Aggregation versus Diffusion in Mathematical Biology

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This course will be devoted to summarize several recent results in nonlocal interaction equations with or without diffusion. These equations have received a lot of attention in the recent years because of their ubiquity in different models and areas of applied and pure mathematics. Collective behavior of animals (swarming), chemotaxis models, and granular media models are some examples. On the other hand, these equations have been studied in connection to entropy-entropy dissipation techniques, optimal transport, and gradient flows with respect to probability measure distances. We will first discuss first-order models of the form:

$$\partial_t \rho = \nabla \cdot (\rho(\nabla U * \rho)) + \epsilon \Delta \rho \tag{1}$$

where $\rho = \rho(t,x)$ is a real function depending on time $t \geq 0$ and space $x \in \mathbb{R}^N$, $U : \mathbb{R}^N \longrightarrow \mathbb{R}$ is an *interaction potential* verifying $U(x) = U(-x)$ without loss of generality.

Without diffusion $\epsilon = 0$, the continuity equation (1) with a singular interaction potential U can lead to very involved dynamics where blow-up can occur, and where Dirac Delta singularities and smooth parts of the solution can coexist even for mildly singular potentials as $U(x) \simeq |x|$ at the origin. This will be one of the issues treated in the course.

The second objective will be to deal with the classical Keller-Segel model corresponding to (1) with $U(x) = -\frac{1}{2\pi} \log |x|$ and $\epsilon = 1$ in two dimensions. Here, we will see that there is a “exact” compensation between the struggle of diffusion and aggregation for some parameter. This particular case will be studied in detail and we will show that some kind of duality with fast diffusion equations is behind the hidden structure. As a byproduct, we give new proofs of related functional inequalities such as the logarithmic-HLS and HLS inequalities with equality cases.

Finally, we will deal with some kinetic models in which the aggregation happens in velocity space. These models have been proposed as second order models for swarming and some issues can be dealt with similar strategies as in the first-order models.

The main tools used along the 3 topics contain transport distances, gradient flows in probability measures and bits of optimal transportation theory. A reading of the basics of these tools in your favorite reference in the subject will be beneficial, although the course is organized in a self-contained manner. This course is a summary of results in collaboration with M. DiFrancesco, A. Figalli, T. Laurent, D. Slepcev (Duke Math. J. 2011); A. Blanchet, E. Carlen (Preprint UAB); E. Carlen, M. Loss (PNAS 2010); J. A. Cañizo, J. Rosado (M3AS 2010); M. Fornasier, J. Rosado, G. Toscani (SIMA 2010).