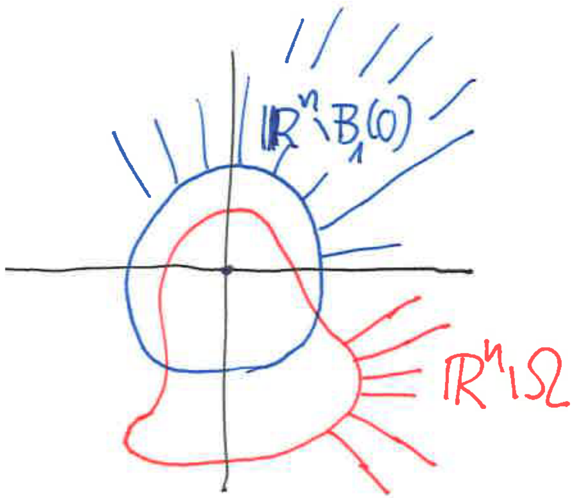
$$|||x||| = \left(|x_1|^p + \dots + |x_n|^p\right)^{1/p}$$
$$1 \le p \le \infty$$

Thm [Cabré–Ros-Serra '12]

$w \in C^1(\mathbb{R}^n)$, $w > 0$ in $\mathbb{R}^n \setminus \{0\}$, homogeneous degree $\alpha < -n$, $w^{1/\alpha}$ convex.

$\Rightarrow \forall\, 0 \in \Omega \subset \mathbb{R}^n$ bdd regular

$$\frac{P(\mathbb{R}^n \setminus \Omega)}{m(\mathbb{R}^n \setminus \Omega)^{\frac{D-1}{D}}} \geq \frac{P(\mathbb{R}^n \setminus B_1(0))}{m(\mathbb{R}^n \setminus B_1(0))^{\frac{D-1}{D}}}, \quad D = \alpha + n.$$



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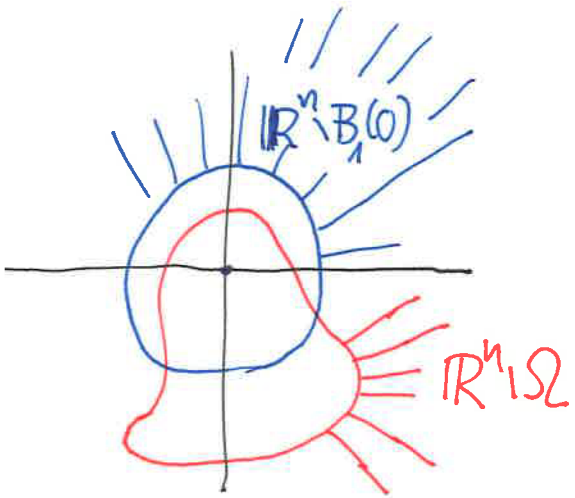

$w(x) = |||x|||^{\alpha}$

$\alpha < -n$

$|||\cdot|||$ a norm in $\mathbb{R}^n$

Examples:

$(|x_1|^p + \ldots + |x_n|^p)^{\frac{\alpha}{p}}$

$1 \leq p \leq \infty$ $\rightarrow$ NONRADIAL if $p \neq 2$

(but exterior of BALLS minimize!!)

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$$\boxed{\begin{array}{c} w(x) = |||x|||^{\alpha} \\ \alpha < -n \\ |||\cdot||| \text{ a norm in } \mathbb{R}^n \end{array}}$$

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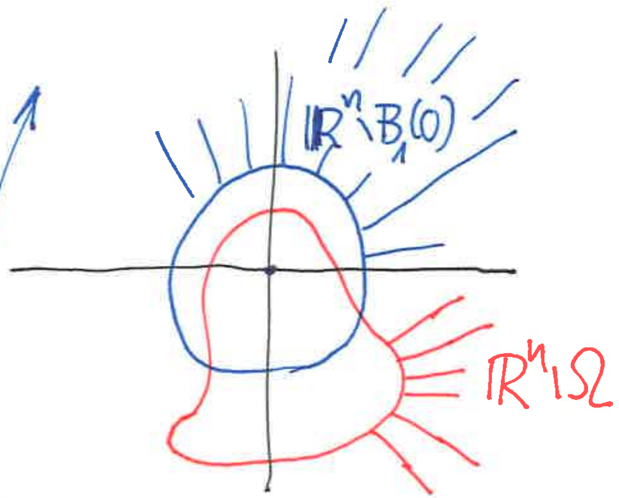
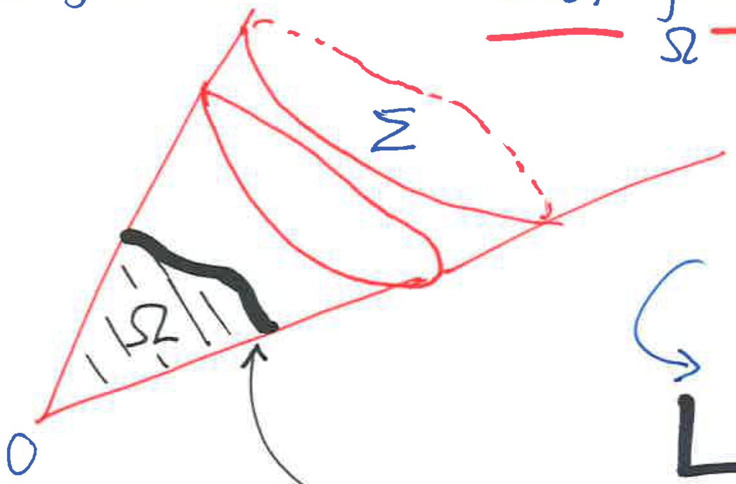


- Known for $w(x) = |x|^{\alpha}$, $\alpha < -2$, $\mathbb{R}^2$
[Carroll et al. '08 & Dahlberg et al '10]

- See also Frank Morgan, "Manifolds with Density", Notices AMS '05.

• Isoperimetric inequalities with homogeneous weights

$w \geq 0$, $\Omega \subset \mathbb{R}^n$ $\longrightarrow$ $m(\Omega) = \int_\Omega w(x)\,dx$, $P(\Omega) = \int_{\partial\Omega} w(x)\,d\sigma(x)$



$\Sigma \subset \mathbb{R}^n$ open convex cone

$$P_\Sigma(\Omega) = \int_{\Sigma \cap \partial\Omega} w(x)\,d\sigma(x)$$

$\Omega \subset \Sigma$

Thm [Cabré – Ros – Serra '12]

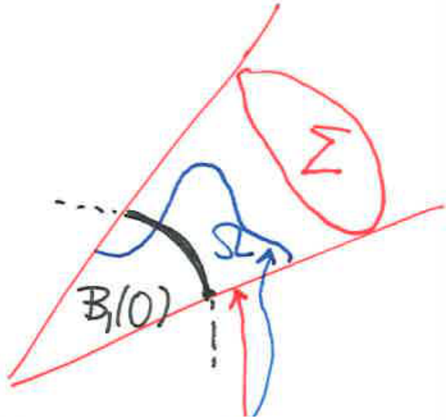
Σ convex open cone of $\mathbb{R}^n$

$w: \Sigma \to \mathbb{R}_+$ homogeneous of degree $\alpha \geq 0$ & $w^{1/\alpha}$ concave.

w non radial
BUT
balls minimize!

$\Rightarrow \forall\, \Omega \subset \Sigma$ regular

$$\frac{P_\Sigma(\Omega)}{m(\Omega)^{\frac{D-1}{D}}} \geq \frac{P_\Sigma(\Sigma \cap B_1(0))}{m(\Sigma \cap B_1(0))^{\frac{D-1}{D}}}, \quad D = n + \alpha.$$



- $w \equiv 1$: Thm due to Lions-Pacella '90

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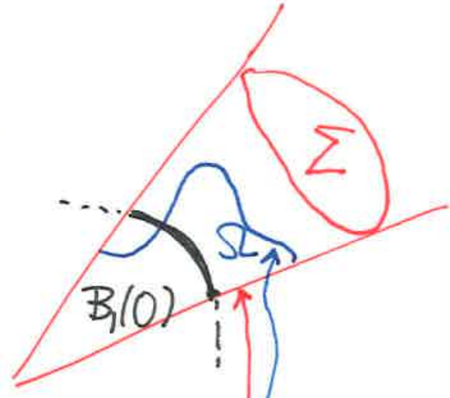
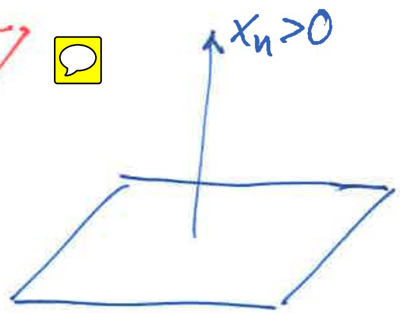


- $W \equiv 1$: Thm due to Lions-Pacella '90
- Berestycki-Pacella '89
  Radial symmetry of positive solu's in some convex cones $\begin{cases} \text{Dirichlet} = 0 \\ \text{Neumann} = 0 \end{cases}$
- Example : $W(x) = x_1^{A_1} \cdots x_n^{A_n}$, $A_i \geq 0$

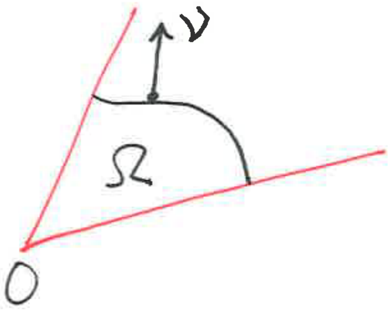
$\boxed{\Sigma = \mathbb{R}^n_*}$ $\hookrightarrow$ Particular case : $\Sigma = \mathbb{R}^n_+$

$W = x_n^{A_n}$

$A_n \geq 0$



• Also true for perimeter $\displaystyle\int_{\partial\Omega\cap\Sigma} w(x)\,H(\nu)\,d\sigma(x)$



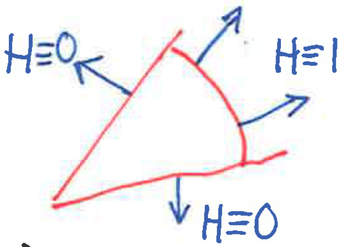
$\to$ minimizer $= \Sigma \cap W$

Wulff shape of H.

• Proof :

$$\begin{cases} \operatorname{div}(w(x)\nabla u) = c\, w(x) & \text{in } \Omega_{(\varepsilon)} \\ \dfrac{\partial u}{\partial \nu} = H_\varepsilon(\nu) \text{ on } \partial\Omega \end{cases}$$

regularize corners!



$W_\varepsilon \subset \nabla u(\Gamma_u)$

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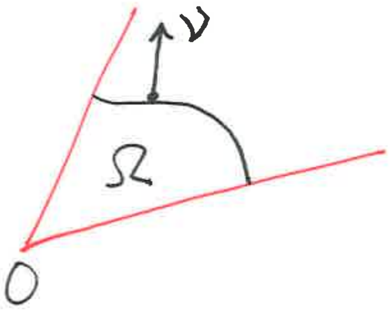

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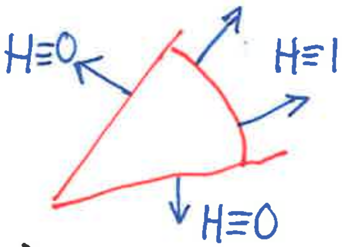

$W_\varepsilon \subset \nabla u(\Gamma_u)$

$$\hookrightarrow m(W_\varepsilon) \le \int_{\nabla u(\Gamma_u)} w(p)\, dp \le \int_{\Gamma_u} \frac{w(\nabla u) \det D^2 u}{w(x)}\, dx \cdot w(x)$$

$w^{1/\alpha}$ concave !!

$$\le \left(\frac{\alpha \left(\frac{w(\nabla u)}{w(x)}\right)^{1/\alpha} + \Delta u}{\alpha+n} \right)^{\alpha+n} \le \left(\frac{\frac{\nabla w(x)\cdot \nabla u}{w(x)} + \Delta u}{\alpha+n} \right)^{\alpha+n}$$

$$= \left(\frac{c}{\alpha+n} \right)^{\alpha+n} \quad \blacksquare$$