# Functional inequalities and the symmetry properties of the extremal functions

Maria J. Esteban

CEREMADE

CNRS & Université Paris-Dauphine

IN COLLABORATION WITH

J. DOLBEAULT, M. LOSS, G. TARANTELLO, A. TERTIKAS

UIMP, Santander, August 2011

# Attainability and value of best constants in functional inequalities

Suppose that we know the existence of a constant $C$ such that in some space space $X$, for all $u \in X$,

$$F(Du, u, x) \leq C\, G(D^2u, Du, u, x)\,.$$

Functional inequalities play an important role in obtaining a priori estimates for solutions of PDEs, in analyzing the long time behavior of solutions of evolution problems, in describing the blow-up profile for finite time blow-up phenomena, etc

Knowing the exact value of $C$ or the properties of the optimal solutions (when they exist) can be crucial in some critical cases.

Important questions :

- Is $C$ attained in $X$?

- If yes, how do the optimal functions $u$ look like?

If we know a priori that the optimal solutions have some symmetry properties, for instance, that they are radially symmetric, then it might be easier to compute the value of $C$.

# Caffarelli-Kohn-Nirenberg (CKN) inequalities

$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx \qquad \forall\, u \in \mathcal{D}_{a,b}$$

with $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d = 2$, and $a \ne \frac{d-2}{2}$

$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

$$\mathcal{D}_{a,b} := \left\{\,|x|^{-b}\,u \in L^p(\mathbb{R}^d, dx)\;:\;|x|^{-a}\,|\nabla u| \in L^2(\mathbb{R}^d, dx)\right\}$$

$$b-a \,\to\, 0 \iff p \to \tfrac{2d}{d-2}$$

$$b-(a+1) \,\to\, 0 \iff p \,\to\, 2_+$$

$$\frac{1}{C_{a,b}} = \inf_{\mathcal{D}_{a,b}} \quad \frac{\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx}{\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{b\,p}}\,dx\right)^{2/p}}$$

# The symmetry issue

$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b}\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx \qquad \forall\, u \in \mathcal{D}_{a,b}$$

$\mathsf{C}_{a,b}$ = best constant for general functions $u$
$\mathsf{C}^*_{a,b}$ = best constant for radially symmetric functions $u$

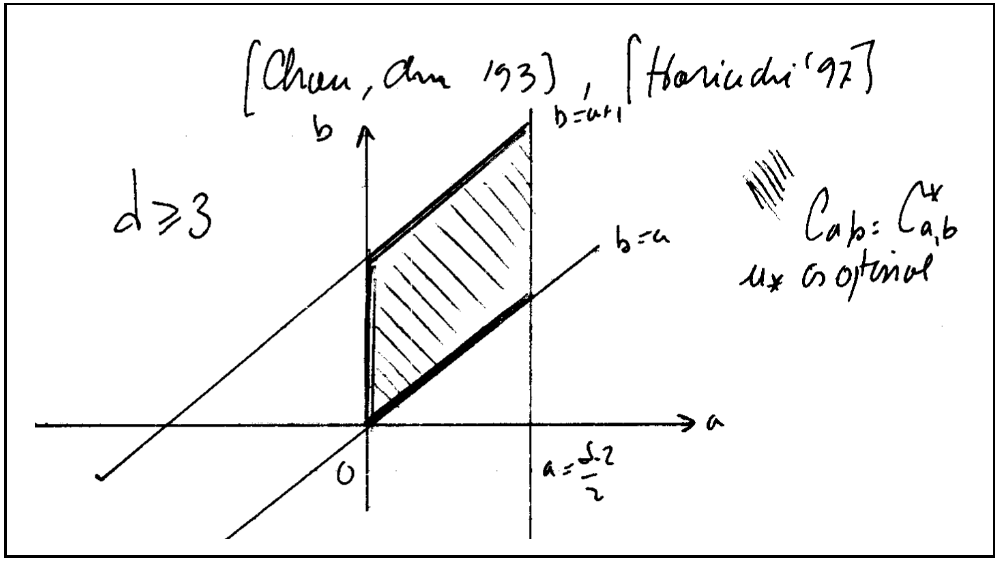
$$\mathsf{C}^*_{a,b} \le C_{a,b}$$

Up to scalar multiplication and dilation, the optimal radial function is

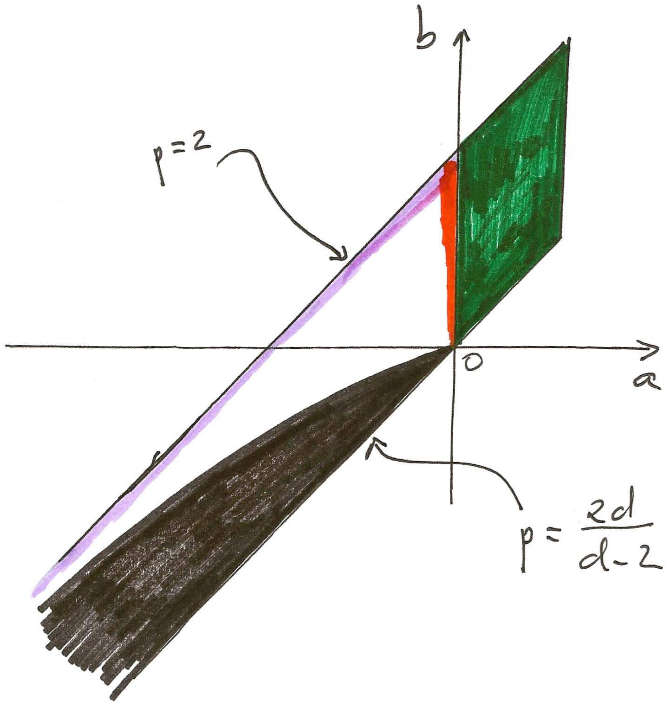
$$u^*_{a,b}(x) = \left(1 + |x|^{-\frac{2a\,(1+a-b)}{b-a}}\right)^{-\frac{b-a}{1+a-b}}$$

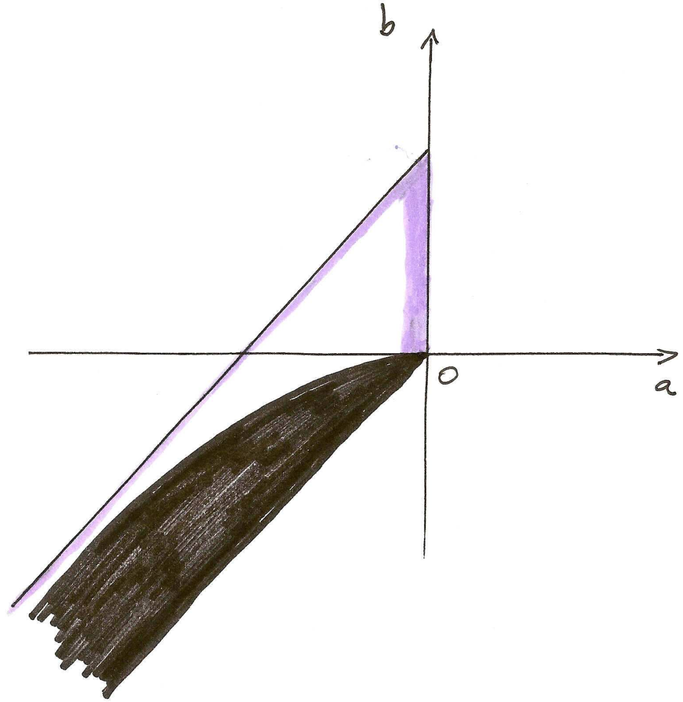
Questions: is optimality (equality) achieved ? do we have $u_{a,b} = u^*_{a,b}$ ?

- Existence inside the half-strip $a < b < a+1\,,\ a < \frac{d-2}{2}$
- Symmetry (and existence) in the zone $a \le b < a+1\,,\ 0 < a < \frac{d-2}{2}$
- Nonexistence for $a < 0$ and $b = a$ or $b = a+1$.



# Symmetry and symmetry breaking




SYMMETRY BREAKING: Catrina-Wang, Felli-Schneider.

Aubin, Talenti, Horiuchi, Lieb, Chou-Chu,...

Lin, Wang; Dolbeault, E., Tarantello (d=2)

Dolbeault, E., Loss, Tarantello

$$b^{FS}(a) = \frac{d(d-2-2a)}{2\sqrt{(d-2-2a)^2 + 4(d-1)}} - \frac{1}{2}(d-2-2a)$$

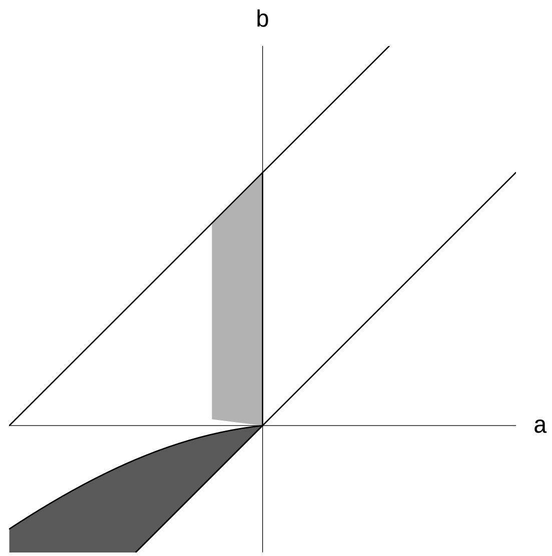
# Relation with Onofri's inequality ($d = 2$) (Dolbeault, E., Tarantello)

A generalized Onofri inequality:

On $\mathbb{R}^2$, consider $d\mu_\alpha = \frac{\alpha+1}{\pi} \frac{|x|^{2\alpha}\,dx}{(1+|x|^{2\,(\alpha+1)})^2}$ with $\alpha > -1$

$$\log\left(\int_{\mathbb{R}^2} e^u \; d\mu_\alpha\right) - \int_{\mathbb{R}^2} u \, d\mu_\alpha \;\leq\; \frac{1}{16\,\pi\,(\alpha+1)} \, \|\nabla u\|^2_{L^2(\mathbb{R}^2,\,dx)}$$

- For $d = 2$, radial symmetry holds if $-\eta < a < 0$ and $-\varepsilon(\eta)\,a \leq b < a+1$



# Emden-Fowler transformation and the cylinder $\mathcal{C} = \mathbb{R} \times S^{d-1}$

$$t = \log|x| \;, \quad \omega = \frac{x}{|x|} \in S^{d-1} \;, \quad v(t,\omega) = |x|^{-a}\, u(x) \;, \quad \Lambda = \frac{1}{4}\,(d-2-2a)^2$$

• Caffarelli-Kohn-Nirenberg inequalities rewritten on the cylinder become standard interpolation inequalities of Gagliardo-Nirenberg type

$$\|v\|^2_{L^p(\mathcal{C})} \le C_{\Lambda,p} \left[\|\nabla v\|^2_{L^2(\mathcal{C})} + \Lambda\, \|v\|^2_{L^2(\mathcal{C})}\right]$$

$$\mathcal{E}_\Lambda[v] := \|\nabla v\|^2_{L^2(\mathcal{C})} + \Lambda\, \|v\|^2_{L^2(\mathcal{C})}$$

$$C_{\Lambda,p}{}^{-1} := \mathsf{C}_{a,b}^{-1} = \inf\left\{\mathcal{E}_\Lambda(v) \;:\; \|v\|^2_{L^p(\mathcal{C})} = 1\right\}$$

$$a < \frac{d-2}{2} \implies \Lambda > 0 \;, \qquad a < 0 \implies \Lambda > \tfrac{1}{4}\,(d-2)^2$$

# Strategy of Catrina-Wang (or Felli-Schneider) for the symmetry breaking:

$$\mathcal{E}''(v^*_{\Lambda,p})(v) = \big(|a|p - (1+|a|^2) + \cdots\big)\,||v||^2_{L^2} < 0$$

when $v$ is orthogonal to all functions which do not depend on the angular variables $\omega$.

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This is done by stuyding the properties of the spectrum of the operator

$$-\Delta + \Lambda - (p-1)\,|v^*_{\Lambda,p}|^{p-2}$$

in $H^1(\mathcal{C})$ and then using the decomposition in spherical harmonics.

# Auxiliary results for the symmetry's proof

Normalize the extremal functions so that they satisfy $-\Delta v + \Lambda v = v^{p-1}$, that is

$$\int_{\mathcal{C}} |\nabla v|^2 \, dx + \Lambda \int_{\mathcal{C}} |v|^2 \, dx = \int_{\mathcal{C}} |v|^p \, dx$$

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Lemma. Let $d \geq 2$ , $p \in (2, 2^*)$ . For all $\Lambda \neq 0$ , we have

$$\|v_{\Lambda,p}\|^p_{L^p(\mathcal{C})} = (\mathsf{C}_{\Lambda,p})^{-\frac{p}{p-2}} \leq \left(\mathsf{C}^*_{\Lambda,p}\right)^{-\frac{p}{p-2}} = \|v^*_{\Lambda,p}\|^p_{L^p(\mathcal{C})} = 4\,|S^{d-1}|\,(2\,\Lambda\,p)^{\frac{p}{p-2}}\;\frac{c_p}{2\,p\,\sqrt{\Lambda}}$$

where $c_p$ in a function increasing in $p$ such that

$$\lim_{p \to 2_+} 2^{\frac{2\,p}{p-2}} \sqrt{p-2}\, c_p = \sqrt{2\pi}\ .$$

# Proof of of the symmetry result for $b$ near $a+1$ $d \geq 3$

By contradiction. Suppose the existence of $(\Lambda_n)_{n\in\mathbb{N}}$ and $(p_n)_{n\in\mathbb{N}}$ s. t.

$$\lim_{n\to+\infty} \Lambda_n = \Lambda \geq (d-2)^2/4\,, \quad \lim_{n\to+\infty} p_n = 2_+\,,$$

and such that the corresponding global minimum $v_n := v_{\Lambda_n\,,\,p_n}$ satisfies

$$\mathcal{F}_{\Lambda,p}[v_{\Lambda_n\,,\,p_n}] < \mathcal{F}_{\Lambda,p}[v^*_{\Lambda_n\,,\,p_n}]\,, \quad -\Delta_y v_n + \Lambda_n\, v_n = v_n^{p-1} \quad \text{in} \quad \mathcal{C}\,.$$

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Define $\quad c_n^2 = (\Lambda_n\, p_n)^{-\frac{p_n}{p_n-2}}\, 2^{\frac{p_n}{p_n-2}}\, \sqrt{p_n-2} \quad$ et $\quad W_n := c_n\, v_n\,.$

$$-\Delta W_n + \Lambda_n\, W_n = c_n^{2-p_n}\, W_n^{p_n-1} \quad \text{in} \quad \mathcal{C}\,, \qquad \int_{\mathcal{C}} |\nabla W_n|^2\, dy + \Lambda_n \int_{\mathcal{C}} W_n^2\, dy = c_n^2 \int_{\mathcal{C}} v_n^{p_n}\, dy\,.$$

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We have : $\lim_{n\to+\infty} c_n^{2-p_n} = \Lambda$ and $\limsup_{n\to+\infty} c_n^2 \int_{\mathcal{C}} v_n^{p_n}\, dy \leq |S^{d-1}|\,\sqrt{2\,\pi/\Lambda}$.
Hence $(W_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathcal{C})$

Elliptic estimates and Harnack's inequality imply that $W_n \to W$ and

$$-\Delta W + \Lambda W = \Lambda W\,.$$

Hence $W_n \to W \equiv 0\,.$

# End of the proof of the symmetry result for $b$ near $a+1$

Differentiating with respect to $\omega$ the equation

$$-\Delta W_n + \Lambda_n\, W_n = c_n^{2-p_n}\, W_n^{p_n-1} \text{ dans } \mathcal{C}\,,$$

we obtain:

$$-\Delta \chi_n + \Lambda_n\, \chi_n = (p_n - 1)\, c_n^{2-p_n}\, W_n^{p_n-2}\, \chi_n \quad , \quad \chi_n := \nabla_\omega v_n := (\sin\omega)^{2-d}\, \frac{\partial}{\partial\omega}\Big((\sin\omega)^{d-2}\, v_n\Big)$$

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$$0 = \int_{\mathcal{C}} |\nabla \chi_n|^2\, dy + \Lambda_n \int_{\mathcal{C}} |\chi_n|^2\, dy - (p_n - 1)\, c_n^{2-p_n} \int_{\mathcal{C}} W_n^{p_n-2}\, |\chi_n|^2\, dy\,.$$

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Since $\int_{S^{d-1}} \nabla_\omega W_n(t,\omega)\, d\omega = 0$, an expansion of $\chi_n$ in spherical harmonics shows that

$$\int_{\mathcal{C}} |\nabla \chi_n|^2\, dy \geq (d-1) \int_{\mathcal{C}} |\chi_n|^2\, dy\ .$$

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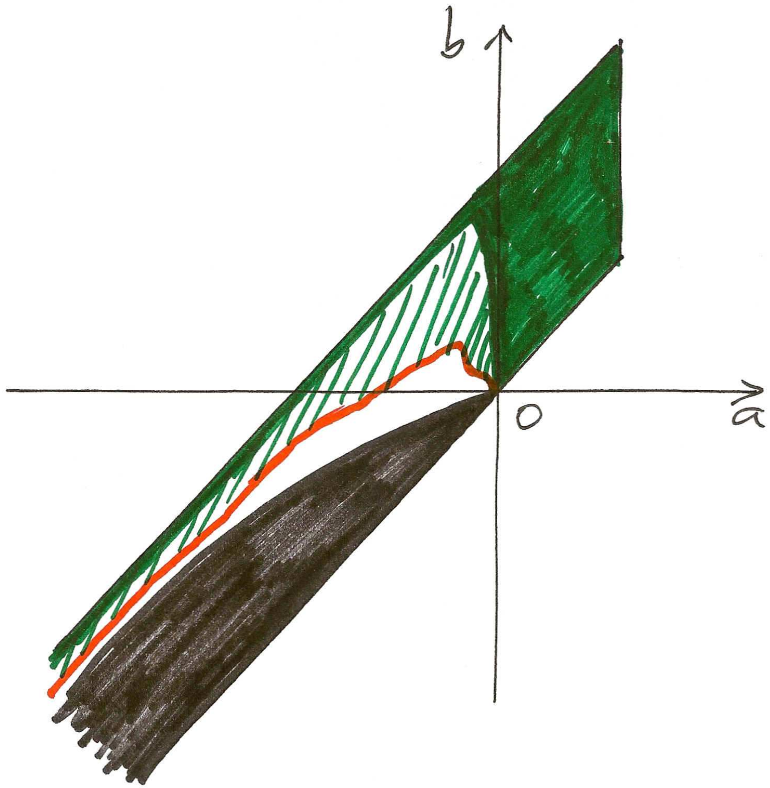
Since the functions $W_n$ are bounded from above by $W_n(0,\omega_0)$, we obtain

$$0 \geq \Big(d-1+\Lambda_n-(p_n-1)\, c_n^{2-p_n}\, W_n(0,\omega_0)^{p_n-2}\Big) \int_{\mathcal{C}} |\chi_n|^2\, dy\;.$$

Contradiction: $\lim_{n\to+\infty}\Lambda_n = \Lambda$, $\limsup\,(p_n-1)\, c_n^{2-p_n}\, W_n(0,\omega_0)^{p_n-2} \leq \Lambda$

# Two simply connected regions separated by a continuous curve

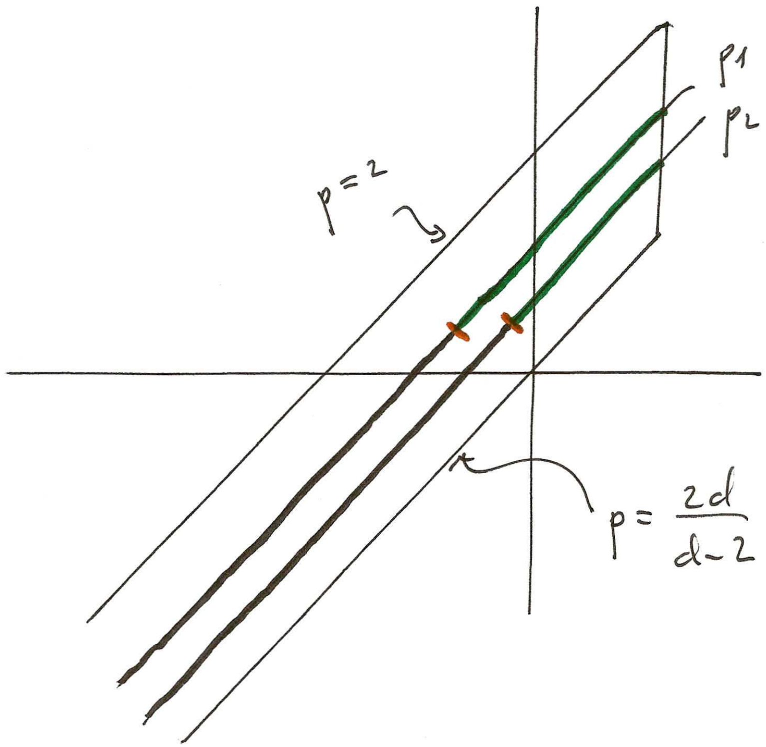
The symmetry and the symmetry breaking zones are simply connected and separated by a continuous curve.

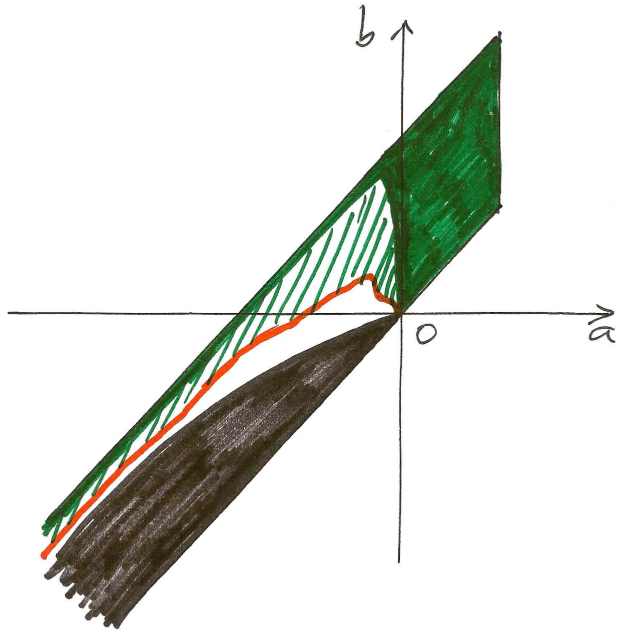


# Scaling and consequences

Let $w_\sigma(t,\omega) := w(\sigma\, t,\omega)$ for any $\sigma > 0$

$$\mathcal{F}_{\sigma^2\Lambda,p}(w_\sigma) = \sigma^{1+2/p}\,\mathcal{F}_{\Lambda,p}(w) - \sigma^{-1+2/p}\,(\sigma^2-1)\,\frac{\int_{\mathcal{C}}|\nabla_\omega w|^2\,dy}{\left(\int_{\mathcal{C}}|w|^p\,dy\right)^{2/p}}$$




Upper semicontinuity of the curve $p \to \bar\Lambda(p)$ is easy to prove.

For continuity, a delicate spectral analysis is needed.

CONJECTURE: $\bar\Lambda$ coincides with $\Lambda_{FS}$.

# End of the proof of the symmetry result for $b$ near $a+1$

Let us how see why the identity

$$\mathcal{F}_{\sigma^2\Lambda,p}(w_\sigma) = \sigma^{1+2/p}\,\mathcal{F}_{\Lambda,p}(w) - \sigma^{-1+2/p}\,(\sigma^2-1)\,\frac{\int_{\mathcal{C}}|\nabla_\omega w|^2\,dy}{\left(\int_{\mathcal{C}}|w|^p\,dy\right)^{2/p}}$$

implies that if for some given $(\Lambda, p)$ there is a non-symmetric minimizer, then for all $\sigma > 1)$, and for $(\sigma^2\Lambda, p)$ no minimizer can be symmetric.

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CONSEQUENCE: $\left(\mathsf{C}^*_{\sigma^2\Lambda,p}\right)^{-1} = \sigma^{1+2/p}\,\left(\mathsf{C}^*_{\Lambda,p}\right)^{-1}$ .

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CONSEQUENCE: $$\left(\mathsf{C}^*_{\sigma^2\Lambda,p}\right)^{-1} = \sigma^{1+2/p}\,\left(\mathsf{C}^*_{\Lambda,p}\right)^{-1}\,.$$

Assume that $w_{\Lambda,p}$ is a non radially symmetric extremal function and apply the above identity with $w = w_{\Lambda,p}$, $w_\sigma(t,\omega) := w(\sigma\,t,\omega)$, $\lambda = \sigma^2\Lambda$ and $\sigma > 1$:

$$\left(\mathsf{C}_{\lambda,p}\right)^{-1} \le \mathcal{F}_{\sigma^2\Lambda,p}(w_\sigma) = \sigma^{1+\frac{2}{p}}\,\left(\mathsf{C}_{\Lambda,p}\right)^{-1} - \sigma^{-1+\frac{2}{p}}\,(\sigma^2-1)\,\frac{\int_{\mathcal{C}} |\nabla_\omega w_{\Lambda,p}|^2\,dy}{\left(\int_{\mathcal{C}} |w_{\Lambda,p}|^p\,dy\right)^{\frac{2}{p}}}$$

$$\le \sigma^{1+\frac{2}{p}}\left(\mathsf{C}^*_{\Lambda,p}\right)^{-1} - \sigma^{-1+\frac{2}{p}}\,(\sigma^2-1)\,\frac{\int_{\mathcal{C}} |\nabla_\omega w_{\Lambda,p}|^2\,dy}{\left(\int_{\mathcal{C}} |w_{\Lambda,p}|^p\,dy\right)^{\frac{2}{p}}} < \left(\mathsf{C}^*_{\lambda,p}\right)^{-1},$$

since $\nabla_\omega w_{\Lambda,p} \not\equiv 0$. This proves the second claim with $\lambda = \sigma^2\,\Lambda$.

# Generalized Caffarelli-Kohn-Nirenberg inequalities (CKN)

Let $d \geq 3$ . For any $p \in [2, p(\theta, d) := \frac{2d}{d-2\theta}]$, there exists a positive constant $\mathsf{C}(\theta, p, a)$ such that

$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{b\,p}}\,dx\right)^{\frac{2}{p}} \leq \mathsf{C}(\theta, p, a)\left(\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx\right)^{\theta}\left(\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}}\,dx\right)^{1-\theta}$$

In the radial case, with $\Lambda = (a - a_c)^2$, the best constant when the inequality is restricted to radial functions is $\mathsf{C}^*_{\rm CKN}(\theta, p, a)$ and (see [Del Pino, Dolbeault, Filippas, Tertikas]):

$$\mathsf{C}_{\rm CKN}(\theta, p, a) \geq \mathsf{C}^*_{\rm CKN}(\theta, p, a) = \mathsf{C}^*_{\rm CKN}(\theta, p)\,\Lambda^{\frac{p-2}{2p}-\theta}$$

$$\mathsf{C}^*_{\rm CKN}(\theta, p) = \left[\frac{2\,\pi^{d/2}}{\Gamma(d/2)}\right]^{2\,\frac{p-1}{p}}\left[\frac{(p-2)^2}{2+(2\,\theta-1)\,p}\right]^{\frac{p-2}{2\,p}}\left[\frac{2+(2\,\theta-1)\,p}{2\,p\,\theta}\right]^{\theta}\left[\frac{4}{p+2}\right]^{\frac{6-p}{2\,p}}\left[\frac{\Gamma\left(\frac{2}{p-2}+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{2}{p-2}\right)}\right]^{\frac{p-2}{p}}$$

---

For $\theta \in (0,1)$ fixed, $p \in [2, p(\theta, d) := \frac{2d}{d-2\theta}]$.

For $p \in [2, \frac{2d}{d-2}]$ fixed, $\theta \in [2, \theta(p, d) := \frac{d(p-2)}{2p}]$.

# Existence results (Dolbeault, E.)

Let $d \geq 2$ and assume that $a \in (-\infty, \frac{d-2}{2})$.

– For any $p \in (2, 2^*)$ and any $\theta \in (\vartheta(p,d), 1)$, the Caffarelli-Kohn-Nirenberg inequality (CKN)

$$\left( \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{b\,p}} \, dx \right)^{\frac{2}{p}} \leq \mathsf{C}(\theta, p, a) \left( \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}} \, dx \right)^{\theta} \left( \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}} \, dx \right)^{1-\theta}$$

admits an extremal function in $\mathcal{D}_a^{1,2}(\mathbb{R}^d)$

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Critical case: there exists a continuous function $a^* : (2, 2^*) \to (-\infty, a_c)$ such that the inequality also admits an extremal function in $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$ if $\theta = \vartheta(p,d)$ and $a \in (a^*(p), a_c)$

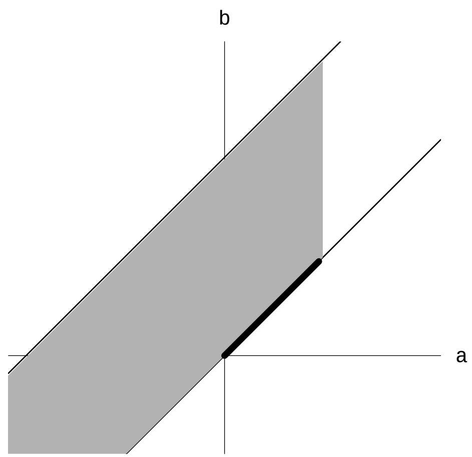
Let $d \geq 2$ and assume that $a \in (-\infty, \frac{d-2}{2})$.

– For any $p \in (2, 2^*)$ and any $\theta \in (\vartheta(p,d), 1)$, the Caffarelli-Kohn-Nirenberg inequality (CKN)

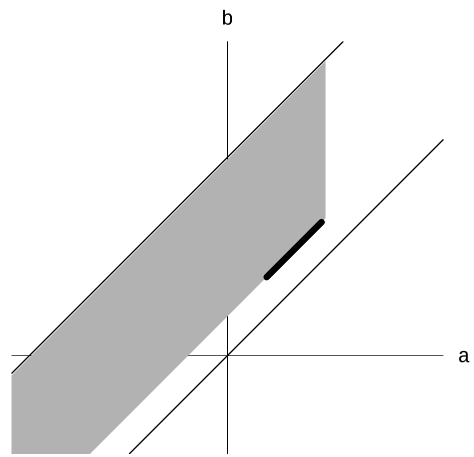
$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{b\,p}}\,dx\right)^{\frac{2}{p}} \leq \mathsf{C}(\theta,p,a)\left(\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx\right)^{\theta}\left(\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}}\,dx\right)^{1-\theta}$$

admits an extremal function in $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$

Critical case: there exists a continuous function $a^* : (2, 2^*) \to (-\infty, a_c)$ such that the inequality also admits an extremal function in $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$ if $\theta = \vartheta(p,d)$ and $a \in (a^*(p), a_c)$




$d = 3$, $\theta = 1$




$d = 3$, $\theta = 0.8$

# Strategy of the proofs

Emden-Fowler transformation and minimization on the cylinder of the functionals

$$\mathcal{E}_\theta[v] := \left(\|\nabla v\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda\,\|v\|^2_{\mathrm{L}^2(\mathcal{C})}\right)^\theta \|v\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathcal{C})}$$

under the constraints $\|v\|_{\mathrm{L}^p(\mathcal{C})} = 1$.

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under the constraints $\|v\|_{\mathrm{L}^p(\mathcal{C})} = 1$.

Existence: Convergence of minimizing sequences if they are bounded in $H^1(\mathcal{C})$: by a more or less classical use of concentration-compactness techniques.

For $p \in (2, p(\theta, d))$ , existence is easy to prove, for all $\Lambda > 0$.

For $p = p(\theta, d)$ , a priori estimates can be obtained by relatively elaborate direct interpolation inequalities or estimating the energy strictly from above.

# A possible loss of compactness

- Gagliardo-Nirenberg interpolation inequalities: if $p \in (2, 2^*)$,

$$\|u\|^2_{\mathrm{L}^p(\mathbb{R}^d)} \le \mathsf{C}_{\mathrm{GN}}(p)\, \|\nabla u\|^{2\,\vartheta(p,d)}_{\mathrm{L}^2(\mathbb{R}^d)}\, \|u\|^{2\,(1-\vartheta(p,d))}_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{R}^d)$$

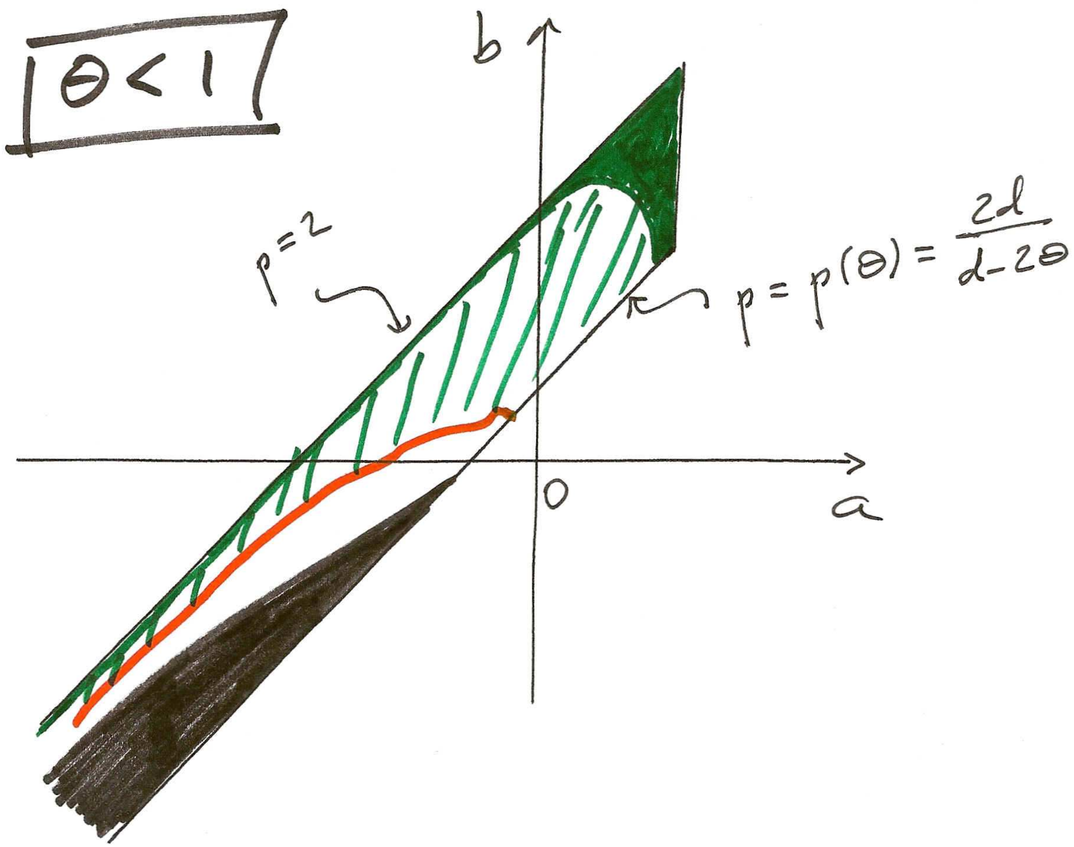
If $u$ is a radial minimizer for $1/\mathsf{C}_{\mathrm{GN}}(p)$ and $u_n(x) := u(x + n\,\mathsf{e})$ , $\mathsf{e} \in \mathbb{S}^{d-1}$

$$\frac{1}{\mathsf{C}_{\mathrm{CKN}}(\vartheta(p,d), p, a)} \le \frac{\||x|^{-a}\, \nabla u_n\|^{2\,\vartheta(p,d)}_{\mathrm{L}^2(\mathbb{R}^d)}\, \||x|^{-(a+1)}\, u_n\|^{2\,(1-\vartheta(p,d))}_{\mathrm{L}^2(\mathbb{R}^d)}}{\||x|^{-b}\, u_n\|^2_{\mathrm{L}^p(\mathbb{R}^d)}}$$

$$= \frac{1}{\mathsf{C}_{\mathrm{GN}}(p)}\, \left(1 + \mathcal{R}\, n^{-2} + O(n^{-4})\right)$$

So, $\mathsf{C}_{\mathrm{GN}} \le \mathsf{C}_{\mathrm{CKN}}$.

A priori estimates (and relative compactness of minimizing sequences) can also be obtained if $\mathsf{C}_{\mathrm{GN}} < \mathsf{C}_{\mathrm{CKN}}$.


θ < 1
b
p = 2
p = p(θ) = 2d / (d − 2θ)
0
a

# New symmetry breaking results

We already know that

$$\mathsf{C}_{\mathrm{GN}} \le \mathsf{C}_{\mathrm{CKN}}\,.$$

If we are able to find a positive $\Lambda$ and a function $g$ such that

$$\frac{1}{\mathsf{C}_{\mathrm{GN}}} \le \mathcal{E}_{GN}[g] < \frac{1}{\mathsf{C}^*_{\mathrm{CKN}}(\vartheta(p,d),p,\Lambda)}\,,$$

then,

$$\frac{1}{\mathsf{C}_{\mathrm{CKN}}(\vartheta(p,d),p,\Lambda)} \le \frac{1}{\mathsf{C}_{\mathrm{GN}}} < \frac{1}{\mathsf{C}^*_{\mathrm{CKN}}(\vartheta(p,d),p,\Lambda)}$$

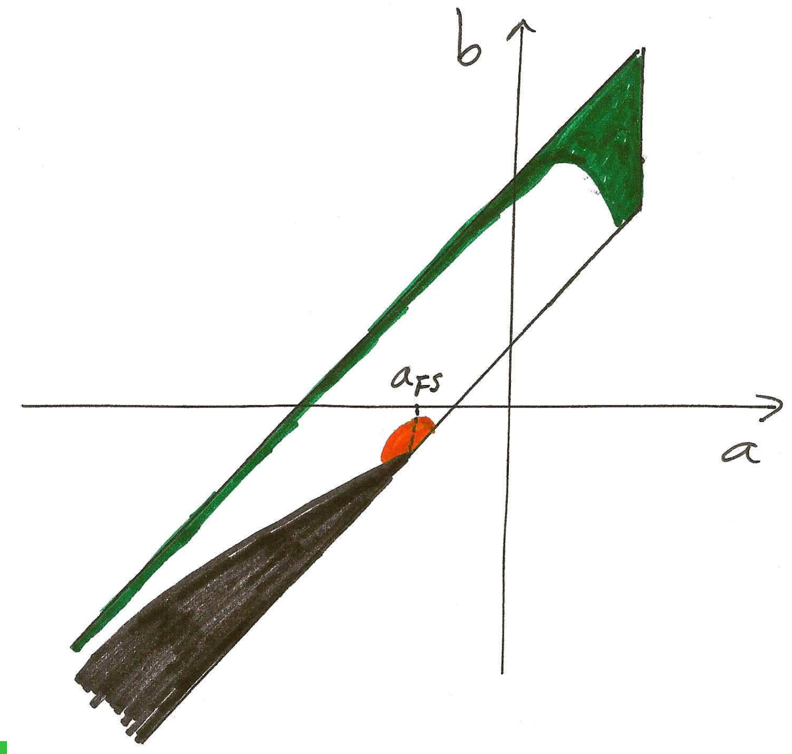
# A new symmetry breaking result (2010, Dolbeault, E., Tarantello, Tertikas)

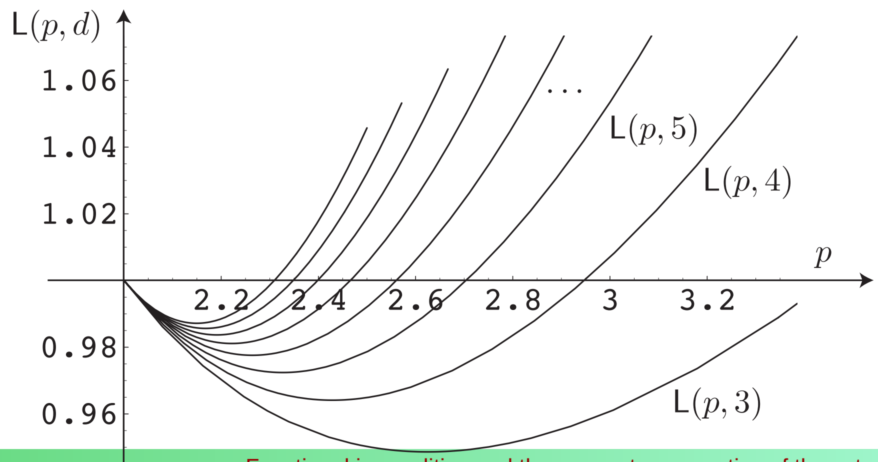
Let $g(x) := (2\pi)^{-d/4} \exp(-|x|^2/4)$. Choose $\Lambda = \Lambda_{FS}(p(\theta, d), d)$

Symmetry breaking occurs if

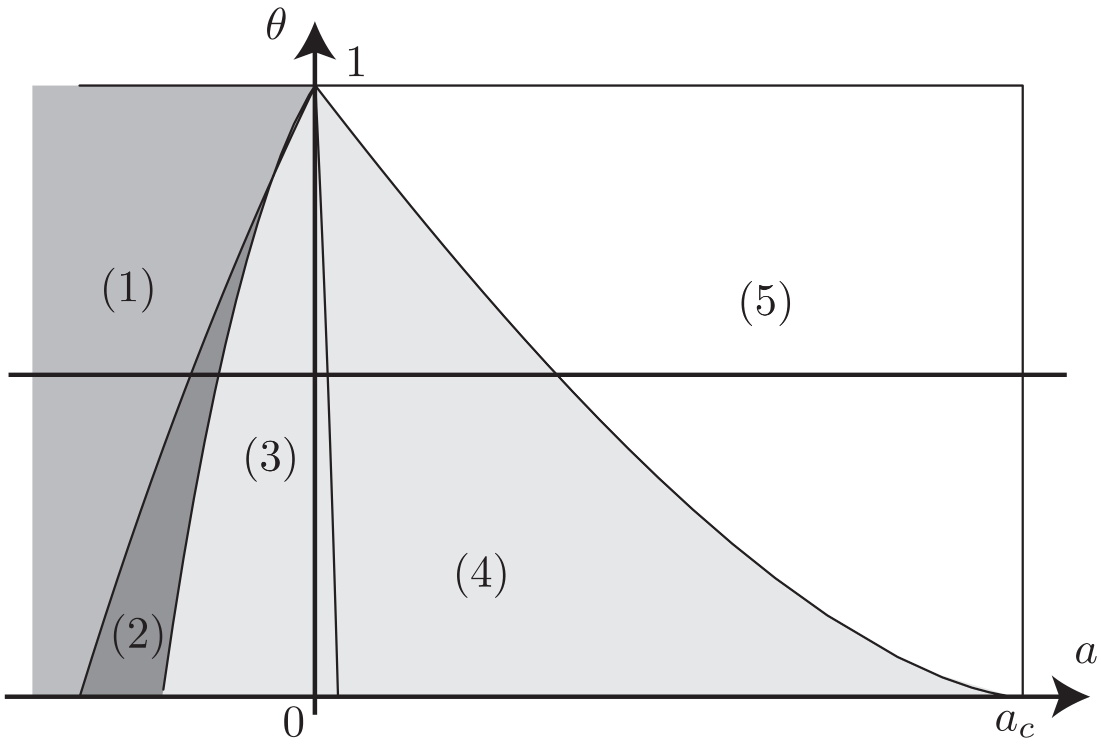
$$\mathsf{L}(p,d) := \frac{\mathcal{E}_{GN}[g]}{\frac{1}{\mathsf{C}^*_{\mathrm{CKN}}(\vartheta(p,d),p,\Lambda)}} < 1$$

We have the following result:



# Analytical and numerical results




(1) Symmetry breaking by instability of the radial extremals.
(2) Symmetry breaking by comparison with GN best constant.

(5) existence obtained because of symmetry by kind of Schwarz' symmetrization.
(4) existence by a priori estimates obtained analytically.
(3) existence obtained by (strict) comparison of best constant with best constant of GN.

# A new symmetry result (June 2011, Dolbeault, E., Loss)

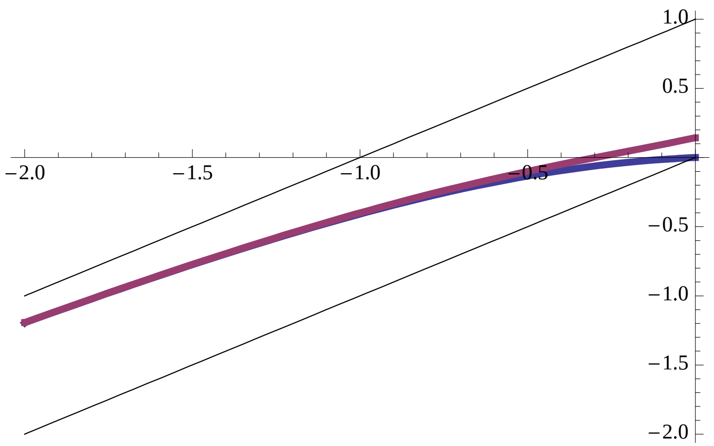
For $\theta = 1$ and $d \geq 2$,

using an elaborate argument involving Lieb-Thirring inequalities, Beckner's inequalities on the sphere and a rigidity argument due to Bidaut-Veron and Veron for positive solutions of $-\Lambda_g v + \mu v = v^q$ on manifolds $(M, g)$,

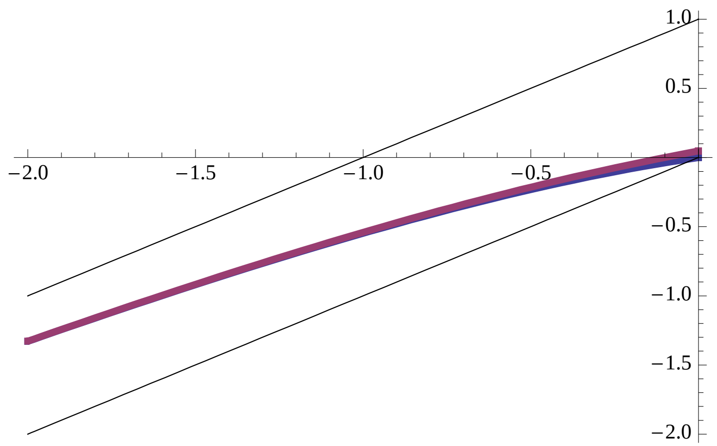
there exists a unique minimizer for the (CKN) problem, and it is symmetric, for all $\Lambda \leq \tilde{\Lambda}(p)$, for all $p \in (2, \frac{2d}{d-2})$.

$$\tilde{\Lambda}(p) := \frac{(d-1)(6-p)}{4(p-2)} < \Lambda_{FS}(p)\,.$$

$d = 3$




$d = 5$



# Strategy of the proofs

Let $L^2$ be the Laplace-Beltrami operator on $S^{d-1}$. So that $-\Delta$ on the cylinder becomes $-\partial_s^2 - L^2$.

THEOREM. Let $d \geq 2$ and let $u$ be a non-negative function on $\mathcal{C} = \mathbb{R} \times S^{d-1}$ that satisfies

$$-\partial_s^2 v - L^2 v + \Lambda\, v = v^{p-1}$$

and consider the symmetric solution $v_*$ . Assume that

$$\int_{\mathcal{C}} |v(s,\omega)|^p \, ds \, d\omega \leq \int_{\mathbb{R}} |v_*(s)|^p \, ds$$

for some $2 < p < 6$ satisfying $p \leq \frac{2\,d}{d-2}$ . If $\Lambda \leq \tilde{\Lambda}(p)$, then for a.e. $\omega \in S^{d-1}$ and $s \in \mathbb{R}$ , we have $v(s,\omega) = v_*(s - C)$ for some constant $C$ .

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REMARK 1. With the above normalization, we have

$$\frac{1}{C_{\Lambda,p}} = \inf \frac{\int_{\mathcal{C}} |\nabla v|^2 + \Lambda\, v^2 \; dx}{\left(\int_{\mathcal{C}} |v|^p \; dx\right)^{2/p}} = \left(\int_{\mathcal{C}} |v(s,\omega)|^p \, ds\, d\omega\right)^{\frac{p-2}{p}} \quad .$$

REMARK 2. We choose $d\omega$ to be a probability mesaure on $S^{d-1}$.

LEMMA. Let $V = V(s)$ be a non-negative real valued potential in $\mathrm{L}^{\gamma+\frac{1}{2}}(\mathbb{R})$ for some $\gamma > 1/2$ and let $-\lambda_1(V)$ be the lowest eigenvalue of the Schrödinger operator $-\frac{d^2}{ds^2} - V$. Define

$$c_{\mathrm{LT}}(\gamma) = \frac{\pi^{-1/2}}{\gamma - 1/2} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1/2)} \left(\frac{\gamma - 1/2}{\gamma + 1/2}\right)^{\gamma+1/2} .$$

Then

$$\lambda_1(V)^\gamma \leq c_{\mathrm{LT}}(\gamma) \int_{\mathbb{R}} V^{\gamma+1/2}(s)\, ds$$

with equality if and only if, up to scalings and translations,

$$V(s) = \frac{\gamma^2 - 1/4}{\cosh^2(s)} =: V_0(s)$$

in which case

$$\lambda_1(V_0) = (\gamma - 1/2)^2 .$$

Furthermore, the corresponding ground state eigenfunction is given by

$$\psi_\gamma(s) = \pi^{-1/4} \left(\frac{\Gamma(\gamma)}{\Gamma(\gamma - 1/2)}\right)^{1/2} \left[\cosh(s)\right]^{-\gamma+1/2} .$$

With $V = v^{p-2}$, the equation $-\Delta v + \Lambda v = v^{p-1}$ can be seen as the “linear" equation $-\Delta v - V v = -\Lambda v$.

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$$-\Lambda \int_{\mathcal{C}} |v(s,\omega)|^2 \, ds \, d\omega = \int_{S^{d-1}} \int_{\mathbb{R}} \left( v_s^2 - v^p \right) \; ds \, d\omega + \int_{\mathcal{C}} |Lv|^2 \; ds \, d\omega$$

$$\int_{S^{d-1}} \int_{\mathbb{R}} \left( v_s^2 - V \, v^2 \right) \; ds \, d\omega + \int_{\mathcal{C}} |Lv|^2 \; ds \, d\omega =: \mathcal{F}[v] \; .$$

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$$\mathcal{F}[v] \geq - c_{\mathrm{LT}}(\gamma)^{1/\gamma} \left( \int_{\mathbb{R}} |v(s,\omega)|^p\, ds \right)^{1/\gamma} |f|^2 + \int_{S^{d-1}} |Lf|^2\, d\omega\, .$$

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Now, setting $D := c_{\rm LT}(\gamma)^{1/\gamma} \left( \int_{\mathcal{C}} v^p \; ds \, d\omega \right)^{\frac{1}{\gamma}}$, by using Hölders's inequality, we obtain

$$\mathcal{F}[v] \geq \int_{S^{d-1}} (Lf)^2 \, d\omega - D \left( \int_{S^{d-1}} f^{\frac{2\,\gamma}{\gamma-1}} \, d\omega \right)^{\frac{\gamma-1}{\gamma}} =: \mathcal{E}[f] \; .$$

The generalized Poincaré inequality on the sphere states that for all $q \in \left(1, \frac{d+1}{d-3}\right]$,

$$\frac{q-1}{d-1} \int_{S^{d-1}} (Lf)^2 \, d\omega \geq \left( \int_{S^{d-1}} f^{q+1} \, d\omega \right)^{\frac{2}{q+1}} - \int_{S^{d-1}} f^2 \, d\omega$$

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Choosing $q+1 = \frac{2\,\gamma}{\gamma-1} = 2\,\frac{p+2}{6-p}$ ,

$$\mathcal{E}[f] \geq \left( \frac{d-1}{q-1} - D \right) \left( \int_{S^{d-1}} f^{q+1} \, d\omega \right)^{\frac{2}{q+1}} - \frac{d-1}{q-1} \int_{S^{d-1}} f^2 \, d\omega \; .$$

To justify this step, we notice that $q \leq \frac{d+1}{d-3}$ is equivalent to $p \leq \frac{2\,d}{d-2}$.

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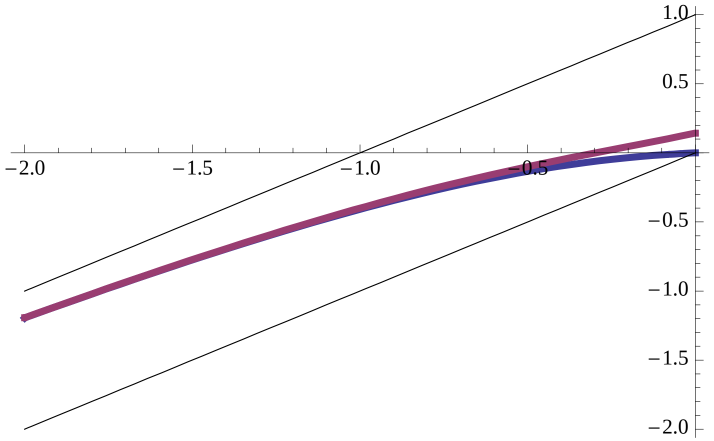
Thus, if $D \leq \frac{d-1}{q-1}$ , and if $\Lambda \leq \tilde{\Lambda}(p)$, we get

$$-\Lambda \int_{S^{d-1}} f^2 \, d\omega \geq \mathcal{E}[f] \geq -D \int_{S^{d-1}} f^2 \, d\omega \geq -\Lambda \int_{S^{d-1}} f^2 \, d\omega \ .$$

So, here we are: for $\theta = 1$,




BUT when $\theta \in (0, \theta_0)$, $\theta_0 < 1$,

we have symmetry beaking outside the Felli-Schneider zone!

# Weighted logarithmic Hardy inequalities (WLH)

Let $d \geq 1$, $a < (d-2)/2$, $\gamma \geq d/4$ and $\gamma > 1/2$ if $d = 2$. Then there exists a positive constant $\mathsf{C}_{\mathrm{WLH}}$ such that, for any $u \in \mathcal{D}^{1,2}_a(\mathbb{R}^d)$ normalized by $\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}}\,dx = 1$,

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}} \log\left(|x|^{d-2-2\,a}\,|u|^2\right) dx \leq 2\,\gamma\,\log\left[\mathsf{C}_{\mathrm{WLH}} \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx\right]$$

(Del Pino, Dolbeault, Filippas, Tertikas)

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(Del Pino, Dolbeault, Filippas, Tertikas)

In the radial case, the best constant when the inequality is restricted to radial functions is $\mathsf{C}^*_{\rm WLH}(\gamma,a)$ and $\mathsf{C}_{\rm WLH}(\gamma,a) \geq \mathsf{C}^*_{\rm WLH}(\gamma,a)$.

$$C^*_{\rm WLH} = \frac{1}{\gamma}\,\frac{\left[\Gamma\left(\frac{d}{2}\right)\right]^{\frac{1}{2\,\gamma}}}{\left(8\,\pi^{d+1}\,e\right)^{\frac{1}{4\,\gamma}}}\left(\frac{4\,\gamma-1}{(d-2-2\,a)^2}\right)^{\frac{4\,\gamma-1}{4\,\gamma}}\;(\gamma > \tfrac{1}{4})\,,\quad C^*_{\rm WLH} = 4\,\frac{\left[\Gamma\left(\frac{d}{2}\right)\right]^2}{8\,\pi^{d+1}\,e}\quad(\gamma = \tfrac{1}{4})$$

If $\gamma > \frac{1}{4}$, equality is achieved by the function

$$u = \frac{\tilde{u}}{\int_{\mathbb{R}^d}\frac{|\tilde{u}|^2}{|x|^2}\,dx}\quad\text{where}\quad \tilde{u}(x) = |x|^{-\frac{d-2-2\,a}{2}}\,\exp\left(-\tfrac{(d-2-2\,a)^2}{4\,(4\,\gamma-1)}\left[\,\log|x|\,\right]^2\right)$$