Nonlinear Integrate & Fire Neuron Models

María del Mar González

Universitat Politècnica de Catalunya

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- Joint work with: J. Carrillo (UAB), M. Gualdani (UT Austin), M. Schonbek (UCSC)

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- Previous work: M. Cáceres, J. Carrillo, B. Perthame (2011)

The model

Unknown: V =membrane potential

Basic LIF model

$$C_m \frac{dV}{dt} = -g_L(V - V_L) + I(t)$$

- $\tau_m = C_m/g_L \approx 10ms$
  $V_L \approx -70mV$.
- $I(t)$: the external input current.

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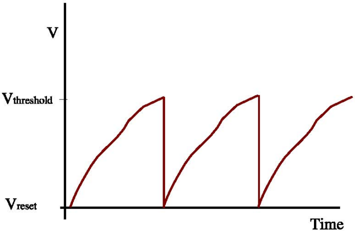
- $I(t)$: the external input current.
- Firing voltage: threshold value $V_F \approx -50mV$.
- Reset voltage: discharged value $V_R \approx -60mV$.

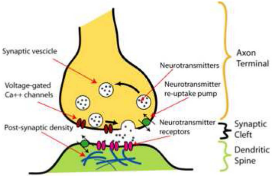


Stochastic Synapse Model

$$I(t) = J_E \sum_{i=1}^{C_E} \sum_j \delta(t - t^i_{Ej}) - J_I \sum_{i=1}^{C_I} \sum_j \delta(t - t^i_{Ij})$$

Parameters

- Two different Neuron-types: Inhibitory (I) or Excitatory (E).
  Strength of the Synapse: J.
  Number of presynaptic neurons: C.
- Spiking times: t^i_j = time of the j^{th}-spike coming from the i^{th}-presynaptic neuron.
- Stochastic Assumption: Neurons spike according to a Poisson process with constant probability of emitting a spike per unit time ν.



Mean and Variance of $I(t)$:

$$\mu_C = b\nu \qquad \text{with} \qquad b = C_E J_E - C_I J_I \qquad \text{and} \qquad \sigma_C^2 = (C_E J_E^2 + C_I J_I^2)\nu.$$

Diffusion Approximation

Several authors (Brunel, Hakim, Renart, Wang, Mattia, del Giudice) propose to approximate the current by

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2. The value of $N(t)$ is then computed as the flux of probability that neurons cross the threshold voltage V_F per unit time.
3. Average-excitatory (-inhibitory resp.) if $b > 0$ ($b < 0$ resp.).

PDE model (special case)

- Ito's rule: PDE for the evolution of the probability density $p(v,t) \geq 0$ of finding neurons at a voltage $v \in (-\infty, V_F]$ at a time $t \geq 0$.

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$$p_t(v,t) + \partial_v \left[h(v, N(t)) p(v,t) \right] - p_{vv}(v,t) = N(t)\delta_{v-V_R},$$

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- Initial data: $p(v,0) = p_I(v)$.

- Conservation of Mass:

$$\int_{-\infty}^{V_F} p(v,t)\,dv = \int_{-\infty}^{V_F} p_I(v)\,dv = 1\,.$$

Preliminaries

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- Excitatory case ($b > 0$): (Cáceres-Carrillo-Perthame)
  - There exists initial data such that develop blow up at finite time
  - There are cases of non-existence or non-uniqueness for the steady state

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    - There exists initial data such that develop blow up at finite time
    - There are cases of non-existence or non-uniqueness for the steady state
- Concentrate, for now, on the inhibitory case ($b < 0$)

PDE analysis

- Change of variables: Fokker-Plank to heat equation
  $y = e^t v, \quad \tau = \frac{1}{2}(e^{2t} - 1), \quad p(v,t) = e^t w\left(e^t v, \frac{1}{2}(e^{2t} - 1)\right)$

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- Get a free boundary Stefan problem with source
- Related to a price formation equation (Lasry-Lions,G.-Gualdani, Chayes-G.-Gualdani-Kim, Caffarelli-Markowich-Pietschmann)

Relation to Stefan problem

$$(P) \quad \begin{cases} u_t = u_{xx} + M(t)\delta_{x=s_1(t)}, & x < s(t), t > 0, \\ s(t) = -b \displaystyle\int_0^t M(s)\alpha^{-1}(s)\, ds, & t > 0, \\ M(t) = -\left.\dfrac{\partial u}{\partial x}\right|_{x=s(t)}, & t > 0, \\ \\ \\ \end{cases}$$

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3. If $b < 0$, the free boundary $s(t)$ is an increasing function of time, and $s(t)$, $s_1(t)$ never cross.

Integral formulation *a la Friedman*

- Green's function: $G(x,t,\xi,\tau)=\frac{1}{[4\pi(t-\tau)]^{1/2}}\exp\left\{-\frac{|x-\xi|^2}{4(t-\tau)}\right\}$

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- Duhamel's formula:

$$u(x,t) = \underbrace{\int_{-\infty}^{0} G(x,t,\xi,0)u_I(\xi)d\xi}_{\text{homogeneous heat equation}}$$

$$\underbrace{-\int_0^t M(\tau)G(x,t,s(\tau),\tau)d\tau}_{\text{Takes care of free boundary}} + \underbrace{\int_0^t M(\tau)G(x,t,s_1(\tau),\tau)d\tau}_{\text{Delta function at } s_1(t)}$$

Integral formula for M

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\begin{aligned}
M(t) &= \partial_x u(s(t), t) \\
&= -2 \int_{-\infty}^{0} G(s(t), t, \xi, 0) u_I'(\xi)\, d\xi \\
&\quad + 2 \int_0^t M(\tau) G_x(s(t), t, s(\tau), \tau) d\tau \\
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Fixed point argument: $\Theta : C_{\sigma,m} \to C_{\sigma,m}$, where

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Contraction for σ small, where $\sigma = \sigma\left(\|u_I'\|, v_R\right)$

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Corollary - short time existence for u

$\exists!$ "regular" solution u for $t \in (0, T)$

Global existence ($b < 0$)

Lemma - no break up time

$\exists \mu > 0$ small enough such that, for any $t_0 > 0$, if

$$\sup_{x \in (-\infty, s(t_0-\mu)]} |u_x(x, t_0 - \mu)| < \infty,$$

then also

$$\sup_{t_0-\mu < t < t_0} M(t) < \infty.$$

Here μ is independent of t_0.

Proof: more careful estimates in the integral formula

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Theorem ($b < 0$)

There exists a unique regular solution u for all $t > 0$.

Blow up when $b > 0$

Criterium

Solution exists up to time T^*, where

$$T^* = \sup\{t < 0 \,:\, M(t) < \infty\}.$$

Proof:
M bounded $\Rightarrow$ u_x bounded $\Rightarrow$ Can extend solution past T^*

Asymptotics for $b = 0$ (linear case - no free boundary)

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- There is a unique equilibrium.
- Solution decays exponentially fast to equilibrium.
- Entropy methods+Poincaré inequality
- Weight does not satisfy Muckenhoupt condition

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Question: decay.... at what rate ????

Equilibria

There exists a one parameter family of equilibria

$$p_\infty(v) = \begin{cases} e^{-v^2/2} & v \in (-\infty, V_R), \\ \alpha_0 e^{-v^2/2} \displaystyle\int_v^0 e^{v^2/2}\, dv & v \in (V_R, 0], \end{cases}$$

for

$$\alpha_0 := \left(\int_{V_R}^0 e^{v^2/2}\, dv \right)^{-1}.$$

Standard Fokker-Planck equation

- Compute the eigenvalues: $\mathcal{L}p = \lambda p$, where

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- Eigenvalues are

$$\lambda = -n, \quad n \in \mathbb{N}.$$

- Eigenfunctions are given by the Hermite polynomials H_n as

$$p_n(v) = H_n(v) e^{-v^2/2}.$$

Back to our model

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Conclusion: the spectral gap may have nothing to do with the exponential decay to equilibrium !!!

Thank you!!