Nonlocal interaction PDEs with repulsion and attraction: the quadratic diffusion case

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Frontiers of Mathematics and Applications - Summer Course UIMP 2011
Santander (Spain), August 15-19, 2011

Table of contents

Table of contents

The evolutionary PDE

We consider

$$\partial_t \rho = \operatorname{div}\left(\rho \nabla \left(\varepsilon \rho - G * \rho\right)\right) \tag{1}$$

posed on $\mathbb{R}^d$ with initial datum in $L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ with $\rho \geq 0$, $\varepsilon > 0$.

Assumptions on the kernel G

- G is *radial*, i. e. $G(x) = g(|x|)$,
- G is *smooth*, i. e. $G \in W^{1,1} \cap L^\infty \cap C^2(\mathbb{R}^d)$,
- G is *strictly attractive*, i. e. $g'(r) < 0$ for all $r > 0$,
- G is *bounded from below*, i. e. (not restrictive) $g \geq 0$,
- G is infinitesimal at infinity, i. e. $\lim_{r \to +\infty} g(r) = 0$,
- $g''(0) < 0$.

Notice that the attractivity and smoothness assumptions imply $g'(0) = 0$.

Basic properties

Conservation of non–negativity

Due to $\rho_0 \geq 0$ we have $\rho(x,t) \geq 0$ almost everywhere for all $t > 0$.

Mass Conservation

$$M := \int \rho(x,t)dx = \int \rho_0(x)dx,$$

for all $t \geq 0$. We shall assume for simplicity that $M = 1$.

Conservation of the center of mass

Let

$$CM[\rho(t)] := \int x\rho(x,t)dx,$$

then $CM[\rho(t)] = CM[\rho_0]$ for all $t \geq 0$. Crucial hypotheses needed: $G(-x) = G(x)$.

Energy functional

We recall the *energy functional*

$$E[\rho] := \frac{\varepsilon}{2}\int_{\mathbb{R}^d} \rho^2(x)dx - \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x-y)\rho(y)\rho(x)dydx. \qquad (2)$$

The following *energy identity* satisfied by the solution to (1) easily follows by formal computation:

$$E[\rho(t)] + \int_0^T \int_{\mathbb{R}^d} \rho\,|\nabla(\varepsilon\rho - G*\rho)|^2\,dxdt = E[\rho_0]. \qquad (3)$$

The identity (3) can be proven rigorously in the context of the *Wasserstein gradient flow* theory developed in [Ambrosio, Gigli, Savaré, Birkhäuser 2003].

Table of contents

Table of contents

Interacting particles

Assume N particles in R^d, located on $X_1(t), \dots, X_N(t)$ respectively and having masses $m_1, \dots, m_N$ respectively, are subject to binary interactions of the form [1] [2] [3]

$$\frac{d}{dt}X_j(t) = -\sum_{j=1, j\neq i}^{N} m_j \nabla I(X_i(t) - X_j(t)), \quad j = 1, \dots, N,$$

where the interaction force ∇I is decomposed into a *repulsive* part and an *attractive* part, namely

$$\nabla I(x) = -\nabla G(x) + \nabla F(x).$$

In order to have attractive and repulsive effects, we require

$$G(x) = g(|x|), \quad g'(r) < 0, \text{ as } r > 0,$$
$$F(x) = f(|x|), \quad f'(r) < 0, \text{ as } r > 0.$$

[1] [Mogilner, Edelstein-Keshet - J. Math. Biol. 1999]
[2] [Morale, Capasso, Oelschlaeger - J. Math. Biol. 2005]
[3] [Topaz, Bertozzi, Lewis - Bull. Math. Biol. 2006]

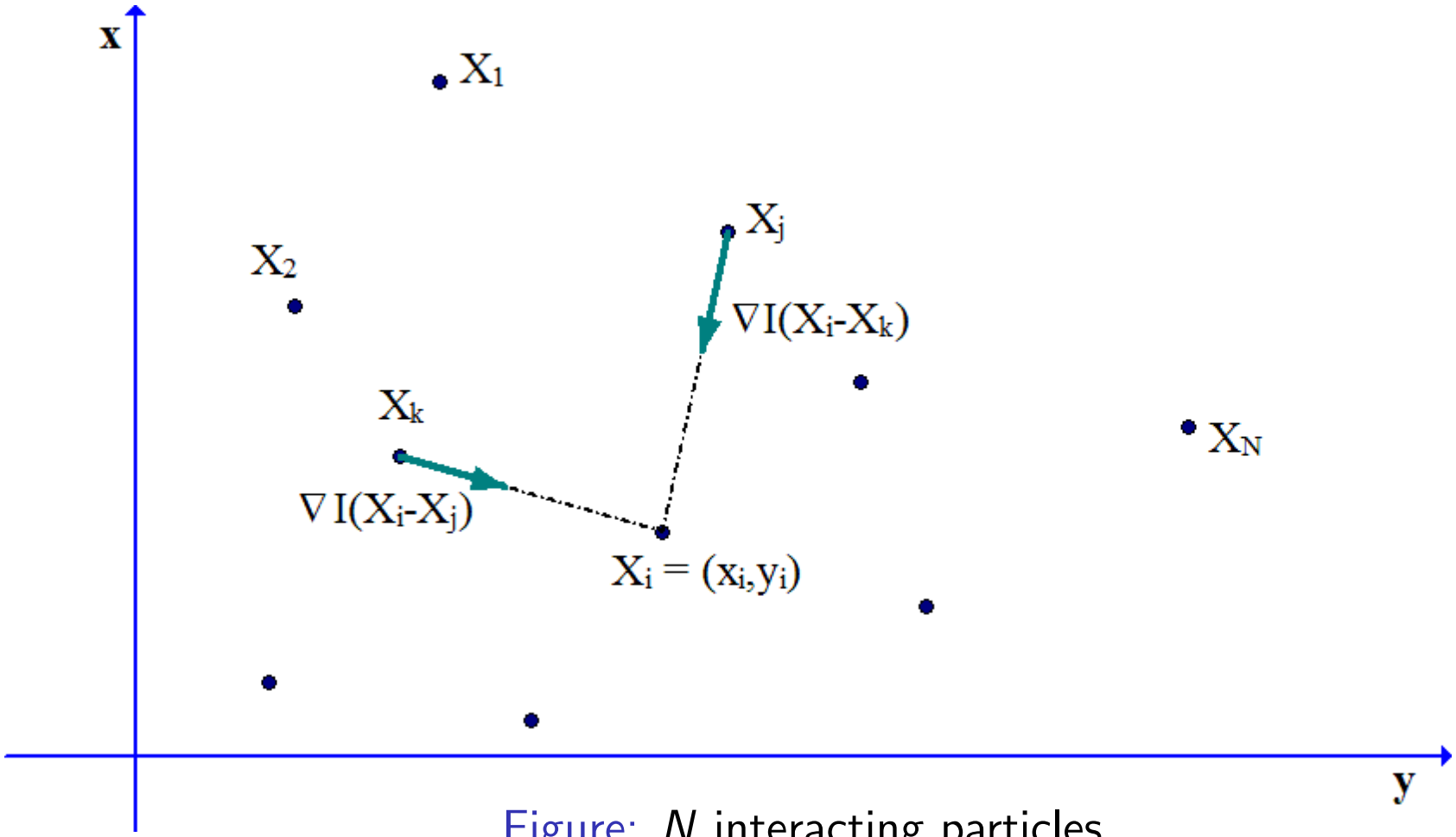

Figure: N interacting particles

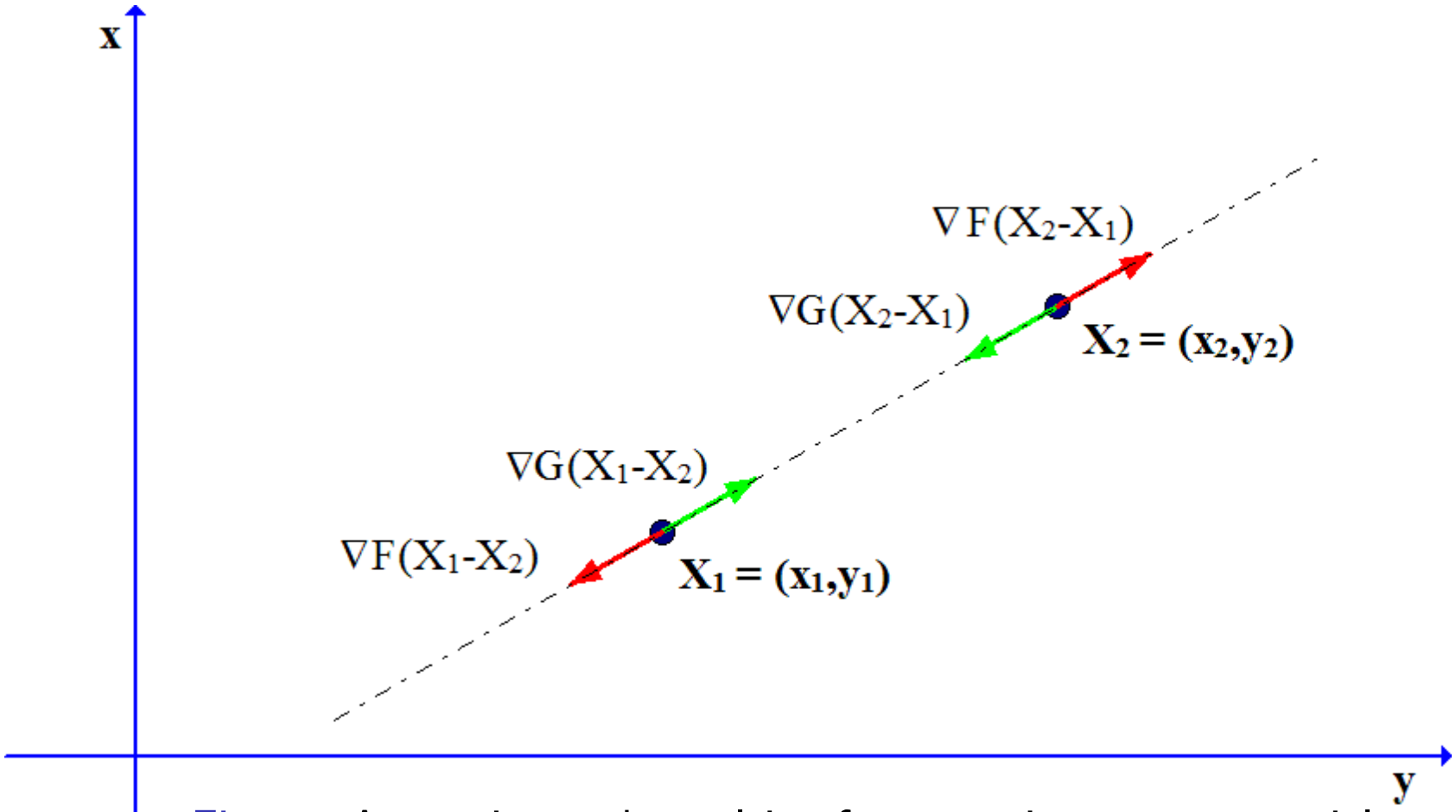

Figure: Attractive and repulsive forces acting on two particles

Repulsion with short range

Since the total mass $M = \sum_{i=1}^{N} m_i$ is preserved along the flow, we shall assume once again $M = 1$.
We assume now (cf. [Burger, Capasso, Morale - Nonlinear Analysis RWA 2006])

$$F(x) := \lambda^d V(\lambda x), \qquad V(x) = v(|x|), \quad v'(r) < 0 \text{ as } r > 0,$$

with $V \in L^1(\mathbb{R}^d)$, $V \geq 0$. By taking $\lambda \gg 1$, it is clear that the range or repulsion gets smaller and smaller (*short repulsion range*).
A typical situation occurs when $\lambda = \lambda(N)$ is an increasing function of the number of particles. Here, the repulsion range gets smaller and smaller as the number of interacting individuals diverges.
The attraction range is taken independent on N. We refer to this as *large attraction range*.

The empirical measure

Let us now consider the *empirical measure*

$$\mu(t) := \sum_{i=1}^{N} m_i \delta_{X_i(t)}.$$

It is an easy exercise to check that μ satisfies the following integro-PDE in the sense of distributions

$$\frac{\partial \mu}{\partial t} = \mathrm{div}(\mu \nabla F * \mu) - \mathrm{div}(\mu \nabla G * \mu).$$

By sending $\lambda \to +\infty$ one obtain (formally) that

$$F \rightharpoonup \varepsilon \delta_0, \qquad \varepsilon := \|V\|_{L^1},$$

and the limiting measure for μ (if it exists) satisfies the integro-PDE

$$\frac{\partial \mu}{\partial t} = \varepsilon \mathrm{div}(\mu \nabla \mu) - \mathrm{div}(\mu \nabla G * \mu).$$

Table of contents

A minimization problem

The minimization problem

$$\mathrm{argmin}_{\rho\in L^1_+(\mathbb{R}^d)}\left\{\int_{\mathbb{R}^d}\Phi(\rho(x))dx-\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\rho(x)\rho(y)G(x-y)dxdy\right\}$$

was first studied in [Lions - Ann. Inst. H. Poincare 1984]. Existence of nontrivial minimizers was proven under the assumptions of

- Total mass sufficiently large,
- $\Phi(tu)\leq t^\nu\Phi(u)$ with $1<\nu<2$,
- G slow decaying at infinity, i. e. $G(tx)\geq t^{-\alpha}G(x)$ with $\alpha\in(0,d)$,
- $\Phi(u)=o(u^{1+\frac{\alpha}{d}})$ as $u\to 0$.

A critical exponent

In [Bedrossian, preprint 2010][4], it is proven that nontrivial minimizers exist under the assumptions

- $G \in L^1_+$,
- $\Phi(u) = cu^2 + o(u^2)$ as $u \to 0$ with $c > 0$,
- either $c = 0$ or $2c < \int G$.

It turns out indeed that $m = 2$ is a *critical* exponent in $\Phi(u) = u^m$. Roughly speaking:

- If $m > 2$, then aggregation dominates and it produces formation of nontrivial stationary patterns,
- If $m < 2$, then diffusion dominates,
- If $m = 2$, then the behavior depends on the 'relative size' of the diffusion and aggregation terms.

[4] see also [Bedrossian, Rodríguez, Bertozzi - Nonlinearity 2011]

A result in one space dimension

Clearly, the minimization problem is related to the existence of steady states of (1).
In the case $d = 1$, a partial result in the case $m = 2$ was recovered in [Burger, DF - NHM 2008], namely:

- If $\varepsilon > \|G\|_{L^1}$, then no nontrivial steady states exist.
- If $\varepsilon \ll 1$ then there exists a nontrivial steady state, this is achieved by perturbing the case $\varepsilon = 0$ with an implicit function theorem approach in the pseudo-inverse variable (cf. one dimensional Wasserstein tools)

Let us also mention [Primi, Stevens, Velazquez - Comm. PDE 2009] (linear diffusion, small perturbation regime).

Table of contents

A key question: large time behavior

How does the solution to (1) behave as $t \to +\infty$? There are (basically) three possibilities:

(i) **Diffusion dominated case:** $\rho(t)$ decays to zero in some L^p norm with $p > 1$. In this case, the repulsive effects dominates.

(ii) **Aggregation dominated case:** $\rho(t)$ concentrates to a singular measure (delta) in finite or infinite time. Here, the aggregation effect dominates.

(iii) **Balanced case:** $\rho(t)$ converges to some (stable) non trivial L^1 *steady state* for large times.

Unlike the Keller-Segel system, here no mass threshold phenomenon occurs, since the equation is quadratically homogeneous.

Stationary states in multiple dimensions.

Threshold phenomenon

Parallel to [Bedrossian],we proved the following general property.

- Let $\varepsilon < \|G\|_{L^1}$. Then, there exists at least one non trivial L^1 steady state for (1), which is also a minimizer for the energy $E[\rho]$.
- Let $\varepsilon \geq \|G\|_{L^1}$. Then, there exist no steady states for (1) except $\rho \equiv 0$.

Finite time concentration is not possible under the present smoothness assumptions on G.

Stationary points of $E[\rho]$ are steady states to (1) and vice-versa

Nonexistence for $\varepsilon > \|G\|_{L^1}$

Second derivative of $E[\rho]$

Let $\rho \in L^2 \cap \mathcal{P}$. Then, the second order Gateaux derivative of E on ρ satisfies

$$\frac{d^2}{d\delta^2} E[\rho + \delta v]|_{\delta=0} = \varepsilon \int_{\mathbb{R}^d} v^2(x)dx - \int v(x) G * v(x)dx, \tag{4}$$

for all $v = \mathrm{div}(\rho V)$ and $V \in C_c^1(\mathbb{R}^d)$.

Lemma

Let $\varepsilon > \|G\|_{L^1}$. Then, there exists no stationary solutions to (1) *in the space $L^2 \cap \mathcal{P}$.*

Proof.

Assume ρ is a minimizer for $E[\rho]$ under the constraint $\rho \in \mathcal{P}$. Young inequality for convolutions implies

$$\frac{\varepsilon}{2}\int \rho^2 dx \geq E[\rho] = \frac{\varepsilon}{2}\int \rho^2 dx - \frac{1}{2}\int \rho G * \rho dx \geq \frac{\varepsilon - \|G\|_{L^1}}{2}\int \rho^2 dx \quad (5)$$

with $\varepsilon - \|G\|_{L^1} > 0$. Take a family $\rho_\lambda(x) \geq 0$ such that $\int \rho_\lambda(x)dx = 1$ and $\int \rho_\lambda^2(x)dx \to 0$ as $\lambda \to +\infty$. Clearly

$$E[\rho_\lambda] \to 0, \qquad \text{as } \lambda \to \infty.$$

Therefore, a minimizer ρ_∞ for $E[\rho]$ in $\mathcal{P}$ would imply that $E[\rho_\infty] > 0$ and we would necessarily have $0 < E[\rho_\lambda] < E[\rho_\infty]$ for λ large enough, which is a contradiction.

Now, assume that ρ is a steady state. Then, due to (4) the functional E is convex, and therefore admits only one stationary point, which coincides with its global minimizer. But this contradicts the non existence of a global minimizer proven above. □

The critical case $\varepsilon = \|G\|_{L^1}$

Lemma

Let $\varepsilon = \|G\|_{L^1}$. Then, there exists no stationary solutions to (1) *in the space $L^2 \cap \mathcal{P}$.*

Proof.

Similarly to the previous case, a stationary solution must be a global minimizer because of the convexity of E. Taking the same family ρ_λ as before, with $0 \le E[\rho_\lambda] \le \frac{\varepsilon}{2} \int \rho_\lambda^2 dx \to 0$, a minimizer ρ_∞ should then satisfy $E[\rho_\infty] = 0$. Now, Young inequality for convolutions says that this is possible only if $\rho_\infty = cG * \rho_\infty$, for some $c > 0$, and integration on $\mathbb{R}^d$ implies

$$\rho_\infty = H * \rho_\infty, \qquad H := \frac{1}{\|G\|_{L^1}} G, \quad \int H(x)dx = 1.$$

Taking the Fourier transform on both side of the above identity, we get

$$\widehat{\rho_\infty}(\xi) = \widehat{H}(\xi)\widehat{\rho_\infty}(\xi), \qquad \xi \in \mathbb{R}^d,$$

and since H is radial, nonnegative and with unit mass, it is easy to check that $\widehat{H}(\xi) < \widehat{H}(0) = 1$ for all $\xi \neq 0$, which yields $\widehat{\rho_\infty} = 0$ on $\mathbb{R}^d \setminus \{0\}$, which is a contradiction. □

Steady states for $\varepsilon < \|G\|_{L^1}$

Theorem (Existence of minimizers)

Let $\varepsilon < \|G\|_{L^1}$. Then, there exists a radially symmetric non-increasing minimizer $\rho_\infty \in \mathcal{P} \cap L^2(\mathbb{R}^d)$ for the entropy functional E restricted to $\mathcal{P}$ with $\rho \neq 0$.

For the proof in any d we refer to [Bedrossian]. We just prove

Lemma

Let $\varepsilon < \|G\|_{L^1}$. Then, $\inf_{\rho \in \mathcal{P} \cap L^2(\mathbb{R}^d)} E[\rho] < 0$.

Proof.

We consider the family $\sigma_\lambda(x) = \frac{1}{2\lambda}\chi_{[-\lambda,\lambda]}(x) \in \mathcal{P} \cap L^2$. We have

$$E[\sigma_\lambda] = \frac{\varepsilon}{4\lambda} - \frac{1}{8\lambda^2}\int_{-\lambda}^{\lambda}\int_{-\lambda}^{\lambda} G(x-y)dydx = \frac{1}{4\lambda}\left(\varepsilon - \int_{-\lambda}^{\lambda} G(z)dz\right),$$

and there exists $\overline{\lambda}$ such that $E[\sigma_{\overline{\lambda}}] < 0$. □

Table of contents

Existence and uniqueness of steady states

With $d = 1$ we can characterize all the steady states as follows. From now one we shall assume $\|G\|_{L^1} = 1$ for simplicity and $\mathrm{supp}(G) = \mathbb{R}$.

Theorem (Burger-DF-Franek - 2011 preprint)

Let $\varepsilon < 1$. Then, there exists a unique $\rho \in L^2 \cap \mathcal{P}$ with zero center of mass which solves

$$\rho \partial_x (\varepsilon \rho - G * \rho) = 0.$$

Moreover,

- *ρ is symmetric and monotonically decreasing on $x > 0$,*
- *$\rho \in C^2(\mathrm{supp}[\rho])$,*
- *$\mathrm{supp}[\rho]$ is a bounded interval in $\mathbb{R}$,*
- *ρ has a global maximum at $x = 0$ and $\rho''(0) < 0$,*
- *ρ is the global minimizer of the energy $E[\rho] = \frac{\varepsilon}{2} \int \rho^2 dx - \frac{1}{2} \int \rho G * \rho dx$.*

Comments

The above result is surprising for the following reasons:

- The functional $E[\rho]$ is not convex when $\varepsilon < \|G\|_{L^1}$.
- Our model is a $\lambda \to +\infty$ limit of repulsive–attractive potentials

$$W_\lambda(x) = \lambda V(\lambda x) - G(x), \qquad V, G \in L^1_+, \quad \lambda \ll 1$$

which generate very complex dynamics (stability vs. instability), cf. [Fellner, Raoul - Carrillo et al.]

Table of contents

Continuity in one space dimension

Lemma (1d regularity)

Let ρ be an $L^2 \cap \mathcal{P}$ steady state to (1) in one space dimension. Then ρ is continuous on $\mathbb{R}$.

Proof.

From the energy identity (3) we obtain (after some computations with the dissipation)

$$\int_{\mathbb{R}^d} \rho \rho_x^2 dx < +\infty.$$

Then, the one dimensional Sobolev embedding theorem implies ρ continuous. □

Connected support

Lemma (Steady states have connected support)

Let ρ solve

$$\rho\partial_x\left(\varepsilon\rho - G * \rho\right) = 0 \qquad a.e.\ on\ \mathbb{R}. \tag{6}$$

Then, $\mathrm{supp}(\rho)$ is a connected set.

Proof.

Let A and B be to consecutive connected components of $\mathrm{supp}[\rho]$, $a = \sup A$, $b = \inf B$, $a < b$. Evaluate the evolution of the energy $E[u(x,s)]$ along the solution to $u_s + (uV)_x = 0$, $u(x,0) = \rho(x)$, with $V \in C^1(\mathbb{R})$ such that $V = -1$ on $(-\infty, a]$ and $V = 1$ on $[b, +\infty)$. $\frac{d}{ds}E[u(x,s)]|_{s=0} = 0$ and some computations imply

$$0 = \varepsilon\int \rho\rho_x V = \int \rho V G' * \rho = -\int_{-\infty}^{a}\int_{b}^{+\infty} \rho(x)G'(x-y)\rho(y)dydx$$
$$+\int_{b}^{+\infty} dx \int_{-\infty}^{a} dy\rho(x)G'(x-y)\rho(y)dydx.$$

The assumptions on G imply then $\rho(x)\rho(y) = 0$ on $(x,y) = A \times B$, which is a contradiction. □

Symmetric rearrangement

Proposition

Let ρ_∞ be a minimizer for the energy

$$E[\rho] = \frac{\varepsilon}{2}\int \rho^2(x)dx - \frac{1}{2}\int\int G(x-y)\rho(x)\rho(y)dydx$$

under the constraint that the center of mass is zero. Then, ρ_∞ is symmetric and monotonically decreasing on $x > 0$.

Proof.

Assume u not symmetric and decreasing on $x > 0$, define $u^*(x) = \sup\{t \geq 0 : \mathrm{meas}(\{u > t\}) > 2|x|\}$. We have to check $E[u^*] < E[u]$. It is easy to check that $\int (u^*)^2 dx = \int u^2 dx$. Then, the result follows from the *Riesz's rearrangement inequality*, see e. g. [Lieb and Loss, AMS - 2007],

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)g(x-y)h(y)dydx \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f^*(x)g^*(x-y)h^*(y)dydx, \tag{7}$$

which holds for all nonnegative functions f, g, h vanishing at infinity. □

An interpretation of the energy level

Lemma

Let $\rho \in \mathcal{P}$ be a 1d steady state, i. e.

$$\varepsilon\rho = G * \rho + C \qquad \text{on } \mathrm{supp}[\rho] \tag{8}$$

for some $C \in \mathbb{R}$. Then, $C = 2E[\rho]$.

Proof.

Multiply (8) by ρ and integrate on $\mathrm{supp}[\rho]$:

$$\varepsilon \int_{\mathrm{supp}[\rho]} \rho^2 dx = \int_{\mathrm{supp}[\rho]} \rho G * \rho + C,$$

and this proves the Lemma. □

Lemma (Translation invariance)

Let $\rho \in \mathcal{P} \cap L^2$ and let $x_0 \in \mathbb{R}$. Let ρ_{x_0} be defined by $\rho_{x_0}(x) := \rho(x + x_0)$. Then, $E[\rho_{x_0}] = E[\rho]$.

Compact support

Lemma

Let ρ be a steady state for (6) with $\varepsilon < 1$. Then, the support of ρ is compact.

Proof.

Suppose the (connected) support of ρ is of the form $(a, +\infty)$. Then

$$\varepsilon\rho = G * \rho + 2E[\rho], \qquad \text{on } \mathrm{supp}[\rho]$$

implies $E[\rho] = 0$, otherwise the integration of the above formula on $\mathrm{supp}[\rho]$ gives a contradiction. Therefore

$$0 = \varepsilon\rho(a) = \int_{\mathbb{R}} G(a-y)\rho(y)dy$$

which is a contradiction since $\mathrm{supp}[G] = \mathbb{R}$. A similar situation occurs if $\mathrm{supp}[\rho] = (-\infty, b)$. Therefore, $\mathrm{supp}[\rho] = \mathbb{R}$ and $\varepsilon\rho = G * \rho$ on $\mathbb{R}$. Integration over $\mathbb{R}$ gives a contradiction. □

Symmetrization

Lemma

Let ρ be a steady state. Then there exists a symmetric st. state $\widetilde{\rho}$ s. t.

$$E[\widetilde{\rho}] = E[\rho].$$

Proof.

From previous lemmas we know that $\mathrm{supp}[\rho] = (a, b)$ for some $a, b \in \mathbb{R}$ and

$$\varepsilon\rho(x) = G * \rho(x) + C \qquad \text{for} \quad x \in (a, b), \tag{9}$$

with $C = -\int_a^b G(a-y)\rho(y)dy = -\int_a^b G(b-y)\rho(y)dy$. Let $\overline{\rho}(x) = \rho(x + x_0)$ with $x_0 = (a+b)/2$. Then $\overline{\rho}$ is still a steady state and it satisfies $E[\overline{\rho}] = E[\rho]$ thanks to a previous Lemma. $\mathrm{supp}[\overline{\rho}]$ is symmetric. Let $\widetilde{\rho}(x) := \frac{1}{2}(\overline{\rho}(x) + \overline{\rho}(-x))$. Clearly, $\mathrm{supp}[\widetilde{\rho}] = \mathrm{supp}[\overline{\rho}]$. Using the symmetry of G one can check that, for all $x \in \mathrm{supp}[\widetilde{\rho}]$,

$$\varepsilon\widetilde{\rho}(x) = \int_{(a-b)/2}^{(b-a)/2} G(x-z)\widetilde{\rho}(z)dz + C.$$

Hence, $E[\widetilde{\rho}] = E[\rho]$. □

Minimizers have maximal support

Lemma (Support of a minimizer)

Let ρ_∞ be a global minimizer to E. Let ρ be a steady state such that

$$\mathrm{meas}(\mathrm{supp}[\rho_\infty]) \leq \mathrm{meas}(\mathrm{supp}[\rho]).$$

Then ρ is also a minimizer.

Proof.

Due to the translation invariance, we can assume $\mathrm{supp}[\rho_\infty] \subset \mathrm{supp}[\rho]$. The computation of the second variation of E around the minimizer ρ_∞ along the direction $\rho_\infty - \rho$ yields

$$0 \leq \frac{d^2}{d\delta^2} E[\rho_\infty + \delta(\rho - \rho_\infty)]|_{\delta=0} = -2E[\rho] + 2E[\rho_\infty],$$

and this completes the proof. □

Table of contents

The Krein–Rutman Theorem

Theorem (Krein–Rutman Theorem, strong version)

Let X be a Banach space, $K \subset X$ a solid cone, i. e. such that $\lambda K \subset K$ for all $\lambda \geq 0$ and such that K has a nonempty interior K_0. Let T be a compact linear operator which is strongly positive with respect to K, i. e. such that $T[u] \in K_0$ if $u \in K$. Then,

(i) *The spectral radius $r(T)$ is strictly positive and $r(T)$ is a simple eigenvalue with an eigenvector $v \in K_0$. There is no other eigenvalue with a corresponding eigenvector $v \in K$.*

(ii) $|\lambda| < r(T)$ *for all other eigenvalues* $\lambda \neq r(T)$.

The functional equation

Assume ρ is a symmetric steady states with unit mass, monotonically decreasing on the positive semi-axis, $\varepsilon < 1$. This is equivalent to

$$\varepsilon\rho(x) = \int_{-L}^{L} G(x-y)\rho(y)dy + C, \qquad C = 2E[\rho]. \tag{10}$$

Taking the x-derivative on $[-L, L]$ we obtain

$$\varepsilon\rho'(x) = \frac{d}{dx}\int_{-L}^{L} G(x-y)\rho(y)dy = \frac{d}{dx}G * \rho(x) = \int_{-L}^{L} G(x-y)\rho'(y)dy$$
$$= -\int_{0}^{L} G(x+y)\rho'(y)dy + \int_{0}^{L} G(x-y)\rho'(y)dy$$
$$= \int_{0}^{L} \left[G(x-y) - G(x+y)\right]\rho'(y)dy.$$

Application of KR Theorem

Assuming that $\rho \in C^1([-L, L])$, finding a steady state with the above assumptions is equivalent to find ρ on $[0, L]$ such that

$$\rho(L) = 0, \quad -\rho'(x) = u(x), \quad x \in [0, L],$$

$$u \geq 0, \quad \text{and } u \text{ solves} \quad \varepsilon u = \int_0^L H(x, y) u(y) dy,$$

$$H(x, y) = G(x - y) - G(x + y).$$

Ingredients

- Banach space $X_L = \left\{ f \in C^1([0, L]) : \ f(0) = 0 \right\}$ with the norm $\|f\|_{X_L} = \|f\|_{L^\infty([0,L])} + \|f'\|_{L^\infty([0,L])}$.
- Compact operator $\mathcal{H}_L[u](x) := \int_0^L H(x, y) u(y) dy = \int_0^L (G(x - y) - G(x + y)) u(y) dy$.
- Solid cone $K := \{f \in X : \ f \geq 0\}$: any function $f \in K$ with $f'(0) > 0$ belongs to the interior K_0 of K.

Strong positivity of the operator

Lemma

The compact operator $\mathcal{H}_L$ is strongly positive, i. e. it maps K into K_0.

Proof.

Since G is decreasing on the half-line $[0, +\infty)$ we get

$$H(x, y) = G(x - y) - G(x + y) \geq 0, \qquad \text{on} \quad x, y \geq 0.$$

Hence, for a given $u \in K$, we have $\mathcal{H}_L[u](x) = \int_0^L H(x, y)u(y)dy \geq 0$ for all $x \in [0, L]$ and

$$\mathcal{H}_L[u](0) = \int_0^L H(0, y)u(y)dy = \int_0^L (G(-y) - G(y))u(y)dy = 0.$$

Therefore $\mathcal{H}_L[u] \in K$. Moreover,

$$(\mathcal{H}_L[u])'(0) = \int_0^L (G'(-y) - G'(y))u(y)dy = -2\int_0^L G'(y)u(y)dy > 0,$$

which proves $\mathcal{H}_L[u] \in K_0$.

□

Existence and uniqueness of eigenfunctions

By applying the KR Theorem, we have then proven what follows:

Proposition

For a fixed $L > 0$ there exists a unique symmetric function $\rho \in C^2([-L, L])$ with unit mass and with $\rho'(x) \leq 0$ on $x \geq 0$ such that ρ solves

$$\varepsilon\rho(x) = \int_{-L}^{L} G(x-y)\rho(y)dy + C, \qquad C = 2E[\rho].$$

for some $\varepsilon = \varepsilon(L) > 0$. Such function ρ also satisfies $\rho''(0) < 0$. Moreover, $\varepsilon(L)$ is the largest eigenvalue of the compact operator $\mathcal{G}_L[\rho](x) := \int_0^L [G(x-y) + G(x+y) - G(L-y) - G(L+y)]\,\rho(y)dy$ on the space Banach $Y_L := \{\rho \in C([0, L]) \ : \ \rho(L) = 0\}$, and any other eigenfunction of $\mathcal{G}_L$ on Y_L with unit mass has the corresponding eigenvalue ε' satisfying $|\varepsilon'| < \varepsilon(L)$.

We observe that the uniqueness is achieved by the unit mass constraint (eigenvalues are defined up to multiplication by a constant).

Behavior of the function $\varepsilon(L)$

Proposition

The simple eigenvalue $\varepsilon(L)$ found in the previous Proposition is uniquely determined as a function of L with the following properties

(i) $\varepsilon(L)$ *is strictly increasing with respect to* L

(ii) $\lim_{L\to+\infty}\varepsilon(L)=1$

(iii) $\varepsilon(0)=0$.

The proof is omitted. (i) is obtained by the formula

$$\varepsilon(L)u_L(x)=\mathcal{H}_L[u_L](x)=\int_0^L H(x,y)u_L(y)dx$$

for $x\in[0,L]$, multiplied by u_L and integrated on $[0,L]$. In order to evaluate the monotonicity of $\varepsilon(L)$ one considers the variation $\varepsilon(L+\delta)-\varepsilon(L)$ and proves that it is positive when $\delta>0$.

(ii) is obtained by contradiction, assuming $\lim_{L\to+\infty}\varepsilon(L)=\varepsilon_0<1$ (using that for $\varepsilon\in(\varepsilon_0,1)$ there exists a minimizer). (iii) is trivial.

Main result

Theorem

Let $\varepsilon < 1$. Then, there exists a unique $\rho \in L^2$ solution to

$$\rho \partial_x (\varepsilon \rho - G * \rho) = 0,$$

with unit mass and zero center of mass. Moreover,

- *ρ is symmetric and monotonically decreasing on $x > 0$,*
- *$\rho \in C^2(\mathrm{supp}[\rho])$,*
- *$\mathrm{supp}[\rho]$ is a bounded interval in $\mathbb{R}$,*
- *ρ has a global maximum at $x = 0$ and $\rho''(0) < 0$,*
- *ρ is the global minimizer of the energy $E[\rho] = \frac{\varepsilon}{2} \int \rho^2 dx - \frac{1}{2} \int \rho G * \rho dx$.*

Proof of the main theorem I

- We know that there exists a minimizer ρ_∞ with unit mass and zero center of mass, which is symmetric and monotonically decreasing on $x > 0$ and compactly supported on a certain $[-L, L]$
- From the two above Propositions, we know that there exists a unique steady state with such properties, because the correspondence $\varepsilon = \varepsilon(L)$ is one-to-one.
- So, the only possibility to violate uniqueness of steady states with unit mass and zero center of mass is to have a steady state which violates either the monotonicity property or the symmetry.
- Suppose first that there exists a steady state with zero center of mass ρ which is not symmetric, $\mathrm{supp}[\rho] = [-L', L']$ (not restrictive).
- Then, we know from one Lemma above that it is possible to construct a symmetric steady state $\widetilde{\rho}$ with the same energy of ρ and with the same support of ρ. Now, there are two possibilities: either $\widetilde{\rho}$ is a minimizer or not.

Proof of the main theorem II

- If $\widetilde{\rho}$ is a minimizer, then so is ρ (they have the same energy!), and this is not possible because ρ would be symmetric (every minimizer is symmetric by the rearrangement property).
- If $\widetilde{\rho}$ is not a minimizer, then the support of $\widetilde{\rho}$ is strictly contained in the support of ρ_∞, and $\widetilde{\rho}$ is not monotonically decreasing on $x > 0$ because otherwise it would be the unique minimizer provided before.
- Therefore, $-\widetilde{\rho}'$ is an eigenfunction for $\mathcal{H}_{L'}$ in the space $X_{L'}$ which is not belonging to the solid cone K
- Therefore, KR and the fact that $\varepsilon(L)$ is increasing imply that $L' > L$, since ρ is an eigenfunction outside the solid cone K and it therefore should have an eigenvalue strictly less than $\varepsilon(L')$. This implies that $\varepsilon(L) < \varepsilon(L')$ and therefore $L < L'$.
- Now this is a clearly a contradiction because we said before that the support or $\widetilde{\rho}$ is strictly contained in the support of ρ_∞, so $L > L'$.
- The case in which ρ is symmetric but not monotonic on $x > 0$ can be covered by repeating the same argument above (assume $\rho = \widetilde{\rho}$!).

Concavity of ρ for small ε

Corollary

There exists a value $\varepsilon_0 \in (0,1)$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ the corresponding stationary solution is concave on the whole interval $[0, L]$.

Proof.

We can differentiate twice w.r.t x in

$$\varepsilon\rho(x) = \int_{-L}^{L} G(x-y)\rho(y)dy + C$$

to obtain

$$\varepsilon\rho''(x) = \int_{-L}^{L} G''(x-y)\rho(y)dy$$

for all $x \in [-L, L]$. Therefore, G'' is evaluated on the interval $[-2L, 2L]$ in the above integral. We know that L is a monotonically increasing function of ε with $\lim_{\varepsilon \searrow 0} L(\varepsilon) = 0$. Since $G''(0) < 0$, and $G \in C^2$, then there exists $L_0 > 0$ such that $G'' < 0$ on $[-2L_0, 2L_0]$. Let ε_0 be the eigenvalue in K corresponding to $L = L_0$. Then, the eigenfunction ρ is concave on its support. $\square$

Table of contents

Future work

1. The case $\mathrm{supp}[G] = [-l, l]$ bounded is of great interest since it allows for multiple steady states. We can prove that a necessary condition is that the distance between two connected components of ρ is at least $2l$.
2. Large time stability of the nontrivial steady states (under preparation).
3. Large time decay in case of non-existence of steady states.
4. In the critical case $\varepsilon = \|G\|_{L^1}$ we conjecture a large time dynamics of fourth order type (thin film).

End of the talk

Thank you for your attention!