Aggregation versus Diffusion in Mathematical Biology

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Lecture 4: Kinetic Models

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Outline

Types of interaction

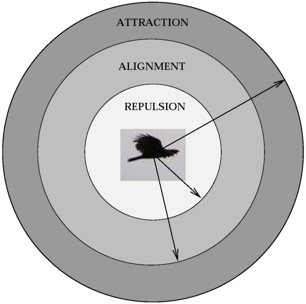
Swarming = Aggregation of agents of similar size and body type generally moving in a coordinated way.

Highly developed social organization: insects (locusts, ants, bees ...), fishes, birds, micro-organisms (myxo-bacteria, ...) and artificial robots for unmanned vehicle operation.

Interaction regions between individuals[a]

[a] Barbaro, Birnir et al. (2008).

- Repulsion Region: R_k.
- Attraction Region: A_k.
- Orientation Region: O_k.



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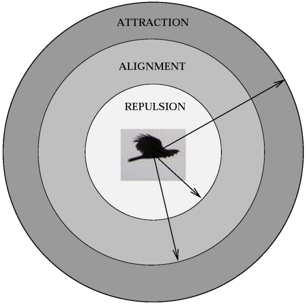

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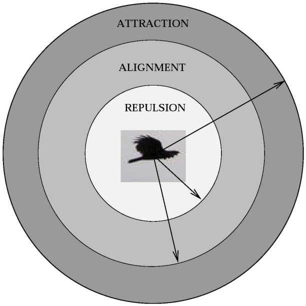


Model with an asymptotic velocity

D'Orsogna, Bertozzi et al. model (PRL 2006):

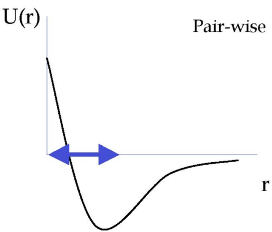
$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = (\alpha - \beta\,|v_i|^2)v_i - \displaystyle\sum_{j\neq i} \nabla U(|x_i - x_j|). \end{cases}$$

$C = C_R/C_A > 1$, $\ell = \ell_R/\ell_A < 1$ and $C\ell^2 < 1$:

Model assumptions:

- Self-propulsion and friction terms determines an asymptotic speed of $\sqrt{\alpha/\beta}$.
- Attraction/Repulsion modeled by an effective pairwise potential $U(x)$.

$$U(r) = -C_A e^{-r/\ell_A} + C_R e^{-r/\ell_R}.$$



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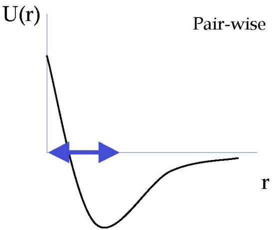

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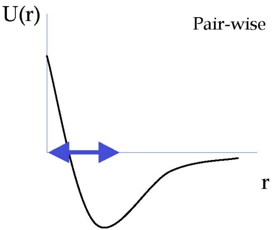

Velocity consensus model

Cucker-Smale Model (IEEE Automatic Control 2007):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j=1}^{N} a_{ij}\left(v_j - v_i\right), \end{cases}$$

with the communication rate, $\gamma \geq 0$:

$$a_{ij} = a(|x_i - x_j|) = \frac{1}{(1 + |x_i - x_j|^2)^{\gamma}}.$$

Unconditional flocking: $\gamma \leq 1/2$; Ha-Tadmor, Ha-Liu, Carrillo-Fornasier-Toscani-Rosado.
Conditional flocking: $\gamma > 1/2$.

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Mesoscopic models

Model with asymptotic velocity + Attraction/Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \mathrm{div}_v[(\alpha - \beta|v|^2)vf] - \mathrm{div}_v\left[(\nabla_x U \star \rho)f\right] = 0.$$

Velocity consensus Model:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot \Bigg[\underbrace{\left(\int_{\mathbb{R}^{2d}} \frac{v-w}{(1+|x-y|^2)^\gamma} f(y,w,t)\,dy\,dw\right)}_{:=\xi(f)(x,v,t)} f(x,v,t)\Bigg]$$

Orientation, Attraction and Repulsion:

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Well-posedness in probability measures[1]

Existence, uniqueness and stability

Take a potential $U \in \mathcal{C}_b^2(\mathbb{R}^d)$, and f_0 a measure on $\mathbb{R}^d \times \mathbb{R}^d$ with compact support. There exists a solution $f \in \mathcal{C}([0,+\infty);\mathcal{P}_1(\mathbb{R}^d))$ in the sense of solving the equation through the characteristics: $f_t := P^t \# f_0$ with P^t the flow map associated to the equation.

Moreover, the solutions remains compactly supported for all time with a possibly growing in time support.

Moreover, given any two solutions f and g with initial data f_0 and g_0, there is an increasing function depending on the size of the support of the solutions and the parameters, such that

$$W_1(f_t, g_t) \le \alpha(t)\, W_1(f_0, g_0)$$

[1] Dobrushin-Hepp-Neunzert, 1977-79 for the Vlasov.

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Convergence of the particle method

- Empirical measures: if $x_i, v_i : [0,T) \to \mathbb{R}^d$, for $i = 1, \ldots, N$, is a solution to the ODE system,

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \overbrace{(\alpha - \beta |v_i|^2) v_i}^{\text{propulsion-friction}} - \overbrace{\sum_{j \neq i} m_j \nabla U(|x_i - x_j|)}^{\text{attraction-repulsion}} + \overbrace{\sum_{j=1}^{N} m_j a_{ij} (v_j - v_i)}^{\text{orientation}}. \end{cases}$$

then the $f : [0,T) \to \mathcal{P}_1(\mathbb{R}^d)$ given by

$$f_N(t) := \sum_{i=1}^{N} m_i \delta_{(x_i(t), v_i(t))}$$

is the solution corresponding to initial atomic measures.

- Convergence of approximations of measures by particles due to the stability at any given time T as an alternative derivation of the kinetic models.

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Mean-Field Limit

Convergence of approximations of measures by particles due to the stability at any given time T as an alternative derivation of the kinetic models.

Just take as many particles as needed in order to have

$$W_1(f_t, f_t^N) \leq \alpha(t)\, W_1(f_0, f_0^N) \to 0 \qquad \text{as } N \to \infty$$

by sampling the initial data in a suitable way.

The sequences of particle solutions becomes a Cauchy sequence with the distance W_1 converging to the solution of the kinetic equation.

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Outline

Proof of the Theorem

Conditions on E:

1. E is continuous on $[0,T] \times \mathbb{R}^d$,
2. For some $C > 0$,

$$|E(t,x)| \leq C_E(1+|x|), \qquad \text{for all } t,x \in [0,T] \times \mathbb{R}^d, \text{ and}$$

3. E is locally Lipschitz with respect to x, i.e., for any compact set $K \subseteq \mathbb{R}^d$ there is some $L_K > 0$ such that

$$|E(t,x) - E(t,y)| \leq L_K|x-y|, \qquad t \in [0,T], \quad x,y \in K.$$

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$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f + \mathrm{div}_v((\alpha - \beta |v|^2) v f) = 0,$$

which is a linear first-order equation. The associated characteristic system of ODE's is

$$\frac{d}{dt} X = V,$$
$$\frac{d}{dt} V = E(t, X) + V(\alpha - \beta |V|^2).$$

Flow Map:

Given $(X_0, V_0) \in \mathbb{R}^d \times \mathbb{R}^d$ there exists a unique solution (X, V) to the ODE system in $C^1([0,T]; \mathbb{R}^d \times \mathbb{R}^d)$ satisfying $X(0) = X_0$ and $V(0) = V_0$. In addition, there exists a constant C which depends only on T, $|X_0|$, $|V_0|$, α, β and the constant C_E, such that

$$|(X(t), V(t))| \le |(X_0, V_0)| \, e^{Ct} \qquad \text{for all } t \in [0, T].$$

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We can thus consider the flow at time $t \in [0, T)$ of ODE's equations

$$\mathcal{T}_E^t : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d.$$

Again by basic results in ode's, the map $(t, x, v) \mapsto \mathcal{T}_E^t(x, v) = (X, V)$ with (X, V) the solution at time t to the ODE system with initial data (x, v), is jointly continuous in (t, x, v).

For a measure $f_0 \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$, the function

$$f : [0, T) \to \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d), \quad t \mapsto f_t := \mathcal{T}_E^t \# f_0$$

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Estimates on characteristics:

- Taking characteristics with initial data inside a fixed ball then there exists $R > 0$ depending on T, in which the whole trajectories are inside a possibly larger ball of radius R for all times $t \in [0, T]$.
- For some constant C which depends only on α, β, R and $\mathrm{Lip}_R(E^i)$, for all P^0 in B_R

$$\left|\mathcal{T}^t_{E^1}(P^0) - \mathcal{T}^t_{E^2}(P^0)\right| \le \frac{e^{Ct}-1}{C} \sup_{s\in[0,T)} \left\|E^1_s - E^2_s\right\|_{L^\infty(B_R)}.$$

- For some constant C as before

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- Taking characteristics with initial data inside a fixed ball then there exists $R > 0$ depending on T, in which the whole trajectories are inside a possibly larger ball of radius R for all times $t \in [0, T]$.
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$$\left|\mathcal{T}^t_{E^1}(P^0) - \mathcal{T}^t_{E^2}(P^0)\right| \le \frac{e^{Ct} - 1}{C} \sup_{s\in[0,T)} \left\|E^1_s - E^2_s\right\|_{L^\infty(B_R)} .$$

- For some constant C as before

$$\left|\mathcal{T}^t_E(P_1) - \mathcal{T}^t_E(P_2)\right| \le |P_1 - P_2|\, e^{C\int_0^t (\mathrm{Lip}_R(E_s)+1)\, ds}, \quad t \in [0, T].$$

Proof of the Theorem

Error on transported measures through different flows:

Let $P_1, P_2 : \mathbb{R}^d \to \mathbb{R}^d$ be two Borel measurable functions. Also, take $f \in \mathcal{P}_1(\mathbb{R}^d)$. Then,

$$W_1(\mathcal{T}_1\#f, \mathcal{T}_2\#f) \leq \|\mathcal{T}_1 - \mathcal{T}_2\|_{L^\infty(\mathrm{supp} f)} .$$

Continuity in time for solutions of the linear transport:

$$W_1(\mathcal{T}_E^s\#f, \mathcal{T}_E^t\#f) \leq C\,|t-s|\,, \qquad \text{for any } t,s \in [0,T].$$

Error on transported measures through different initial data:

Take a locally Lipschitz map $\mathcal{T} : \mathbb{R}^d \to \mathbb{R}^d$ and $f, g \in \mathcal{P}_1(\mathbb{R}^d)$, both with compact support contained in the ball B_R. Then,

$$W_1(\mathcal{T}\#f, \mathcal{T}\#g) \leq L\, W_1(f,g),$$

where L is the Lipschitz constant of $\mathcal{T}$ on the ball B_R.

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&\leq \left\| P_f^t - P_g^t \right\|_{L^\infty(\operatorname{supp} f_0)} + L_t\, W_1(f_0, g_0) \\
&\leq C_2 \int_0^t e^{C_2(t-s)} \left\| E[f_s] - E[g_s] \right\|_{L^\infty(B_R)} ds + L_t\, W_1(f_0, g_0) \\
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\end{aligned}
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Outline

Asymptotic Flocking

Let us consider the N_p-particle system:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i & , \ x_i(0) = x_i^0 \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j=1}^{N_p} m_j a(|x_i - x_j|)\,(v_j - v_i) & , \ v_i(0) = v_i^0, \end{cases} .$$

Due to translation invariancy, w.l.o.g. the mean velocity is zero and thus the center of mass is preserved along the evolution, i.e.,

$$\sum_{i=1}^{N_p} m_i v_i(t) = 0 \qquad \text{and} \qquad \sum_{i=1}^{N_p} m_i x_i(t) = x_c$$

for all $t \geq 0$ and $x_c \in \mathbb{R}^d$.

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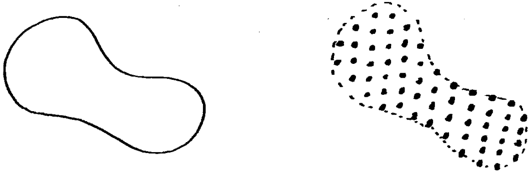
Asymptotic Flocking

Find a bound independent of the number of particles for the time it takes for all the particles to travel at the mean velocity.



Asymptotic Flocking

Unconditional Non-universal Flocking Result for Particles

The unique measure-valued solution for the CS kinetic model with $\gamma \leq 1/2$, with a finite number of particles given by

$$\tilde{\mu}(t) = \sum_{i=1}^{N_p} m_i \, \delta(x - x_i(t)) \, \delta(v - v_i(t)),$$

satisfies that

$$\lim_{t\to\infty} W_1 \left(\tilde{\mu}(t), \tilde{\mu}^\infty\right) = 0$$

with

$$\tilde{\mu}^\infty = \sum_{i=1}^{N_p} m_i \, \delta(x - x_i^\infty - mt) \, \delta(v - m)$$

with m the initial mean velocity of the particles.

Asymptotic Flocking

Unconditional Non-universal Flocking Result for general measures

Given $\mu_0 \in \mathcal{M}(\mathbb{R}^{2d})$ compactly supported, then the unique measure-valued solution to the CS kinetic model with $\gamma \leq 1/2$, satisfies the following bounds on their supports:

$$\text{supp } \mu(t) \subset B(x_c(0) + mt, R^x(t)) \times B(m, R^v(t))$$

for all $t \geq 0$, with $R^x(t) \leq \bar{R}$ and $R^v(t) \leq R_0\, e^{-\lambda t}$ with $\bar{R}^x$ depending only on the initial support radius.

Asymptotic Flocking

Let us fix any $R_0^x > 0$ and $R_0^v > 0$, such that all the initial velocities lie inside the ball $B(0, R_0^v)$ and all positions inside $B(x_c, R_0^x)$.

Let us define the function $R^v(t)$ to be

$$R^v(t) := \max\left\{|v_i(t)|\,,\, i = 1, \ldots, N_p\right\}.$$

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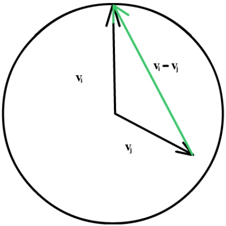
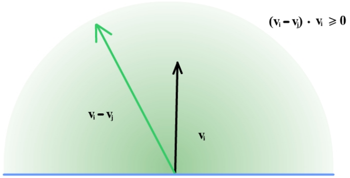
Asymptotic Flocking

Choosing the label i to be the one achieving the maximum, we get

$$\frac{d}{dt}R^v(t)^2 = \frac{d}{dt}|v_i|^2 = -2\sum_{j\neq i} m_j\left[(v_i - v_j)\cdot v_i\right]\, a(|x_i - x_j|)\ .$$

Because of the choice of the label i, we have that $(v_i - v_j)\cdot v_i \geq 0$ for all $j \neq i$ that together with $a \geq 0$ imply $R^v(t) \leq R^v_0$ for all $t \geq 0$.





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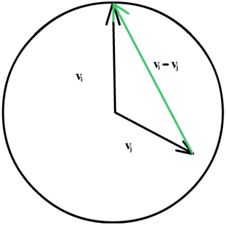

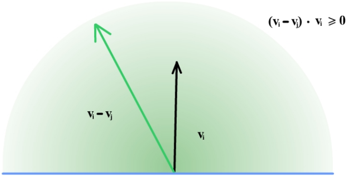

Asymptotic Flocking

Coming back to the equation for the positions,

$$|x_i(t) - x_i^0| \leq R_0^v t \quad \text{for all } t \geq 0 \text{ and all } i = 1, \dots, N_p.$$

$$a(|x_i - x_j|) \geq \frac{1}{[1 + 4R_0^2(1+t)^2]^\gamma} \quad \text{for all } t \geq 0 \text{ and all } i, j = 1, \dots, N_p,$$

with $R_0 = \min(R_0^x, R_0^v)$.

Coming back to the equation for the maximal velocity

$$\begin{aligned} \frac{d}{dt} R^v(t)^2 &= -2 \sum_{j \neq i} m_j \left[(v_i - v_j) \cdot v_i\right] a(|x_i - x_j|) \\ &\leq -\frac{2}{[1 + 4R_0^2(1+t)^2]^\gamma} \sum_{j \neq i} m_j \left[(v_i - v_j) \cdot v_i\right] \\ &= -\frac{2}{[1 + 4R_0^2(1+t)^2]^\gamma} R^v(t)^2 := -f(t)\, R^v(t)^2, \end{aligned}$$

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Asymptotic Flocking

Gronwall's lemma:

$$R^v(t) \leq R_0^v \exp\left\{-\frac{1}{2}\int_0^t f(s)\,ds\right\}.$$

For $\gamma \leq 1/2$, the function $f(t)$ is not integrable at ∞ and therefore

$$\lim_{t\to\infty} \int_0^t f(s)\,ds = +\infty$$

and $R^v(t) \to 0$ as $t \to \infty$ giving the convergence to a single point, its mean velocity, of the support for the velocity.
Again for the position variables, we get

$$\begin{cases} \displaystyle\int_0^t |v_i(s)|\,ds \leq C_1 \int_0^t (1+s)^{-1-\epsilon}\,ds & \gamma < 1/2 \\ \displaystyle\int_0^t |v_i(s)|\,ds \leq C \int_0^t \frac{1}{1+s}\,ds = C\log(1+t) & \gamma = 1/2, \end{cases}.$$

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There exists $R_1^x > 0$ such that

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Now, $a(|x_i(t) - x_j(t)|) \geq a(2\bar{R}^x)$,

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Outline

Leadership, Geometrical Constraints, and Cone of Influence

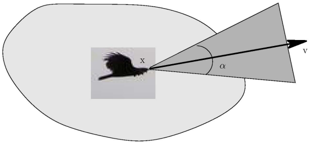
Cucker-Smale with local influence regions:

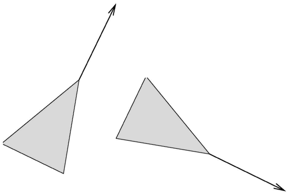
$$\begin{cases} \dfrac{dx_i}{dt} = v_i \ , \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j \in \Sigma_i(t)} a(|x_i - x_j|)(v_j - v_i) \ , \end{cases}$$

where $\Sigma_i(t) \subset \{1, \dots, N\}$ is the set of dependence, given by

$$\Sigma_i(t) := \left\{ 1 \le \ell \le N : \frac{(x_\ell - x_i) \cdot v_i}{|x_\ell - x_i||v_i|} \ge \alpha \right\} .$$

Cone of Vision:





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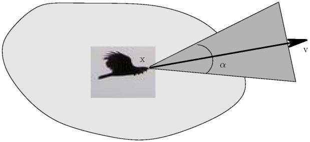
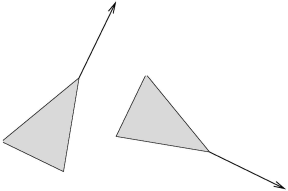




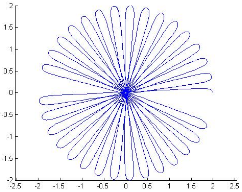
Roosting Forces

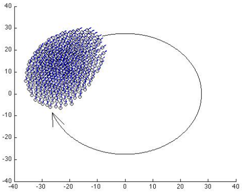
Adding a roosting area to the model:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\[2ex] \dfrac{dv_i}{dt} = (\alpha - \beta\,|v_i|^2)v_i - \displaystyle\sum_{j \neq i} \nabla U(|x_i - x_j|) - v_i^{\perp}\, \nabla_{x_i}\left[\phi(x_i)\cdot v_i^{\perp}\right]. \end{cases}$$

with the roosting potential ϕ given by $\phi(x) := \dfrac{b}{4}\left(\dfrac{|x|}{R_{\text{Roost}}}\right)^4$.

Roosting effect: milling flocks $N = 400, R_{\text{roost}} = 20.$





Adding Noise

Self-Propelling/Friction/Interaction with Noise Particle Model:

$$\begin{cases} \dot{x}_i = v_i, \\ dv_i = \left[(\alpha - \beta\,|v_i|^2)v_i - \nabla_{x_i} \sum_{j\neq i} U(|x_i - x_j|) \right] dt + \sqrt{2\sigma}\, d\Gamma_i(t)\ , \end{cases}$$

where $\Gamma_i(t)$ are N independent copies of standard Wiener processes with values in $\mathbb{R}^d$ and $\sigma > 0$ is the noise strength. The Cucker–Smale Particle Model with Noise:

$$\begin{cases} dx_i = v_i dt\ , \\ dv_i = \sum_{j=1}^{N} a(|x_j - x_i|)(v_j - v_i)\, dt + \sqrt{2\sigma \sum_{j=1}^{m} a(|x_j - x_i|)}\, d\Gamma_i(t)\ . \end{cases}$$

Mean-field: Growing at infinity Lipschitz constants treated by moment bounds.

Adding Noise

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- Millings can be understood as kinetic measure solutions concentrated on certain velocities. Geometric constraints: velocities on a sphere. Stability of these patterns?
- Mean field limit for less singular potentials than Newtonian and less smooth than locally Lipschitz potentials allowing for more "generic" initial data.
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- References:

1. C.-D'Orsogna-Panferov (KRM 2008).
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