Pattern formation, energy landscapes, scaling laws

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In spatially extended systems, the average size of a (statistically homogeneous) pattern depends in a reproducible way on model parameters or on time — typically described by some exponents, thus giving rise to a "scaling law". Typically, such a scaling law is a robust feature — at least in terms of experimental observation and numerical simulation. Mathematically, it can be captured in terms of a priori bounds on the solutions of the underlying partial differential equation or variational problem. We have shown in a number of applications how the well-developed theory of a priori estimates for nonlinear or non-convex problems can be used to make mathematically rigorous and physically meaningful statements on such scaling laws.

In this mini course, we'll treat one example: We consider demixing processes of binary mixtures and are interested in the coarsening of the spatial phase distribution, i. e. in how the average length scale ℓ of the first phase, say, increases with time t. The underlying partial differential equation is the Cahn-Hilliard equation that models the thermodynamically driven, diffusion-limited demixing of two components. We are also interested in the effect of flow that typically speeds up the coarsening ("Siggia's growth"); here, the underlying partial differential equation is the Cahn-Hilliard equation coupled to the Stokes system.

The key idea is to use the gradient flow structure of these equations: The energy E is given by a Ginzburg-Landau-type free functional and the metric tensor g encodes the dissipation mechanisms by diffusion (outer friction) and viscosity (inner friction). If the (highly non-convex) energy landscape is not too steep, then it is intuitive that coarsening does not proceed too fast — this intuition can be quantified and in principe provides rigorous upper bounds on coarsening (that turn out to be optimal for many examples — but not for all).

In order to carry out this program, two issues have to be solved:

- In order to measure "steepness" of the energy landscape, one has to characterize the distance d induced by the metric tensor g — that is, one has to pass from a description "in the small" to a description "in the large". At least in the presence of viscosity, d cannot be equated

to a simple expression like a transportation distance, but at least it controls a transportation distance (with a logarithmic cost).

- In order to characterize "steepness" of the energy landscape, one has to relate E and the induced distance $d(\cdot, 0)$ to the well-mixed state 0. This typically is contained in a suitable (functional) interpolation inequality. In the two-dimensional diffusion-dominated case however, this relies on an interesting strengthening of an interpolation inequality.

This is joint work (over the years) with R. V. Kohn, S. Conti & B. Niethammer, Y. Brenier & C. Seis and yet unpublished work with C. Seis & D. Slepcev, and E. Cinti.

References

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