Aggregation versus Diffusion in Mathematical Biology

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Lecture 3: Aggregation versus Diffusion Equations

UIMP, Santander, Spain, 2011

Outline

1. Macroscopic Models: PKS system
   - Modelling Chemotaxis: First Properties
   - PKS as Gradient Flow
   - Critical Fast Diffusion as Gradient Flow
   - New Liapunov Functionals
2. Ideas of the Rigorous Proof
   - Concentration-Control Inequalities
   - One-step JKO
3. A byproduct: A new proof of HLS inequalities
   - Log HLS via Fast Diffusion Flows
   - A New Proof of the HLS inequality with Equality Cases
4. Conclusions

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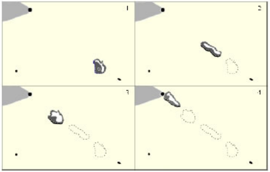
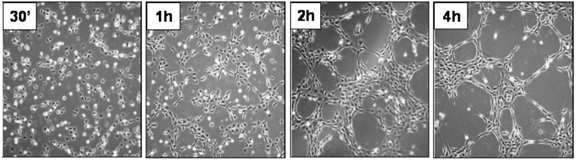
Chemotaxis





Cell movement and aggregation by chemical interaction.

PKS System

$$\begin{cases} \dfrac{\partial \rho}{\partial t}(x,t) = \Delta \rho(x,t) - \nabla \cdot (\rho(x,t) \nabla c(x,t)) & x \in \mathbb{R}^2 \,, \; t > 0 \,, \\ c(x,t) = -\dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}^2} \log|x-y| \, \rho(y,t) \, dy & x \in \mathbb{R}^2 \,, \; t > 0 \,, \\ \rho(x, t=0) = \rho_0 \geq 0 & x \in \mathbb{R}^2 \,. \end{cases}$$

Huge Literature: Horstmann reviews (2003& 2004), Perthame review (2004).

Conservations:

- Conservation of mass:

$$M := \int_{\mathbb{R}^2} \rho_0(x) \, dx = \int_{\mathbb{R}^2} \rho(x,t) \, dx$$

- Conservation of center of mass:

$$M_1 := \int_{\mathbb{R}^2} x \, \rho_0(x) \, dx = \int_{\mathbb{R}^2} x \, \rho(x,t) \, dx \,.$$

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Second Moment

Distributional Solution:

We shall say that $\rho \in C^0\left([0,T); L^1_{\text{weak}}(\mathbb{R}^2)\right)$ is a weak solution to the PKS system if for all test functions $\psi \in \mathcal{D}(\mathbb{R}^2)$,

$$\frac{d}{dt}\int_{\mathbb{R}^2} \psi(x)\,\rho(x,t)\,dx =$$

$$\int_{\mathbb{R}^2} \Delta\psi(x)\,\rho(x,t)\,dx - \frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} [\nabla\psi(x) - \nabla\psi(y)] \cdot \frac{x-y}{|x-y|^2}\,\rho(x,t)\,\rho(y,t)\,dx\,dy$$

holds in the distributional sense in $(0,T)$ and $\rho(0) = \rho_0$.

Evolution of second moment:

$$\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2\,\rho(x,t)\,dx = 4M - \frac{1}{2\pi}M^2,$$

Struggle between diffusion and aggregation. Balance between these two mechanisms happens precisely at the critical mass $M = 8\,\pi$.

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Cases

PKS Cases:

- **Subcritical Case, $M < 8\pi$:** Global existence: Jägger-Luckhaus (1992), Dolbeault-Perthame (2004), Blanchet-Dolbeault-Perthame (2006), Blanchet-Calvez-C. (2008).
- **Supercritical Case, $M > 8\pi$:** Blow-up: Herrero-Velazquez (1996), Velazquez (2002-2004), Dolbeault-Schmeiser (2009).
- **Critical Case, $M = 8\pi$:** Infinite-time aggregation, infinitely many stationary states: Biler-Karch-Laurençot-Nadzieja (2006), Blanchet-C.-Masmoudi (2008), Blanchet-Carlen-C. (2011), Carlen-C.-Loss (2010), Carlen-Figalli (preprint).

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Log HLS Inequality by Carlen& Loss

Let f be a non-negative function in $L^1(\mathbb{R}^2)$ with mass M such that $f \log f$ and $f \log(e + |x|^2)$ belong to $L^1(\mathbb{R}^2)$. Then

$$\int_{\mathbb{R}^2} f(x) \log f(x)\, dx + \frac{2}{M} \iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x) f(y) \log|x-y|\, dx\, dy \geq -C(M)$$

with $C(M) := M(1 + \log \pi - \log M)$.

Equality cases:

There is equality if and only if $f(x) = \bar\rho_\lambda(x - x_0)$ for some $\lambda > 0$ and some $x_0 \in \mathbb{R}^2$, where

$$\bar\rho_\lambda(x) := \frac{M}{\pi} \frac{\lambda}{(\lambda + |x|^2)^2}$$

Natural Liapunov Functional:

Free energy:

$$\mathcal{F}_{PKS}[\rho] := \int_{\mathbb{R}^2} \rho(x) \log \rho(x)\, dx + \frac{1}{4\pi} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \rho(x)\, \rho(y)\, \log|x-y|\, dx\, dy$$

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Formal Gradient Flow

A formal calculation shows that for all $u \in C_c^\infty(\mathbb{R}^2)$ with zero mean,

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\mathcal{F}_{\mathrm{PKS}}[\rho + \epsilon u] - \mathcal{F}_{\mathrm{PKS}}[\rho] \right) = \int_{\mathbb{R}^2} \frac{\delta \mathcal{F}_{\mathrm{PKS}}(\rho)}{\delta \rho}(x)\, u(x)\, \mathrm{d}x$$

where

$$\frac{\delta \mathcal{F}_{\mathrm{PKS}}(\rho)}{\delta \rho}(x) := \log \rho(x) + \frac{1}{2\pi} \int_{\mathbb{R}^2} \log |x-y| \rho(y)\, \mathrm{d}y = \log \rho(x) - G * \rho(x)\,.$$

The PKS equation can be rewritten as

$$\frac{\partial \rho}{\partial t}(t,x) = \mathrm{div}\left(\rho(t,x) \nabla \left[\frac{\delta \mathcal{F}_{\mathrm{PKS}}(\rho(t))}{\delta \rho}(t,x) \right] \right)\,.$$

with entropy dissipation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}_{\mathrm{PKS}}[\rho(t)] = - \int_{\mathbb{R}^2} \rho(t,x) \left| \nabla \frac{\delta \mathcal{F}_{\mathrm{PKS}}(\rho(t))}{\delta \rho}(t,x) \right|^2 \mathrm{d}x\,.$$

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Critical Case: Stationary States & Main Result

The critical case $M = 8\pi$ has a family of explicit stationary solutions of the form

$$\bar{\rho}_\lambda(x) = \frac{8\lambda}{(\lambda + |x|^2)^2}$$

with $\lambda > 0$.

- All of these stationary solutions have critical mass and infinite second moment.
- They are the cases of equality functions for the Log HLS inequality.

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Critical Fast Diffusion

The nonlinear Fokker-Planck equation in $\mathbb{R}^2$ with exponent $1/2$:

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) = \Delta\left(\sqrt{v(t,x)}\right) + \dfrac{1}{\sqrt{2\lambda}}\,\mathrm{div}(x\,v(t,x)) & t>0\,,\ x\in\mathbb{R}^2\,, \\ v(0,x) = v_0(x) \geq 0 & x\in\mathbb{R}^2\,, \end{cases}$$

corresponding to the *fast diffusion equation* $\frac{\partial u}{\partial t} = \Delta\left(\sqrt{u}\right)$ by a self-similar change of variable.

It is critical for the direct Bakry-Emery method to the fast-diffusion equation, meaning by a direct entropy-entropy dissipation inequality and a rate of convergence coming only from displacement convexity.

For $\lambda > 0$, define the functional $\mathcal{H}_\lambda$ on the non-negative functions in $L^1(\mathbb{R}^2)$ by

$$\mathcal{H}_\lambda[v] := \int_{\mathbb{R}^2}\left(\sqrt{v(x)} - \sqrt{\bar\rho_\lambda(x)}\right)^2 \bar\rho_\lambda^{-1/2}(x)\ \mathrm{d}x$$

This functional is the relative entropy of the fast diffusion equation with respect to the stationary solution $\bar\rho_\lambda$. The unique minimizer of $\mathcal{H}_\lambda$ is $\bar\rho_\lambda$.

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Formal Gradient Flow

A formal calculation as before shows,

$$\frac{\delta \mathcal{H}_\lambda[v]}{\delta v} = \frac{1}{\sqrt{\bar{\rho}_\lambda}} - \frac{1}{\sqrt{v}} \,,$$

and the critical fast diffusion equation can be rewritten as

$$\frac{\partial v}{\partial t}(t,x) = \operatorname{div}\left(v(t,x) \left[\nabla \frac{\delta \mathcal{H}_\lambda[v]}{\delta v} \right] \right) \,,$$

with entropy dissipation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{H}_\lambda[v(t)] = - \int_{\mathbb{R}^2} v(t,x) \left| \nabla \frac{\delta \mathcal{H}_\lambda[v]}{\delta v} \right|^2 \, \mathrm{d}x.$$

Let us point out that the functional $\mathcal{H}_\lambda[v]$ can be written as

$$\mathcal{H}_\lambda[u] := \int_{\mathbb{R}^2} \left[\Phi(v(x)) - \Phi(\bar{\rho}_\lambda(x)) - \Phi'(\bar{\rho}_\lambda)(v(x) - \bar{\rho}_\lambda(x)) \right] \, \mathrm{d}x$$

with $\Phi(s) = -2\sqrt{s}$ a convex function.

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Let us point out that the functional $\mathcal{H}_\lambda[v]$ can be written as

$$\mathcal{H}_\lambda[u] := \int_{\mathbb{R}^2} \left[\Phi(v(x)) - \Phi(\bar\rho_\lambda(x)) - \Phi'(\bar\rho_\lambda)(v(x) - \bar\rho_\lambda(x)) \right] \,\mathrm{d}x$$

with $\Phi(s) = -2\sqrt{s}$ a convex function.

Formal Gradient Flow

A formal calculation as before shows,

$$\frac{\delta \mathcal{H}_\lambda[v]}{\delta v} = \frac{1}{\sqrt{\bar\rho_\lambda}} - \frac{1}{\sqrt{v}}\,,$$

and the critical fast diffusion equation can be rewritten as

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Outline

1. Macroscopic Models: PKS system
   - Modelling Chemotaxis: First Properties
   - PKS as Gradient Flow
   - Critical Fast Diffusion as Gradient Flow
   - New Liapunov Functionals
2. Ideas of the Rigorous Proof
   - Concentration-Control Inequalities
   - One-step JKO
3. A byproduct: A new proof of HLS inequalities
   - Log HLS via Fast Diffusion Flows
   - A New Proof of the HLS inequality with Equality Cases
4. Conclusions

New Liapunov Functional

Claim: The critical fast diffusion functional is also a Liapunov functional for the critical mass PKS. Formal Computation:

$$\frac{d}{dt}\mathcal{H}_\lambda[\rho(t)] = \int_{\mathbb{R}^2} \frac{\delta \mathcal{H}_\lambda[\rho]}{\delta\rho} \operatorname{div}\left(\rho(t,x)\nabla\left[\frac{\delta\mathcal{F}_{\text{PKS}}[\rho]}{\delta\rho}\right]\right)\,dx$$

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Also, $\int_{\mathbb{R}^2} x \cdot \nabla\rho \, \mathrm{d}x = -2M$ and, making the same symmetrization that led to the evolution of the second moment,

$$\int_{\mathbb{R}^2} \rho(x)\, x \cdot \nabla G * \rho(x)\, \mathrm{d}x = \frac{1}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \rho(t,x)\, (x-y) \cdot \frac{x-y}{|x-y|^2}\, \rho(t,y)\, \mathrm{d}x\, \mathrm{d}y = -\frac{M^2}{4\pi} .$$

Collecting the above computations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_\lambda[\rho(t)] = -\frac{1}{2}\int_{\mathbb{R}^2} \frac{|\nabla\rho|^2}{\rho^{3/2}}\, \mathrm{d}x + \int_{\mathbb{R}^2} \rho^{3/2}\, \mathrm{d}x + 4\sqrt{\frac{M\pi}{\lambda}}\left(1 - \frac{M}{8\pi}\right) .$$

Notice that the constant term vanishes in critical mass case $M = 8\pi$. Thus, in the critical mass case, formal calculation yields that for all $T > 0$,

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The missing link to exploit the previous relation is a particular case of the Gagliardo-Nirenberg-Sobolev inequalities[1].

Gagliardo-Nirenberg-Sobolev inequality

For all functions f in $\mathbb{R}^2$ with a square integrable distributional gradient ∇f,

$$\pi \int_{\mathbb{R}^2} |f|^6 \, dx \leq \int_{\mathbb{R}^2} |\nabla f|^2 \, dx \int_{\mathbb{R}^2} |f|^4 \, dx$$

and there is equality if and only if f is a multiple of a translate of $\bar{\rho}_\lambda^{1/4}$, $\lambda > 0$.

Dissipation of $\mathcal{H}_\lambda$

Applying the GNS to $f = \rho^{1/4}$: For all densities ρ of mass $M = 8\pi$,

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and moreover, there is equality if and only ρ is a translate of $\bar{\rho}_\lambda$ for some $\lambda > 0$.

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Bassin of attractions in the Critical Mass PKS

Given any density ρ_0 in $\mathbb{R}^2$ with total mass 8π such that there exists $\lambda > 0$ with

$$\mathcal{H}_\lambda[\rho_0] < \infty$$

Then there exists $\rho \in \mathcal{AC}^0([0,T],\mathcal{P}_2(\mathbb{R}^2))$, with $\rho(t) \in L^1(\mathbb{R}^2)$ for all $t \geq 0$ being a global-in-time weak solution of the critical mass PKS. Moreover, the solutions constructed satisfy that

$$\mathcal{H}_\lambda[\rho(t)] \leq \mathcal{H}_\lambda[\rho(s)] \qquad \text{and} \qquad \mathcal{F}_{\text{PKS}}[\rho(t)] \leq \mathcal{F}_{\text{PKS}}[\rho(s)]\,,$$

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Mathematical Difficulties

- For a density of mass 8π, an upper bound on $\mathcal{F}_{\text{PKS}}[\rho]$ does not provide any upper bound on the entropy $\mathcal{E}[\rho]$.

  Indeed, $\mathcal{F}_{\text{PKS}}$ takes its minimum value for $\rho = \bar{\rho}_\mu$ for all $\mu > 0$, but $\mathcal{E}[\bar{\rho}_\mu] \to \infty$ as $\mu \to 0$ since $\bar{\rho}_\mu$ converges weakly-* as measures towards a Dirac delta at the origin with mass 8π.

- The dissipation functional $\mathcal{D}[\rho]$ is well defined as long as $\|\rho\|_{3/2} < \infty$, but an upper bound on $\mathcal{D}[\rho]$ does not give an upper bound on either $\|\nabla(\rho^{1/4})\|_2$ or $\|\rho\|_{3/2}$ since it is the difference of both terms.

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- Stability properties of $\mathcal{H}_\lambda$ over $\mathcal{F}_{\text{PKS}}$. In fact, not much was known about stability for $\mathcal{F}_{\text{PKS}}$ till very recently (E. Carlen, A. Figalli, preprint).

  We do know that if some density ρ with mass 8π satisfies $\mathcal{F}_{\text{PKS}}[\rho] = \mathcal{F}_{\text{PKS}}[\bar{\rho}_\lambda]$, then, up to translation, $\rho = \bar{\rho}_\mu$ for some $\mu > 0$. Even knowing a quantitative estimate for the error in $\mathcal{F}_{\text{PKS}}$, it mainly helps to quantify the decay rate.

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Concentration-Control for the Entropy I

In summary, we can show there exists $R > 1$ so that for some $0 < a < 8\pi$,

$$\int_{|x|>R-1} \rho \, \mathrm{d}x \le 8\pi - a \quad \text{and} \quad \int_{|x|<R+1} \rho \, \mathrm{d}x \le 8\pi - a \, .$$

The densities lying in sublevel sets of $\mathcal{F}_{\mathrm{PKS}}[\rho]$ and $\mathcal{H}_\lambda[\rho]$ are compact.

Entropy bound

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The densities lying in sublevel sets of $\mathcal{F}_{\text{PKS}}[\rho]$, $\mathcal{H}_\lambda[\rho]$ and $\mathcal{D}[\rho]$ are compact.

Concentration control for $\mathcal{D}$

Let ρ be any density with mass 8π, $\mathcal{F}_{\text{PKS}}[\rho]$ finite, and $\mathcal{H}_\lambda[\rho]$ finite for some $\lambda > 0$. Then there exist positive computable constants γ_2 and C_{CCD} depending only on λ, $\mathcal{H}_\lambda[\rho]$ and $\mathcal{F}_{\text{PKS}}[\rho]$ such that

$$\gamma_2 \int_{\mathbb{R}^2} |\nabla \rho^{1/4}|^2 \, dx \leq \pi \mathcal{D}[\rho] + C_{\text{CCD}} \ .$$

This allows to control the $L^{3/2}$-norm of ρ too.

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Set-up

Regularization

For all $0 < \epsilon \leq 1$, define

$$\mathcal{S} := \left\{ \rho \in L^1(\mathbb{R}^2, \log(e + |x|^2)\ \mathrm{d}x) : \int_{\mathbb{R}^2} \rho(x) \log \rho(x)\ \mathrm{d}x < \infty,\ \int_{\mathbb{R}^2} \rho(x)\ \ \mathrm{d}x = 8\,\pi \right\}$$

For any $\rho \in \mathcal{S}$ define

$$\mathcal{F}^{\epsilon}_{\mathrm{PKS}}[\rho] = \int_{\mathbb{R}^2} \rho(x) \log \rho(x)\ \mathrm{d}x - \frac{1}{2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \rho(x)\, G_\epsilon(x-y)\, \rho(y)\ \ \mathrm{d}x\, \mathrm{d}y\ .$$

Here, the regularized Green's function is

$$G_\epsilon = \gamma_\epsilon * G * \gamma_\epsilon$$

where $*$ denotes convolution, $G(x) = -1/(2\pi) \log |x|$, and $\gamma_\epsilon(x) = \epsilon^{-2} \gamma\,(x/\epsilon)$. with γ a standard gaussian.

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Difficulties in the JKO scheme + Passing to the Limit

- Regularized Problem: we can show now existence of minimizers in the JKO scheme and have the right regularity to produce estimates for one step:

$$\arg\min_{\rho \in \mathcal{S}} \left\{ \frac{W_2^2(\rho, \rho_0)}{2\tau} + \mathcal{F}_{\mathrm{PKS}}^{\epsilon}[\rho] \right\} .$$

- However, the regularization destroys the decay of the functional $\mathcal{H}_\lambda$. Then we have to cope with error terms in the dissipation inequality for $\mathcal{H}_\lambda$.
- Cure: take step-dependent regularization parameter ϵ_k such that the errors are summable and thus negligible, in the limit of small $\tau \to 0$.
- For the cure and the compactness, the concentration controlled inequalities and a L^p-smoothing effect are crucial.
- The dissipation on $\mathcal{H}_\lambda$ in the limit identifies the limiting density ϱ_λ. A qualitative stability estimate on $\mathcal{F}_{\mathrm{PKS}}$ implies the L^1-strong convergence.
- Rate of Convergence: (Carlen-Figalli, preprint) Stability estimates for GNS imply stability estimates for the Log-HLS via the fast diffusion flow. This can be applied to decay rates in L^1 for the solutions constructed to the PKS.

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$$\arg\min_{\rho\in\mathcal{S}}\left\{\frac{\mathrm{W}_2^2(\rho,\rho_0)}{2\tau}+\mathcal{F}^{\epsilon}_{\mathrm{PKS}}[\rho]\right\}.$$

- However, the regularization destroys the decay of the functional $\mathcal{H}_\lambda$. Then we have to cope with error terms in the dissipation inequality for $\mathcal{H}_\lambda$.
- Cure: take step-dependent regularization parameter ϵ_k such that the errors are summable and thus negligible, in the limit of small $\tau\to 0$.
- For the cure and the compactness, the concentration controlled inequalities and a L^p-smoothing effect are crucial.
- The dissipation on $\mathcal{H}_\lambda$ in the limit identifies the limiting density ϱ_λ. A qualitative stability estimate on $\mathcal{F}_{\mathrm{PKS}}$ implies the L^1-strong convergence.
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"Duality" between CFD and PKS

Observation of Matthes, McCann & Savaré, CPDE 2009

Consider two smooth functions Φ and Ψ on $\mathbb{R}^d$, and consider the two ordinary differential equations describing gradient flow:

$$\dot{x}(t) = -\nabla\Phi[x(t)] \qquad \text{and} \qquad \dot{y}(t) = -\nabla\Psi[y(t)]\ .$$

Then of course $\Phi[(x(t)]$ and $\Psi[(y(t)]$ are monotone decreasing. Now differentiate each function along the others flow:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi[y(t)] = -\langle\nabla\Phi[y(t)], \nabla\Psi[y(t)]\rangle$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi[x(t)] = -\langle\nabla\Psi[x(t)], \nabla\Phi[x(t)]\rangle\ .$$

Thus, Φ is decreasing along the gradient flow of Ψ for any initial data if and only if Ψ is decreasing along the gradient flow of Φ for any initial data.

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- An analog of this holds for well-behaved gradient flows in the 2-Wasserstein sense as used in (Matthes, McCann & Savare, CPDE 2009).
- Our case: Apply it to the Log-HLS functional in $d = 2$.- Since $\mathcal{H}_\lambda$ is decreasing along the 2-Wasserstein gradient flow for $\mathcal{F}_{\text{PKS}}$, i.e., the Patlak-Keller-Segel equation, one can expect that $\mathcal{F}_{\text{PKS}}$ of the Log-HLS functional is decreasing along the 2-Wasserstein gradient flow for $\mathcal{H}_\lambda$, i.e., the critical fast diffusion.

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The connection from FD perspective

The log HLS functional $\mathcal{F}$ is defined by

$$\mathcal{F}[f] := \int_{\mathbb{R}^2} f(x) \log f(x)\, dx + \frac{2}{\int_{\mathbb{R}^2} f(x)\, dx} \iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x) \log|x-y| f(y)\, dx\, dy\,.$$

The critical fast diffusion is

$$\frac{\partial}{\partial t} u(x,t) = \Delta u^{1/2}(x,t)\,,$$

with associated Fokker-Planck equation

$$\frac{\partial}{\partial t} v(x,t) = \Delta v^{1/2}(x,t) + \nabla\cdot[x v(x,t)]\,.$$

which has as stationary solution

$$h(x) = \frac{M}{\pi}\left(\frac{1}{1+|x|^2}\right)^2$$

with mass M.

Scaling: $f_{(a)} := a^2 f(ax)$. Then, $\mathcal{F}[f_{(a)}] = \mathcal{F}[f]$ for all a. $v(x,t) := e^{2t} u(e^t x, e^t)$ and thus $\mathcal{F}[v(\cdot,t)] = \mathcal{F}[u(\cdot,e^t)]$.

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Log HLS via fast diffusion flow

Let f be a non-negative measurable functions on $\mathbb{R}^2$ such that $f \log f$ and $f \log(e+|x|^2)$ belong to $L^1(\mathbb{R}^2)$. Suppose also that

$$\int_{\mathbb{R}^2} f(x)\, \mathrm{d}x = \int_{\mathbb{R}^2} h(x)\, \mathrm{d}x = M$$

and

$$\sup_{|x|>R} f(x)|x|^4 < \infty\,.$$

Let $u(x,t)$ be the solution of critical fast diffusion with $u(x,1) = f(x)$. Then

$$\mathcal{F}[f] = \mathcal{F}[h] + \int_1^\infty \mathcal{D}[u^{1/4}(\cdot, e^t)]\, \mathrm{d}t \geq \mathcal{F}[h]\,,$$

where

$$\mathcal{D}[g] = \int_{\mathbb{R}^2} |\nabla g(x)|^2\, \mathrm{d}x \int_{\mathbb{R}^2} g^4(x)\, \mathrm{d}x - \pi \int_{\mathbb{R}^2} g^6(x)\, \mathrm{d}x\,,$$

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Proof: Let $v(x,t)$ be the solution of the nonlinear FP with $v(x,0) = f(x)$.

Then it is known that $\lim_{t\to\infty} v(x,t) = h(x)$ strongly in $L^1(\mathbb{R}^2)$ and moreover[2], we have the uniform bounds:

$$C\,h(x) \le v(x,t) \le \frac{1}{C}\,h(x)\;,$$

under the assumption on $f(x)$ after some time.

Dominated convergence, implies that $\lim_{t\to\infty} \mathcal{F}[v(\cdot,t)] = \mathcal{F}[h]$. Thus,

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Two Inequalities, One Equation

Sharp Hardy-Littlewood-Sobolev inequality

[a] It states that for all non-negative measurable functions f on $\mathbb{R}^d$, and all $0 < \lambda < d$,

$$\frac{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)\frac{1}{|x-y|^\lambda} f(y)\,\mathrm{d}x\,\mathrm{d}y}{\|f\|_p^2} \le \frac{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h(x)\frac{1}{|x-y|^\lambda} h(y)\,\mathrm{d}x\,\mathrm{d}y}{\|h\|_p^2}$$

where

$$h(x) = \left(\frac{1}{1+|x|^2}\right)^{(2d-\lambda)/2}.$$

and $p = \frac{2d}{2d-\lambda}$.

Moreover, there is equality if and only if for some $x_0 \in \mathbb{R}^d$ and $s \in \mathbb{R}_+$, f is a non-zero multiple of $h(x/s - x_0)$.

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Two Inequalities, One Equation

Sharp Gagliardo-Nirenberg-Sobolev inequality

[a] It states that for all locally integrable functions f on $\mathbb{R}^d$, $d \geq 2$, with a square integrable distributional gradient, and p with $1 < p < d/(d-2)$

$$\frac{\|\nabla f\|_2^{\theta} \|f\|_{p+1}^{1-\theta}}{\|f\|_{2p}} \geq \frac{\|\nabla g\|_2^{\theta} \|g\|_{p+1}^{1-\theta}}{\|g\|_{2p}}$$

where

$$g(x) = \left(\frac{1}{1+|x|^2}\right)^{1/(p-1)} .$$

and

$$\theta = \frac{d(p-1)}{p(d+2-(d-2)p)} .$$

Moreover, there is equality if and only if for some $x_0 \in \mathbb{R}^d$ and $s \in \mathbb{R}_+$, f is a non-zero multiple of $g(x/s - x_0)$.

[a] DelPino & Dolbeault, JMPA, 2002.

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The Fast Diffusion Equation

The equality cases are steady states for nonlinear Fokker-Planck equations related to the fast diffusion equation:

$$\frac{\partial}{\partial t}u(x,t) = \Delta u^m(x,t)\,.$$

$u(x,t)$ solves the fast diffusion equation if and only if

$$v(x,t) := e^{td}u(e^t x, e^{\alpha t})$$

with $\alpha = 2 - d(1-m)$ satisfies

$$\alpha\frac{\partial}{\partial t}v(x,t) = \Delta v^m(x,t) + \nabla\cdot[xv(x,t)]\,.$$

$$v_\infty(x) := \left(\frac{2m}{(1-m)}\right)^{1/(1-m)}\left(\frac{1}{1+|x|^2}\right)^{1/(1-m)}$$

is a stationary solution with certain mass M.

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The connection

- The relation between the HLS optimizers, the GNS optimizers and the FD equation is more than a superficial coincidence, at least for certain λ, p and m.
- For $d \geq 3$, the $\lambda = d - 2$ case of the sharp HLS inequality can be proved by using the fast diffusion flow for $m = d/(d+2)$ to deform any reasonably nice trial function into a multiple of h.

Motivation: We choose m for the fast diffusion equation and λ for the HLS inequality in order to have h as stationary solution/equality case. Thus,

$$\frac{1}{1-m} = \frac{d+2}{2} \qquad \Leftrightarrow \qquad m = \frac{d}{d+2}.$$

The exact value of the exponent for the GNS inequality

$$1 < p = \frac{d+1}{d-1} < \frac{d}{d-2}$$

will be dictated by the evolution of the HLS functional along the flow of the FD equation. Note p is in the correct range of the sharp GNS inequalities.

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The proof

The *HLS functional*

For $\lambda = d - 2$, the HLS inequality can be rewritten as $\mathcal{F}[f] \geq 0$ for all $f \in L^{2d/(d+2)}$ where

$$\mathcal{F}[f] := C_S \left(\int_{\mathbb{R}^d} f^{2d/(d+2)}(x) \mathrm{d}x \right)^{(d+2)/d} - \int_{\mathbb{R}^d} f(x) \left[(-\Delta)^{-1} f \right](x) \, \mathrm{d}x \,,$$

with

$$C_S := \frac{4}{d(d-2)} |S^d|^{-2/d} \,.$$

The *GNS inequality*

The GNS inequality with $p = (d+1)/(d-1)$ can be written as $\mathcal{D}[g] \geq 0$ for all g with a square integrable distributional gradient on $\mathbb{R}^d$ where

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The proof

Dissipation of the HLS functional by the FD equation

Let $f \in L^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^{2d/(d+2)})$ be non-negative, and suppose that

$$\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} h(x)\,\mathrm{d}x = M$$

Let $u(x,t)$ be the solution of the FD equation with $u(x,1) = f(x)$. Then, for all $t > 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[u(\cdot,t)] = -2\mathcal{D}[u^{(d-1)/(d+2)}(\cdot,t)]\ .$$

HLS inequality via FD flow

Let $f \in L^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^{2d/(d+2)})$ be non-negative, and suppose that

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Then $\mathcal{F}[f] \geq 0$.

Conclusions

- A new Liapunov functional unveiled for the critical mass PKS: a bassin of attraction determined for each stationary state.
- Optimal Transportation Tools are crucially used to construct the solutions.
- A technique of Concentration Controlled inequalities developed to cope with the lack of compactness.
- The duality between the PKS in the critical mass case and the critical nonlinear fast diffusion equation in 2D leads to the discovery of new proofs of log-HLS and HLS inequalities.
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