# The Theories of Nonlinear Diffusion

*Juan Luis Vázquez*

Departamento de Matemáticas
Universidad Autónoma de Madrid
Madrid, Spain

*Frontiers of Mathematics and Applications*

Summer Course UIMP 2010
Santander (Spain), August 9-13, 2010

# Outline

1. **Theories of Diffusion**
   - Diffusion
   - Heat equation
   - Linear Parabolic Equations
   - Nonlinear equations

2. **Degenerate Diffusion**
   - Introduction
   - The basics
   - Generalities

3. **Fast Diffusion Equation**
   - Fast Diffusion Ranges
   - Regularity through inequalities. Aronson–Caffarelli Estimates
   - Local Boundedness
   - Flows on manifolds

# Outline

1. **Theories of Diffusion**
   - Diffusion
   - Heat equation
   - Linear Parabolic Equations
   - Nonlinear equations
2. **Degenerate Diffusion**
   - Introduction
   - The basics
   - Generalities
3. **Fast Diffusion Equation**
   - Fast Diffusion Ranges
   - Regularity through inequalities. Aronson–Caffarelli Estimates
   - Local Boundedness
   - Flows on manifolds

# Diffusion

Populations diffuse, substances (like particles in a solvent) diffuse, heat propagates, electrons and ions diffuse, the momentum of a viscous (Newtonian) fluid diffuses (linearly), there is diffusion in the markets, ...

- *what is diffusion anyway?*
- *how to explain it with mathematics?*

• *A main question is: how much of it can be explained with linear models, how much is essentially nonlinear?*

• *The stationary states of diffusion belong to an important world, elliptic equations. Elliptic equations, linear and nonlinear, have many relatives: diffusion, fluid mechanics, waves of all types, quantum mechanics, ...*

*The Laplacian $\Delta$ is the King of Differential Operators.*

# Diffusion

Populations diffuse, substances (like particles in a solvent) diffuse, heat propagates, electrons and ions diffuse, the momentum of a viscous (Newtonian) fluid diffuses (linearly), there is diffusion in the markets, ...

- *what is diffusion anyway?*
- *how to explain it with mathematics?*
- *A main question is: how much of it can be explained with linear models, how much is essentially nonlinear?*
- *The stationary states of diffusion belong to an important world, elliptic equations. Elliptic equations, linear and nonlinear, have many relatives: diffusion, fluid mechanics, waves of all types, quantum mechanics, ...*

*The Laplacian $\Delta$ is the King of Differential Operators.*

# Diffusion

Populations diffuse, substances (like particles in a solvent) diffuse, heat propagates, electrons and ions diffuse, the momentum of a viscous (Newtonian) fluid diffuses (linearly), there is diffusion in the markets, ...

- *what is diffusion anyway?*
- *how to explain it with mathematics?*

- *A main question is: how much of it can be explained with linear models, how much is essentially nonlinear?*

- *The stationary states of diffusion belong to an important world, elliptic equations. Elliptic equations, linear and nonlinear, have many relatives: diffusion, fluid mechanics, waves of all types, quantum mechanics, ...*

*The Laplacian $\Delta$ is the King of Differential Operators.*

# Diffusion

Populations diffuse, substances (like particles in a solvent) diffuse, heat propagates, electrons and ions diffuse, the momentum of a viscous (Newtonian) fluid diffuses (linearly), there is diffusion in the markets, ...

- *what is diffusion anyway?*
- *how to explain it with mathematics?*
- *A main question is: how much of it can be explained with linear models, how much is essentially nonlinear?*
- *The stationary states of diffusion belong to an important world, elliptic equations. Elliptic equations, linear and nonlinear, have many relatives: diffusion, fluid mechanics, waves of all types, quantum mechanics, ...*

*The Laplacian $\Delta$ is the King of Differential Operators.*

# Diffusion

Populations diffuse, substances (like particles in a solvent) diffuse, heat propagates, electrons and ions diffuse, the momentum of a viscous (Newtonian) fluid diffuses (linearly), there is diffusion in the markets, ...

- *what is diffusion anyway?*
- *how to explain it with mathematics?*

• *A main question is: how much of it can be explained with linear models, how much is essentially nonlinear?*

• *The stationary states of diffusion belong to an important world, elliptic equations. Elliptic equations, linear and nonlinear, have many relatives: diffusion, fluid mechanics, waves of all types, quantum mechanics, ...*

*The Laplacian $\Delta$ is the King of Differential Operators.*

# Diffusion in Wikipedia

- Diffusion. The spreading of any quantity that can be described by the diffusion equation or a random walk model (e.g. concentration, heat, momentum, ideas, price) can be called diffusion.
- Some of the most important examples are listed below.
  ** Atomic diffusion*
  ** Brownian motion, for example of a single particle in a solvent*
  ** Collective diffusion, the diffusion of a large number of (possibly interacting) particles * Effusion of a gas through small holes.*
  ** Electron diffusion, resulting in electric current*
  ** Facilitated diffusion, present in some organisms.*
  ** Gaseous diffusion, used for isotope separation*
  ** Heat flow * Ito- diffusion * Knudsen diffusion*
  ** Momentum diffusion, ex. the diffusion of the hydrodynamic velocity field*
  ** Osmosis is the diffusion of water through a cell membrane. * Photon diffusion*
  ** Reverse diffusion * Self-diffusion * Surface diffusion*

# Diffusion in Wikipedia

- Diffusion. The spreading of any quantity that can be described by the diffusion equation or a random walk model (e.g. concentration, heat, momentum, ideas, price) can be called diffusion.
- Some of the most important examples are listed below.
  ** Atomic diffusion*
  ** Brownian motion, for example of a single particle in a solvent*
  ** Collective diffusion, the diffusion of a large number of (possibly interacting) particles * Effusion of a gas through small holes.*
  ** Electron diffusion, resulting in electric current*
  ** Facilitated diffusion, present in some organisms.*
  ** Gaseous diffusion, used for isotope separation*
  ** Heat flow * Ito- diffusion * Knudsen diffusion*
  ** Momentum diffusion, ex. the diffusion of the hydrodynamic velocity field*
  ** Osmosis is the diffusion of water through a cell membrane. * Photon diffusion*
  ** Reverse diffusion * Self-diffusion * Surface diffusion*

# The heat equation origins

- We begin our presentation with the Heat Equation $u_t = \Delta u$ and the analysis proposed by Fourier, 1807, 1822 (Fourier decomposition, spectrum). The mathematical models of heat propagation and diffusion have made great progress both in theory and application.

  They have had a strong influence on 5 areas of Mathematics: PDEs, Functional Analysis, Inf. Dim. Dyn. Systems, Diff. Geometry and Probability. And on and from Physics.

- The heat flow analysis is based on two main techniques: integral representation (convolution with a Gaussian kernel) and mode separation:

$$u(x,t) = \sum T_i(t) X_i(x)$$

where the $X_i(x)$ form the spectral sequence

$$-\Delta X_i = \lambda_i X_i.$$

This is the famous linear eigenvalue problem, Spectral Theory.

# The heat equation origins

- We begin our presentation with the Heat Equation $u_t = \Delta u$
and the analysis proposed by Fourier, 1807, 1822
(Fourier decomposition, spectrum). The mathematical models of heat propagation and diffusion have made great progress both in theory and application.
They have had a strong influence on 5 areas of Mathematics: PDEs, Functional Analysis, Inf. Dim. Dyn. Systems, Diff. Geometry and Probability. And on and from Physics.

- The heat flow analysis is based on two main techniques: integral representation (convolution with a Gaussian kernel) and mode separation:

$$u(x,t) = \sum T_i(t) X_i(x)$$

where the $X_i(x)$ form the spectral sequence

$$-\Delta X_i = \lambda_i X_i.$$

This is the famous linear eigenvalue problem, Spectral Theory.

# The heat equation origins

- We begin our presentation with the Heat Equation $u_t = \Delta u$ and the analysis proposed by Fourier, 1807, 1822 (Fourier decomposition, spectrum). The mathematical models of heat propagation and diffusion have made great progress both in theory and application.
  They have had a strong influence on 5 areas of Mathematics: PDEs, Functional Analysis, Inf. Dim. Dyn. Systems, Diff. Geometry and Probability. And on and from Physics.

- The heat flow analysis is based on two main techniques: integral representation (convolution with a Gaussian kernel) and mode separation:

$$u(x,t) = \sum T_i(t) X_i(x)$$

where the $X_i(x)$ form the spectral sequence

$$-\Delta X_i = \lambda_i X_i.$$

This is the famous linear eigenvalue problem, Spectral Theory.

# The heat equation origins

- We begin our presentation with the Heat Equation $\boxed{u_t = \Delta u}$ and the analysis proposed by Fourier, 1807, 1822 (Fourier decomposition, spectrum). The mathematical models of heat propagation and diffusion have made great progress both in theory and application.
  They have had a strong influence on 5 areas of Mathematics: PDEs, Functional Analysis, Inf. Dim. Dyn. Systems, Diff. Geometry and Probability. And on and from Physics.

- The heat flow analysis is based on two main techniques: integral representation (convolution with a Gaussian kernel) and mode separation:

$$u(x,t) = \sum T_i(t) X_i(x)$$

  where the $X_i(x)$ form the spectral sequence

$$-\Delta X_i = \lambda_i X_i.$$

  This is the famous linear eigenvalue problem, Spectral Theory.

# Linear heat flows

From 1822 until 1950 the heat equation has motivated
(i) Fourier analysis decomposition of functions (and set theory),
(ii) development of other linear equations
$\Longrightarrow$ Theory of Parabolic Equations

$$u_t = \sum a_{ij}\partial_i\partial_j u + \sum b_i \partial_i u + cu + f$$

Main inventions in Parabolic Theory:
(1) $a_{ij}, b_i, c, f$ regular $\Rightarrow$ Maximum Principles, Schauder estimates, Harnack inequalities; $C^\alpha$ spaces (Hölder); potential theory; generation of semigroups.
(2) coefficients only continuous or bounded $\Rightarrow$ $W^{2,p}$ estimates, Calderón-Zygmund theory, weak solutions; Sobolev spaces.

The probabilistic approach: Diffusion as an stochastic process: Bachelier, Einstein, Smoluchowski, Wiener, Levy, Ito,...

$$dX = b\,dt + \sigma\,dW$$

# Linear heat flows

From 1822 until 1950 the heat equation has motivated
(i) Fourier analysis decomposition of functions (and set theory),
(ii) development of other linear equations
$\Longrightarrow$ Theory of Parabolic Equations

$$u_t = \sum a_{ij}\partial_i\partial_j u + \sum b_i\partial_i u + cu + f$$

Main inventions in Parabolic Theory:
(1) $a_{ij}, b_i, c, f$ regular $\Rightarrow$ Maximum Principles, Schauder estimates, Harnack inequalities; $C^\alpha$ spaces (Hölder); potential theory; generation of semigroups.
(2) coefficients only continuous or bounded $\Rightarrow$ $W^{2,p}$ estimates, Calderón-Zygmund theory, weak solutions; Sobolev spaces.

The probabilistic approach: Diffusion as an stochastic process: Bachelier, Einstein, Smoluchowski, Wiener, Levy, Ito,...

$$dX = bdt + \sigma dW$$

# Linear heat flows

From 1822 until 1950 the heat equation has motivated
(i) Fourier analysis decomposition of functions (and set theory),
(ii) development of other linear equations
$\Longrightarrow$ Theory of Parabolic Equations

$$u_t = \sum a_{ij}\partial_i\partial_j u + \sum b_i\partial_i u + cu + f$$

Main inventions in Parabolic Theory:
(1) $a_{ij}, b_i, c, f$ regular $\Rightarrow$ Maximum Principles, Schauder estimates, Harnack inequalities; $C^\alpha$ spaces (Hölder); potential theory; generation of semigroups.
(2) coefficients only continuous or bounded $\Rightarrow$ $W^{2,p}$ estimates, Calderón-Zygmund theory, weak solutions; Sobolev spaces.

The probabilistic approach: Diffusion as an stochastic process: Bachelier, Einstein, Smoluchowski, Wiener, Levy, Ito,...

$$dX = bdt + \sigma dW$$

# Linear heat flows

From 1822 until 1950 the heat equation has motivated
(i) Fourier analysis decomposition of functions (and set theory),
(ii) development of other linear equations
$\Longrightarrow$ Theory of Parabolic Equations

$$u_t = \sum a_{ij}\partial_i\partial_j u + \sum b_i\partial_i u + cu + f$$

Main inventions in Parabolic Theory:
(1) $a_{ij}, b_i, c, f$ regular $\Rightarrow$ Maximum Principles, Schauder estimates, Harnack inequalities; $C^\alpha$ spaces (Hölder); potential theory; generation of semigroups.
(2) coefficients only continuous or bounded $\Rightarrow$ $W^{2,p}$ estimates, Calderón-Zygmund theory, weak solutions; Sobolev spaces.

The probabilistic approach: Diffusion as an stochastic process: Bachelier, Einstein, Smoluchowski, Wiener, Levy, Ito,...

$$dX = bdt + \sigma dW$$

# Nonlinear heat flows

- In the last 50 years emphasis has shifted towards the Nonlinear World. Maths more difficult, more complex, and more realistic.
  My group works in the areas of Nonlinear Diffusion and Reaction Diffusion.
  I will talk about the theory mathematically called Nonlinear Parabolic PDEs. General formula

$$u_t = \sum \partial_i A_i(u, \nabla u) + \sum B(x, u, \nabla u)$$

- Typical nonlinear diffusion: Stefan Problem, Hele-Shaw Problem, PME: $u_t = \Delta(u^m)$, EPLE: $u_t = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$.
- Typical reaction diffusion: Fujita model $u_t = \Delta u + u^p$.
- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. *Any singularities?*
- The geometrical models: the Ricci flow, movement by curvature.

# Nonlinear heat flows

- In the last 50 years emphasis has shifted towards the Nonlinear World. Maths more difficult, more complex, and more realistic.
  My group works in the areas of Nonlinear Diffusion and Reaction Diffusion.
  I will talk about the theory mathematically called Nonlinear Parabolic PDEs. General formula

$$u_t = \sum \partial_i A_i(u, \nabla u) + \sum B(x, u, \nabla u)$$

- Typical nonlinear diffusion: Stefan Problem, Hele-Shaw Problem, PME: $u_t = \Delta(u^m)$, EPLE: $u_t = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$.
- Typical reaction diffusion: Fujita model $u_t = \Delta u + u^p$.
- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. *Any singularities?*
- The geometrical models: the Ricci flow, movement by curvature.

# Nonlinear heat flows

- In the last 50 years emphasis has shifted towards the Nonlinear World. Maths more difficult, more complex, and more realistic.
  My group works in the areas of Nonlinear Diffusion and Reaction Diffusion.
  I will talk about the theory mathematically called Nonlinear Parabolic PDEs. General formula

$$u_t = \sum \partial_i A_i(u, \nabla u) + \sum B(x, u, \nabla u)$$

- Typical nonlinear diffusion: Stefan Problem, Hele-Shaw Problem, PME: $u_t = \Delta(u^m)$, EPLE: $u_t = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$.
- Typical reaction diffusion: Fujita model $u_t = \Delta u + u^p$.
- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. *Any singularities?*
- The geometrical models: the Ricci flow, movement by curvature.

# Nonlinear heat flows

- In the last 50 years emphasis has shifted towards the Nonlinear World. Maths more difficult, more complex, and more realistic.
  My group works in the areas of Nonlinear Diffusion and Reaction Diffusion.
  I will talk about the theory mathematically called Nonlinear Parabolic PDEs. General formula

$$u_t = \sum \partial_i A_i(u, \nabla u) + \sum B(x, u, \nabla u)$$

- Typical nonlinear diffusion: Stefan Problem, Hele-Shaw Problem, PME: $u_t = \Delta(u^m)$, EPLE: $u_t = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$.
- Typical reaction diffusion: Fujita model $u_t = \Delta u + u^p$.
- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. *Any singularities?*
- The geometrical models: the Ricci flow, movement by curvature.

# Nonlinear heat flows

- In the last 50 years emphasis has shifted towards the Nonlinear World. Maths more difficult, more complex, and more realistic.
  My group works in the areas of Nonlinear Diffusion and Reaction Diffusion.
  I will talk about the theory mathematically called Nonlinear Parabolic PDEs. General formula

$$u_t = \sum \partial_i A_i(u, \nabla u) + \sum B(x, u, \nabla u)$$

- Typical nonlinear diffusion: Stefan Problem, Hele-Shaw Problem, PME: $u_t = \Delta(u^m)$, EPLE: $u_t = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$.
- Typical reaction diffusion: Fujita model $u_t = \Delta u + u^p$.
- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. *Any singularities?*
- The geometrical models: the Ricci flow, movement by curvature.

# The Nonlinear Diffusion Models

- The Stefan Problem (Lamé and Clapeyron, 1833; Stefan 1880)

$$SE : \begin{cases} u_t = k_1 \Delta u & \text{for } u > 0, \\ u_t = k_2 \Delta u & \text{for } u < 0. \end{cases} \quad TC : \begin{cases} u = 0, \\ \mathbf{v} = L(k_1 \nabla u_1 - k_2 \nabla u_2). \end{cases}$$

  Main feature: the free boundary or moving boundary where $u = 0$. TC= Transmission conditions at $u = 0$.

- The Hele-Shaw cell (Hele-Shaw, 1898; Saffman-Taylor, 1958)

$$u > 0,\ \Delta u = 0 \quad \text{in} \quad \Omega(t); \qquad u = 0,\ \mathbf{v} = L\partial_n u \quad \text{on} \quad \partial\Omega(t).$$

- The Porous Medium Equation →*(hidden free boundary)*

$$u_t = \Delta u^m, \qquad m > 1.$$

- The *p*-Laplacian Equation, $u_t = \mathrm{div}\,(|\nabla u|^{p-2} \nabla u)$.

  Recent interest in $p = 1$ (images) or $p = \infty$ (geometry and transport)

# The Reaction Diffusion Models

- The Standard Blow-Up model (Kaplan, 1963; Fujita, 1966)

$$u_t = \Delta u + u^p$$

Main feature: If $p > 1$ the norm $\|u(\cdot,t)\|_\infty$ of the solutions goes to infinity in finite time. Hint: Integrate $u_t = u^p$.
Problem: *what is the influence of diffusion / migration?*

- General scalar model

$$u_t = \mathcal{A}(u) + f(u)$$

- The system model: $\overrightarrow{\mathbf{u}} = (u_1, \cdots, u_m) \to$ chemotaxis system.
- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. Quadratic nonlinear, Mixed type *Any singularities?*
- The geometrical models: the Ricci flow: $\partial_t g_{ij} = -R_{ij}$. This is a nonlinear heat equation. Posed in the form of PDEs by R Hamilton, 1982. Solved by G Perelman 2003.

# The Reaction Diffusion Models

- The Standard Blow-Up model (Kaplan, 1963; Fujita, 1966)

$$u_t = \Delta u + u^p$$

  Main feature: If $p > 1$ the norm $\|u(\cdot, t)\|_\infty$ of the solutions goes to infinity in finite time. Hint: Integrate $u_t = u^p$.
  Problem: *what is the influence of diffusion / migration?*

- General scalar model

$$u_t = \mathcal{A}(u) + f(u)$$

- The system model: $\overrightarrow{\mathbf{u}} = (u_1, \cdots, u_m) \rightarrow$ chemotaxis system.
- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. Quadratic nonlinear, Mixed type *Any singularities?*
- The geometrical models: the Ricci flow: $\partial_t g_{ij} = -R_{ij}$. This is a nonlinear heat equation. Posed in the form of PDEs by R Hamilton, 1982. Solved by G Perelman 2003.

# The Reaction Diffusion Models

- The Standard Blow-Up model (Kaplan, 1963; Fujita, 1966)

$$u_t = \Delta u + u^p$$

Main feature: If $p > 1$ the norm $\|u(\cdot, t)\|_\infty$ of the solutions goes to infinity in finite time. Hint: Integrate $u_t = u^p$.
Problem: *what is the influence of diffusion / migration?*

- General scalar model

$$u_t = \mathcal{A}(u) + f(u)$$

- The system model: $\vec{\mathbf{u}} = (u_1, \cdots, u_m) \to$ chemotaxis system.
- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. Quadratic nonlinear, Mixed type *Any singularities?*
- The geometrical models: the Ricci flow: $\partial_t g_{ij} = -R_{ij}$. This is a nonlinear heat equation. Posed in the form of PDEs by R Hamilton, 1982. Solved by G Perelman 2003.

# The Reaction Diffusion Models

- The Standard Blow-Up model (Kaplan, 1963; Fujita, 1966)

$$u_t = \Delta u + u^p$$

Main feature: If $p > 1$ the norm $\|u(\cdot,t)\|_\infty$ of the solutions goes to infinity in finite time. Hint: Integrate $u_t = u^p$.
Problem: *what is the influence of diffusion / migration?*

- General scalar model

$$u_t = \mathcal{A}(u) + f(u)$$

- The system model: $\vec{\mathbf{u}} = (u_1, \cdots, u_m) \to$ chemotaxis system.
- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. Quadratic nonlinear, Mixed type *Any singularities?*
- The geometrical models: the Ricci flow: $\partial_t g_{ij} = -R_{ij}$. This is a nonlinear heat equation. Posed in the form of PDEs by R Hamilton, 1982. Solved by G Perelman 2003.

# The Reaction Diffusion Models

- The Standard Blow-Up model (Kaplan, 1963; Fujita, 1966)

$$u_t = \Delta u + u^p$$

Main feature: If $p > 1$ the norm $\|u(\cdot, t)\|_\infty$ of the solutions goes to infinity in finite time. Hint: Integrate $u_t = u^p$.
Problem: *what is the influence of diffusion / migration?*

- General scalar model

$$u_t = \mathcal{A}(u) + f(u)$$

- The system model: $\vec{\mathbf{u}} = (u_1, \cdots, u_m) \to$ chemotaxis system.
- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. Quadratic nonlinear, Mixed type *Any singularities?*
- The geometrical models: the Ricci flow: $\partial_t g_{ij} = -R_{ij}$. This is a nonlinear heat equation. Posed in the form of PDEs by R Hamilton, 1982. Solved by G Perelman 2003.

*An opinion by John Nash, 1958:*

The open problems in the area of nonlinear p.d.e. are very relevant to applied mathematics and science as a whole, perhaps more so that the open problems in any other area of mathematics, and the field seems poised for rapid development. It seems clear, however, that fresh methods must be employed...

Little is known about the existence, uniqueness and smoothness of solutions of the general equations of flow for a viscous, compressible, and heat conducting fluid...

*"Continuity of solutions of elliptic and parabolic equations",*
*paper published in Amer. J. Math, 80, no 4 (1958), 931-954*

# Outline

# The Porous Medium - Fast Diffusion Equation

- The simplest model of nonlinear diffusion equation is maybe

$$u_t = \Delta u^m = \nabla \cdot (c(u)\nabla u)$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = mu^{m-1} [= m|u|^{m-1}]$$

- If $m > 1$ it degenerates at $u = 0$ , $\Longrightarrow$ slow diffusion
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0 \Longrightarrow$ Fast Diffusion.
- But power functions are tricky:
  - $c(u) \to 0$ as $u \to \infty$ if $m > 1$ ("slow case")
  - $c(u) \to \infty$ as $u \to \infty$ if $m < 1$ ("fast case")

# The Porous Medium - Fast Diffusion Equation

- The simplest model of nonlinear diffusion equation is maybe

$$u_t = \Delta u^m = \nabla \cdot (c(u)\nabla u)$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = mu^{m-1} [= m|u|^{m-1}]$$

- If $m > 1$ it degenerates at $u = 0$ , $\Longrightarrow$ slow diffusion
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0 \Longrightarrow$ Fast Diffusion.
- But power functions are tricky:

  - $c(u) \to 0$ as $u \to \infty$ if $m > 1$ ("slow case")

  - $c(u) \to \infty$ as $u \to \infty$ if $m < 1$ ("fast case")

# The Porous Medium - Fast Diffusion Equation

- The simplest model of nonlinear diffusion equation is maybe

$$u_t = \Delta u^m = \nabla \cdot (c(u)\nabla u)$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = mu^{m-1}[= m|u|^{m-1}]$$

- If $m > 1$ it degenerates at $u = 0$ , $\Longrightarrow$ slow diffusion
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0 \Longrightarrow$ Fast Diffusion.
- But power functions are tricky:

  - $c(u) \to 0$ as $u \to \infty$ if $m > 1$ ("slow case")

  - $c(u) \to \infty$ as $u \to \infty$ if $m < 1$ ("fast case")

# The Porous Medium - Fast Diffusion Equation

- The simplest model of nonlinear diffusion equation is maybe

$$u_t = \Delta u^m = \nabla \cdot (c(u)\nabla u)$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = mu^{m-1}[= m|u|^{m-1}]$$

- If $m > 1$ it degenerates at $u = 0$ , $\Longrightarrow$ slow diffusion
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0 \Longrightarrow$ Fast Diffusion.
- But power functions are tricky:

  - $c(u) \to 0$ as $u \to \infty$ if $m > 1$ ("slow case")

  - $c(u) \to \infty$ as $u \to \infty$ if $m < 1$ ("fast case")

# The Porous Medium - Fast Diffusion Equation

- The simplest model of nonlinear diffusion equation is maybe

$$u_t = \Delta u^m = \nabla \cdot (c(u)\nabla u)$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = mu^{m-1}[= m|u|^{m-1}]$$

- If $m > 1$ it degenerates at $u = 0$ , $\Longrightarrow$ slow diffusion
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0 \Longrightarrow$ Fast Diffusion.
- But power functions are tricky:

  - $c(u) \to 0$ as $u \to \infty$ if $m > 1$ ("slow case")

  - $c(u) \to \infty$ as $u \to \infty$ if $m < 1$ ("fast case")

# The Porous Medium - Fast Diffusion Equation

- The simplest model of nonlinear diffusion equation is maybe

$$u_t = \Delta u^m = \nabla \cdot (c(u)\nabla u)$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = mu^{m-1}[= m|u|^{m-1}]$$

- If $m > 1$ it degenerates at $u = 0$ , $\Longrightarrow$ slow diffusion
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0 \Longrightarrow$ Fast Diffusion.
- But power functions are tricky:
  - $c(u) \to 0$ as $u \to \infty$ if $m > 1$ ("slow case")
  - $c(u) \to \infty$ as $u \to \infty$ if $m < 1$ ("fast case")

# The basics

- For for $m = 2$ the equation is re-written as

$$\boxed{\frac{1}{2}u_t = u\Delta u + |\nabla u|^2}$$

and you can see that for $u \sim 0$ it looks like the eikonal equation

$$\boxed{u_t = |\nabla u|^2}$$

*This is not parabolic, but hyperbolic (propagation along characteristics). Mixed type, mixed properties.*

- No big problem when $m > 1$, $m \neq 2$. The pressure transformation gives:

$$\boxed{v_t = (m-1)v\Delta v + |\nabla v|^2}$$

where $\boxed{v = cu^{m-1}}$ is the pressure; normalization $c = m/(m-1)$.
This separates $m > 1$ PME - from - $m < 1$ FDE

# The basics

- For for $m = 2$ the equation is re-written as

$$\frac{1}{2}u_t = u\Delta u + |\nabla u|^2$$

and you can see that for $u \sim 0$ it looks like the eikonal equation

$$u_t = |\nabla u|^2$$

*This is not parabolic, but hyperbolic (propagation along characteristics). Mixed type, mixed properties.*

- No big problem when $m > 1$, $m \neq 2$. The pressure transformation gives:

$$v_t = (m-1)v\Delta v + |\nabla v|^2$$

where $v = cu^{m-1}$ is the pressure; normalization $c = m/(m-1)$.

This separates $m > 1$ PME - from - $m < 1$ FDE

# The basics

- For for $m = 2$ the equation is re-written as

$$\frac{1}{2}u_t = u\Delta u + |\nabla u|^2$$

and you can see that for $u \sim 0$ it looks like the eikonal equation

$$u_t = |\nabla u|^2$$

*This is not parabolic, but hyperbolic (propagation along characteristics). Mixed type, mixed properties.*

- No big problem when $m > 1$, $m \neq 2$. The pressure transformation gives:

$$v_t = (m-1)v\Delta v + |\nabla v|^2$$

where $v = cu^{m-1}$ is the pressure; normalization $c = m/(m-1)$.
This separates $m > 1$ PME - from - $m < 1$ FDE

# The basics

- For for $m = 2$ the equation is re-written as

$$\frac{1}{2}u_t = u\Delta u + |\nabla u|^2$$

and you can see that for $u \sim 0$ it looks like the eikonal equation

$$u_t = |\nabla u|^2$$

*This is not parabolic, but hyperbolic (propagation along characteristics). Mixed type, mixed properties.*

- No big problem when $m > 1$, $m \neq 2$. The pressure transformation gives:

$$v_t = (m-1)v\Delta v + |\nabla v|^2$$

where $v = cu^{m-1}$ is the pressure; normalization $c = m/(m-1)$.
This separates $m > 1$ PME - from - $m < 1$ FDE

# The basics

- For for $m = 2$ the equation is re-written as

$$\boxed{\frac{1}{2}u_t = u\Delta u + |\nabla u|^2}$$

and you can see that for $u \sim 0$ it looks like the eikonal equation

$$\boxed{u_t = |\nabla u|^2}$$

*This is not parabolic, but hyperbolic (propagation along characteristics). Mixed type, mixed properties.*

- No big problem when $m > 1$, $m \neq 2$. The pressure transformation gives:

$$\boxed{v_t = (m-1)v\Delta v + |\nabla v|^2}$$

where $\boxed{v = cu^{m-1}}$ is the pressure; normalization $c = m/(m-1)$.

This separates $m > 1$ PME - from - $m < 1$ FDE

# The basics

- For for $m = 2$ the equation is re-written as

$$\boxed{\tfrac{1}{2} u_t = u\Delta u + |\nabla u|^2}$$

and you can see that for $u \sim 0$ it looks like the eikonal equation

$$\boxed{u_t = |\nabla u|^2}$$

*This is not parabolic, but hyperbolic (propagation along characteristics). Mixed type, mixed properties*.

- No big problem when $m > 1$, $m \neq 2$. The pressure transformation gives:

$$\boxed{v_t = (m-1)v\Delta v + |\nabla v|^2}$$

where $\boxed{v = cu^{m-1}}$ is the pressure; normalization $c = m/(m-1)$.
This separates $m > 1$ PME - from - $m < 1$ FDE

# Planning of the Theory

These are the main topics of mathematical analysis (1958-2006):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: *is there a limit? $C^k$ for some $k$?*
- Regularity and movement of interfaces: *$C^k$ for some $k$?*.
- Asymptotic behaviour: *patterns and rates? universal?*
- The probabilistic approach. *Nonlinear process. Wasserstein estimates*
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

# Planning of the Theory

These are the main topics of mathematical analysis (1958-2006):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: *is there a limit?* $C^k$ *for some* $k$*?*
- Regularity and movement of interfaces: $C^k$ *for some* $k$*?*.
- Asymptotic behaviour: *patterns and rates? universal?*
- The probabilistic approach. *Nonlinear process. Wasserstein estimates*
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

# Planning of the Theory

These are the main topics of mathematical analysis (1958-2006):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: *is there a limit? $C^k$ for some $k$?*
- Regularity and movement of interfaces: *$C^k$ for some $k$?.*
- Asymptotic behaviour: *patterns and rates? universal?*
- The probabilistic approach. *Nonlinear process. Wasserstein estimates*
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

# Planning of the Theory

These are the main topics of mathematical analysis (1958-2006):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: *is there a limit? $C^k$ for some $k$?*
- Regularity and movement of interfaces: *$C^k$ for some $k$?*.
- Asymptotic behaviour: *patterns and rates? universal?*
- The probabilistic approach. *Nonlinear process. Wasserstein estimates*
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

# Planning of the Theory

These are the main topics of mathematical analysis (1958-2006):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: *is there a limit? $C^k$ for some $k$?*
- Regularity and movement of interfaces: *$C^k$ for some $k$?*.
- Asymptotic behaviour: *patterns and rates? universal?*
- The probabilistic approach. *Nonlinear process. Wasserstein estimates*
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

# Planning of the Theory

These are the main topics of mathematical analysis (1958-2006):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: *is there a limit?* $C^k$ *for some* $k$*?*
- Regularity and movement of interfaces: $C^k$ *for some* $k$*?*.
- Asymptotic behaviour: *patterns and rates? universal?*
- The probabilistic approach. *Nonlinear process. Wasserstein estimates*
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

# Planning of the Theory

These are the main topics of mathematical analysis (1958-2006):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: *is there a limit? $C^k$ for some $k$?*
- Regularity and movement of interfaces: *$C^k$ for some $k$?*.
- Asymptotic behaviour: *patterns and rates? universal?*
- The probabilistic approach. *Nonlinear process. Wasserstein estimates*
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

# Planning of the Theory

These are the main topics of mathematical analysis (1958-2006):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: *is there a limit?* $C^k$ *for some* $k$*?*
- Regularity and movement of interfaces: $C^k$ *for some* $k$*?*.
- Asymptotic behaviour: *patterns and rates? universal?*
- The probabilistic approach. *Nonlinear process. Wasserstein estimates*
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

# Planning of the Theory

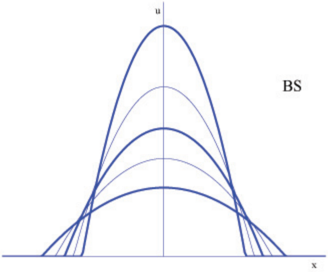
These are the main topics of mathematical analysis (1958-2006):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: *is there a limit?* $C^k$ *for some* $k$*?*
- Regularity and movement of interfaces: $C^k$ *for some* $k$*?*.
- Asymptotic behaviour: *patterns and rates? universal?*
- The probabilistic approach. *Nonlinear process. Wasserstein estimates*
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

# Barenblatt profiles (ZKB)

- These profiles are the alternative to the Gaussian profiles.
  They are source solutions. *Source means that* $u(x,t) \to M\,\delta(x)$ *as* $t \to 0$.
- Explicit formulas (1950):

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \left(C - k\xi^2\right)_+^{1/(m-1)}$$




$\alpha = \frac{n}{2+n(m-1)}$

$\beta = \frac{1}{2+n(m-1)} < 1/2$

Height $u = Ct^{-\alpha}$

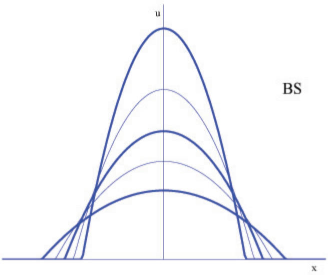
Free boundary at distance $|x| = ct^{\beta}$

Scaling law; anomalous diffusion versus Brownian motion

# Barenblatt profiles (ZKB)

- These profiles are the alternative to the Gaussian profiles.
  They are source solutions. *Source means that* $u(x,t) \to M\,\delta(x)$ *as* $t \to 0$*.*
- Explicit formulas (1950):

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \left(C - k\xi^2\right)_+^{1/(m-1)}$$




$$\alpha = \frac{n}{2+n(m-1)}$$

$$\beta = \frac{1}{2+n(m-1)} < 1/2$$

Height $u = Ct^{-\alpha}$

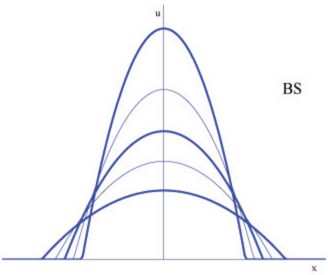
Free boundary at distance $|x| = ct^{\beta}$

Scaling law; anomalous diffusion versus Brownian motion

# Barenblatt profiles (ZKB)

- These profiles are the alternative to the Gaussian profiles. They are source solutions. *Source means that* $u(x,t) \to M\,\delta(x)$ *as* $t \to 0$.
- Explicit formulas (1950):

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \left(C - k\xi^2\right)_+^{1/(m-1)}$$




$\alpha = \frac{n}{2+n(m-1)}$

$\beta = \frac{1}{2+n(m-1)} < 1/2$

Height $u = Ct^{-\alpha}$

Free boundary at distance $|x| = ct^{\beta}$

Scaling law; anomalous diffusion versus Brownian motion

# Concept of solution

There are many concepts of generalized solution of the PME:

- Classical solution: only in non-degenerate situations, $u > 0$.
- Limit solution: physical, but depends on the approximation (?).
- Weak solution Test against smooth functions and eliminate derivatives on the unknown function; it is the mainstream; (Oleinik, 1958)

$$\int\int (u\,\eta_t - \nabla u^m \cdot \nabla\eta)\, dxdt + \int u_0(x)\,\eta(x,0)\, dx = 0.$$

Very weak

$$\int\int (u\,\eta_t + u^m\,\Delta\eta)\, dxdt + \int u_0(x)\,\eta(x,0)\, dx = 0.$$

# More on concepts of solution

Solutions are not always, not only weak:

- Strong solution. More regular than weak but not classical: weak derivatives are $L^p$ functions. *Big benefit: usual calculus is possible.*
- Semigroup solution / mild solution. The typical product of functional discretization schemes: $u = \{u_n\}_n$, $u_n = u(\cdot, t_n)$,

$$u_t = \Delta\Phi(u), \qquad \frac{u_n - u_{n-1}}{h} - \Delta\Phi(u_n) = 0$$

Now put $f := u_{n-1}$, $u := u_n$, and $v = \Phi(u)$, $u = \beta(v)$:

$$-h\Delta\Phi(u) + u = f, \qquad \boxed{-h\Delta v + \beta(v) = f.}$$

"Nonlinear elliptic equations"; Crandall-Liggett Theorems Ambrosio, Savarè, Nochetto

# More on concepts of solution

Solutions are not always, not only weak:

- Strong solution. More regular than weak but not classical: weak derivatives are $L^p$ functions. *Big benefit: usual calculus is possible.*
- Semigroup solution / mild solution. The typical product of functional discretization schemes: $u = \{u_n\}_n$, $u_n = u(\cdot, t_n)$,

$$u_t = \Delta\Phi(u), \qquad \frac{u_n - u_{n-1}}{h} - \Delta\Phi(u_n) = 0$$

Now put $f := u_{n-1}$, $u := u_n$, and $v = \Phi(u)$, $u = \beta(v)$:

$$-h\Delta\Phi(u) + u = f, \qquad \boxed{-h\Delta v + \beta(v) = f.}$$

"Nonlinear elliptic equations"; Crandall-Liggett Theorems

Ambrosio, Savarè, Nochetto

# More on concepts of solution

Solutions are not always, not only weak:

- Strong solution. More regular than weak but not classical: weak derivatives are $L^p$ functions. *Big benefit: usual calculus is possible.*
- Semigroup solution / mild solution. The typical product of functional discretization schemes: $u = \{u_n\}_n$, $u_n = u(\cdot, t_n)$,

$$u_t = \Delta\Phi(u), \qquad \frac{u_n - u_{n-1}}{h} - \Delta\Phi(u_n) = 0$$

Now put $f := u_{n-1}$, $u := u_n$, and $v = \Phi(u)$, $u = \beta(v)$:

$$-h\Delta\Phi(u) + u = f, \qquad \boxed{-h\Delta v + \beta(v) = f.}$$

"Nonlinear elliptic equations"; Crandall-Liggett Theorems    Ambrosio, Savarè, Nochetto

# More on concepts of solution II

Solutions of more complicated diffusion-convection equations need new concepts:

- Viscosity solution  Two ideas: (1) add artificial viscosity and pass to the limit; (2) viscosity concept of Crandall-Evans-Lions (1984); adapted to PME by Caffarelli-Vazquez (1999).
- Entropy solution (Kruzhkov, 1968).  Invented for conservation laws; it identifies unique physical solution from spurious weak solutions. It is useful for general models degenerate diffusion-convection models;
- Renormalized solution (Di Perna - PLLions).
- BV solution (Volpert-Hudjaev).
- Kinetic solutions  (Perthame,...).

# More on concepts of solution II

Solutions of more complicated diffusion-convection equations need new concepts:

- Viscosity solution Two ideas: (1) add artificial viscosity and pass to the limit; (2) viscosity concept of Crandall-Evans-Lions (1984); adapted to PME by Caffarelli-Vazquez (1999).
- Entropy solution (Kruzhkov, 1968). Invented for conservation laws; it identifies unique physical solution from spurious weak solutions. It is useful for general models degenerate diffusion-convection models;
- Renormalized solution (Di Perna - PLLions).
- BV solution (Volpert-Hudjaev).
- Kinetic solutions (Perthame,...).

# More on concepts of solution II

Solutions of more complicated diffusion-convection equations need new concepts:

- Viscosity solution Two ideas: (1) add artificial viscosity and pass to the limit; (2) viscosity concept of Crandall-Evans-Lions (1984); adapted to PME by Caffarelli-Vazquez (1999).
- Entropy solution (Kruzhkov, 1968). Invented for conservation laws; it identifies unique physical solution from spurious weak solutions. It is useful for general models degenerate diffusion-convection models;
- Renormalized solution (Di Perna - PLLions).
- BV solution (Volpert-Hudjaev).
- Kinetic solutions (Perthame,...).

# More on concepts of solution II

Solutions of more complicated diffusion-convection equations need new concepts:

- Viscosity solution  Two ideas: (1) add artificial viscosity and pass to the limit; (2) viscosity concept of Crandall-Evans-Lions (1984); adapted to PME by Caffarelli-Vazquez (1999).
- Entropy solution (Kruzhkov, 1968). Invented for conservation laws; it identifies unique physical solution from spurious weak solutions. It is useful for general models degenerate diffusion-convection models;
- Renormalized solution (Di Perna - PLLions).
- BV solution (Volpert-Hudjaev).
- Kinetic solutions (Perthame,...).

# Regularity results

- The universal estimate holds (Aronson-Bénilan, 79):

$$\boxed{\Delta v \geq -C/t.}$$

  $v \sim u^{m-1}$ is the pressure.

- (Caffarelli-Friedman, 1982) $C^\alpha$ regularity: there is an $\alpha \in (0,1)$ such that a bounded solution defined in a cube is $C^\alpha$ continuous.
- If there is an interface $\Gamma$, it is also $C^\alpha$ continuous in space time.
- How far can you go?
  Free boundaries are stationary (metastable) if initial profile is quadratic near $\partial\Omega$: $u_0(x) = O(d^2)$. This is called waiting time. Characterized by JLV in 1983. *Visually interesting in thin films spreading on a table.*

  Existence of corner points possible when metastable, $\Rightarrow$ *no* $C^1$ Aronson-Caffarelli-V. Regularity stops here in $n = 1$

# Regularity results

- The universal estimate holds (Aronson-Bénilan, 79):

$$\boxed{\Delta v \geq -C/t.}$$

  $v \sim u^{m-1}$ is the pressure.

- (Caffarelli-Friedman, 1982) $C^\alpha$ regularity: there is an $\alpha \in (0,1)$ such that a bounded solution defined in a cube is $C^\alpha$ continuous.
- If there is an interface $\Gamma$, it is also $C^\alpha$ continuous in space time.
- How far can you go?
  Free boundaries are stationary (metastable) if initial profile is quadratic near $\partial\Omega$: $u_0(x) = O(d^2)$. This is called waiting time. Characterized by JLV in 1983. *Visually interesting in thin films spreading on a table.*

  Existence of corner points possible when metastable, $\Rightarrow$ *no* $C^1$ Aronson-Caffarelli-V. Regularity stops here in $n = 1$

# Regularity results

- The universal estimate holds (Aronson-Bénilan, 79):

$$\boxed{\Delta v \geq -C/t.}$$

  $v \sim u^{m-1}$ is the pressure.
- (Caffarelli-Friedman, 1982) $C^\alpha$ regularity: there is an $\alpha \in (0,1)$ such that a bounded solution defined in a cube is $C^\alpha$ continuous.
- If there is an interface $\Gamma$, it is also $C^\alpha$ continuous in space time.
- How far can you go?
  Free boundaries are stationary (metastable) if initial profile is quadratic near $\partial\Omega$: $u_0(x) = O(d^2)$. This is called waiting time. Characterized by JLV in 1983. *Visually interesting in thin films spreading on a table.*

  Existence of corner points possible when metastable, $\Rightarrow$ *no* $C^1$ Aronson-Caffarelli-V. Regularity stops here in $n = 1$

# Regularity results

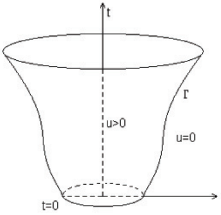
- The universal estimate holds (Aronson-Bénilan, 79):

$$\boxed{\Delta v \geq -C/t.}$$

  $v \sim u^{m-1}$ is the pressure.
- (Caffarelli-Friedman, 1982) $C^\alpha$ regularity: there is an $\alpha \in (0,1)$ such that a bounded solution defined in a cube is $C^\alpha$ continuous.
- If there is an interface $\Gamma$, it is also $C^\alpha$ continuous in space time.
- How far can you go?
  Free boundaries are stationary (metastable) if initial profile is quadratic near $\partial\Omega$: $u_0(x) = O(d^2)$. This is called waiting time. Characterized by JLV in 1983. *Visually interesting in thin films spreading on a table.*

  Existence of corner points possible when metastable, $\Rightarrow$ *no* $C^1$
  Aronson-Caffarelli-V. Regularity stops here in $n = 1$

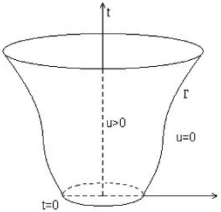
# Free Boundaries in several dimensions




A regular free boundary in n-D

- (Caffarelli-Vazquez-Wolanski, 1987) If $u_0$ has compact support, then after some time $T$ the interface and the solutions are $C^{1,\alpha}$.
- (Koch, thesis, 1997) If $u_0$ is transversal then FB is $C^\infty$ after $T$. Pressure is “laterally” $C^\infty$. *when it moves, it is always a broken profile* .
- A free boundary with a hole in 2D, 3D is the way of showing that focusing accelerates the viscous fluid so that the speed becomes infinite. This is blow-up for $\mathbf{v} \sim \nabla u^{m-1}$. The setup is a viscous fluid on a table occupying an annulus of radii $r_1$ and $r_2$. As time passes $r_2(t)$ grows and $r_1(t)$ goes to the origin. As $t \to T$, the time the hole disappears.
- There is a solution displaying that behaviour Aronson et al., 1993,... $u(x,t) = (T-t)^\alpha F(x/(T-t)^\beta)$. It is proved that

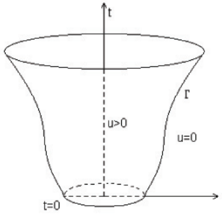
# Free Boundaries in several dimensions




A regular free boundary in n-D

- (Caffarelli-Vazquez-Wolanski, 1987) If $u_0$ has compact support, then after some time $T$ the interface and the solutions are $C^{1,\alpha}$.
- (Koch, thesis, 1997) If $u_0$ is transversal then FB is $C^\infty$ after $T$. Pressure is "laterally" $C^\infty$. *when it moves, it is always a broken profile .*
- A free boundary with a hole in 2D, 3D is the way of showing that focusing accelerates the viscous fluid so that the speed becomes infinite. This is blow-up for $\mathbf{v} \sim \nabla u^{m-1}$. The setup is a viscous fluid on a table occupying an annulus of radii $r_1$ and $r_2$. As time passes $r_2(t)$ grows and $r_1(t)$ goes to the origin. As $t \to T$, the time the hole disappears.
- There is a solution displaying that behaviour Aronson et al., 1993,... $u(x,t) = (T-t)^\alpha F(x/(T-t)^\beta)$. It is proved that

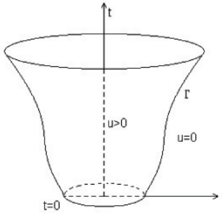
# Free Boundaries in several dimensions




A regular free boundary in n-D

- (Caffarelli-Vazquez-Wolanski, 1987) If $u_0$ has compact support, then after some time $T$ the interface and the solutions are $C^{1,\alpha}$.
- (Koch, thesis, 1997) If $u_0$ is transversal then FB is $C^\infty$ after $T$. Pressure is "laterally" $C^\infty$. *when it moves, it is always a broken profile .*
- A free boundary with a hole in 2D, 3D is the way of showing that focusing accelerates the viscous fluid so that the speed becomes infinite. This is blow-up for $\mathbf{v} \sim \nabla u^{m-1}$. The setup is a viscous fluid on a table occupying an annulus of radii $r_1$ and $r_2$. As time passes $r_2(t)$ grows and $r_1(t)$ goes to the origin. As $t \to T$, the time the hole disappears.
- There is a solution displaying that behaviour Aronson et al., 1993,... $u(x,t) = (T-t)^\alpha F(x/(T-t)^\beta)$. It is proved that

# Free Boundaries in several dimensions




A regular free boundary in n-D

- (Caffarelli-Vazquez-Wolanski, 1987) If $u_0$ has compact support, then after some time $T$ the interface and the solutions are $C^{1,\alpha}$.
- (Koch, thesis, 1997) If $u_0$ is transversal then FB is $C^\infty$ after $T$. Pressure is "laterally" $C^\infty$. *when it moves, it is always a broken profile .*
- A free boundary with a hole in 2D, 3D is the way of showing that focusing accelerates the viscous fluid so that the speed becomes infinite. This is blow-up for $\mathbf{v} \sim \nabla u^{m-1}$. The setup is a viscous fluid on a table occupying an annulus of radii $r_1$ and $r_2$. As time passes $r_2(t)$ grows and $r_1(t)$ goes to the origin. As $t \to T$, the time the hole disappears.
- There is a solution displaying that behaviour Aronson et al., 1993,... $u(x,t) = (T-t)^\alpha F(x/(T-t)^\beta)$. It is proved that $\beta < 1$.

# Asymptotic behaviour I
## Nonlinear Central Limit Theorem

Choice of domain: $\mathbb{R}^n$. Choice of data: $u_0(x) \in L^1(\mathbb{R}^n)$. We can write

$$u_t = \Delta(|u|^{m-1}u) + f$$

Let us put $f \in L^1_{x,t}$. Let $M = \int u_0(x)\,dx + \iint f\,dxdt$.

Asymptotic Theorem [Kamin and Friedman, 1980; V. 2001] *Let $B(x,t;M)$ be the Barenblatt with the asymptotic mass $M$; $u$ converges to $B$ after renormalization*

$$\boxed{t^{\alpha}|u(x,t) - B(x,t)| \to 0}$$

*For every $p \geq 1$ we have*

$$\boxed{\|u(t) - B(t)\|_p = o(t^{-\alpha/p'}), \quad p' = p/(p-1).}$$

Note: $\alpha$ *and* $\beta = \alpha/n = 1/(2 + n(m-1))$ *are the zooming exponents as in* $B(x,t)$.

- Starting result by FK takes $u_0 \geq 0$, compact support and $f = 0$.

# Asymptotic behaviour I

## Nonlinear Central Limit Theorem

Choice of domain: $\mathbb{R}^n$. Choice of data: $u_0(x) \in L^1(\mathbb{R}^n)$. We can write

$$u_t = \Delta(|u|^{m-1}u) + f$$

Let us put $f \in L^1_{x,t}$. Let $M = \int u_0(x)\,dx + \iint f\,dxdt$.

Asymptotic Theorem [Kamin and Friedman, 1980; V. 2001] *Let $B(x,t;M)$ be the Barenblatt with the asymptotic mass $M$; $u$ converges to $B$ after renormalization*

$$t^{\alpha}|u(x,t) - B(x,t)| \to 0$$

*For every $p \geq 1$ we have*

$$\|u(t) - B(t)\|_p = o(t^{-\alpha/p'}), \quad p' = p/(p-1).$$

Note: $\alpha$ *and* $\beta = \alpha/n = 1/(2 + n(m-1))$ *are the zooming exponents as in* $B(x,t)$.

- Starting result by FK takes $u_0 \geq 0$, compact support and $f = 0$.

# Asymptotic behaviour I

## Nonlinear Central Limit Theorem

Choice of domain: $\mathbb{R}^n$. Choice of data: $u_0(x) \in L^1(\mathbb{R}^n)$. We can write

$$u_t = \Delta(|u|^{m-1}u) + f$$

Let us put $f \in L^1_{x,t}$. Let $M = \int u_0(x)\,dx + \iint f\,dxdt$.

Asymptotic Theorem [Kamin and Friedman, 1980; V. 2001] *Let $B(x,t;M)$ be the Barenblatt with the asymptotic mass $M$; u converges to B after renormalization*

$$\boxed{t^\alpha |u(x,t) - B(x,t)| \to 0}$$

*For every $p \geq 1$ we have*

$$\boxed{\|u(t) - B(t)\|_p = o(t^{-\alpha/p'}), \quad p' = p/(p-1).}$$

Note: $\alpha$ *and* $\beta = \alpha/n = 1/(2 + n(m-1))$ *are the zooming exponents as in* $B(x,t)$.

- Starting result by FK takes $u_0 \geq 0$, compact support and $f = 0$.

# Asymptotic behaviour I

## Nonlinear Central Limit Theorem

Choice of domain: $\mathbb{R}^n$. Choice of data: $u_0(x) \in L^1(\mathbb{R}^n)$. We can write

$$u_t = \Delta(|u|^{m-1}u) + f$$

Let us put $f \in L^1_{x,t}$. Let $M = \int u_0(x)\,dx + \iint f\,dxdt$.

Asymptotic Theorem [Kamin and Friedman, 1980; V. 2001] *Let $B(x,t;M)$ be the Barenblatt with the asymptotic mass $M$; u converges to B after renormalization*

$$\boxed{t^{\alpha}|u(x,t) - B(x,t)| \to 0}$$

*For every $p \geq 1$ we have*

$$\boxed{\|u(t) - B(t)\|_p = o(t^{-\alpha/p'}), \quad p' = p/(p-1).}$$

Note: $\alpha$ *and* $\beta = \alpha/n = 1/(2 + n(m-1))$ *are the zooming exponents as in* $B(x,t)$.

- Starting result by FK takes $u_0 \geq 0$, compact support and $f = 0$.

# Parabolic to Elliptic

- Semigroup solution / mild solution. The typical product of functional discretization schemes: $u = \{u_n\}_n$, $u_n = u(\cdot, t_n)$,

$$u_t = \Delta\Phi(u), \qquad \frac{u_n - u_{n-1}}{h} - \Delta\Phi(u_n) = 0$$

Now put $f := u_{n-1}$, $u := u_n$, and $v = \Phi(u)$, $u = \beta(v)$:

$$-h\Delta\Phi(u) + u = f, \qquad \boxed{-h\Delta v + \beta(v) = f.}$$

"Nonlinear elliptic equations"; Crandall-Liggett Theorems    Ambrosio, Savarè, Nochetto

- Separation of variables. Put $u(x,t) = F(x)G(t)$. Then PME gives

$$F(x)G'(t) = G^m(t)\Delta F^m(x),$$

so that $G'(t) = -G^m(t)$, i.e., $G(t) = (m-1)t^{-1/(m-1)}$ if $m > 1$ and

$$-\Delta F^m(x) = F(x), \qquad -\Delta v(x) = v^p(x),\ p = 1/m.$$

This is more interesting for $m < 1$, specially for $m = (n-2)/(n-2)$.

# Calculations of entropy rates

- We rescale the function as $u(x,t) = r(t)^n \rho(y\, r(t), s)$ where $r(t)$ is the Barenblatt radius at $t+1$, and “new time” is $s = \log(1+t)$. Equation becomes
$$\rho_s = \mathrm{div}\,(\rho(\nabla \rho^{m-1} + \frac{c}{2}\nabla y^2)).$$
- Then define the entropy
$$E(u)(t) = \int (\frac{1}{m}\rho^m + \frac{c}{2}\rho y^2)\, dy$$
*The minimum of entropy is identified as the Barenblatt profile.*
- Calculate
$$\frac{dE}{ds} = -\int \rho |\nabla \rho^{m-1} + cy|^2\, dy = -D$$
Moreover,
$$\frac{dD}{ds} = -R, \quad R \sim \lambda D.$$
*We conclude exponential decay of $D$ and $E$ in new time $s$, which is potential in real time $t$.*

# References

*References.* 1903: Boussinesq, ∼1930: Liebenzon.Muskat, ∼1950: Zeldovich, Barenblatt, 1958: Oleinik,...

Classical work after ∼1970 by Aronson, Benilan, Brezis, Crandall, Caffarelli, Friedman, Kamin, Kenig, Peletier, JLV, ... Recent by Carrillo, Toscani, MacCann, Markowich, Dolbeault, Lee, Daskalopoulos, ...

*Books. About the PME*


- J. L. Vázquez, "The Porous Medium Equation. Mathematical Theory", Oxford Univ. Press, 2007, xxii+624 pages.

*About estimates and scaling*

- J. L. Vázquez, "Smoothing and Decay Estimates for Nonlinear Parabolic Equations of Porous Medium Type", Oxford Univ. Press, 2006, 234 pages.

*About asymptotic behaviour. (Following Lyapunov and Boltzmann)*

- J. L. Vázquez. *Asymptotic behaviour for the Porous Medium Equation posed in the whole space*. Journal of Evolution Equations **3** (2003), 67–118.

*On Nonlinear Diffusion*

- J. L. Vázquez. *Perspectives in Nonlinear Diffusion. Between Analysis, Physics and Geometry*. Proceedings of the International Congress of Mathematicians, **ICM Madrid 2006**. Volume 1, (Marta Sanz-Solé et al. eds.), Eur. Math. Soc. Pub. House, 2007. Pages 609–634.

# References

*References.* 1903: Boussinesq, ∼1930: Liebenzon.Muskat, ∼1950: Zeldovich, Barenblatt, 1958: Oleinik,...

Classical work after ∼1970 by Aronson, Benilan, Brezis, Crandall, Caffarelli, Friedman, Kamin, Kenig, Peletier, JLV, ... Recent by Carrillo, Toscani, MacCann, Markowich, Dolbeault, Lee, Daskalopoulos, ...

*Books. About the PME*


- J. L. Vázquez, ”The Porous Medium Equation. Mathematical Theory”, Oxford Univ. Press, 2007, xxii+624 pages.


*About estimates and scaling*


- J. L. Vázquez, “Smoothing and Decay Estimates for Nonlinear Parabolic Equations of Porous Medium Type”, Oxford Univ. Press, 2006, 234 pages.


*About asymptotic behaviour. (Following Lyapunov and Boltzmann)*


- J. L. Vázquez. *Asymptotic behaviour for the Porous Medium Equation posed in the whole space.* Journal of Evolution Equations **3** (2003), 67–118.


*On Nonlinear Diffusion*


- J. L. Vázquez. *Perspectives in Nonlinear Diffusion. Between Analysis, Physics and Geometry.* Proceedings of the International Congress of Mathematicians, **ICM Madrid 2006**. Volume 1, (Marta Sanz-Solé et al. eds.), Eur. Math. Soc. Pub. House, 2007. Pages 609–634.

# References

*References.* 1903: Boussinesq, ∼1930: Liebenzon.Muskat, ∼1950: Zeldovich, Barenblatt, 1958: Oleinik,...

Classical work after ∼1970 by Aronson, Benilan, Brezis, Crandall, Caffarelli, Friedman, Kamin, Kenig, Peletier, JLV, ... Recent by Carrillo, Toscani, MacCann, Markowich, Dolbeault, Lee, Daskalopoulos, ...

*Books. About the PME*

- J. L. Vázquez, "The Porous Medium Equation. Mathematical Theory", Oxford Univ. Press, 2007, xxii+624 pages.

*About estimates and scaling*

- J. L. Vázquez, "Smoothing and Decay Estimates for Nonlinear Parabolic Equations of Porous Medium Type", Oxford Univ. Press, 2006, 234 pages.

*About asymptotic behaviour. (Following Lyapunov and Boltzmann)*

- J. L. Vázquez. *Asymptotic behaviour for the Porous Medium Equation posed in the whole space.* Journal of Evolution Equations **3** (2003), 67–118.

*On Nonlinear Diffusion*

- J. L. Vázquez. *Perspectives in Nonlinear Diffusion. Between Analysis, Physics and Geometry.* Proceedings of the International Congress of Mathematicians, **ICM Madrid 2006**. Volume 1, (Marta Sanz-Solé et al. eds.), Eur. Math. Soc. Pub. House, 2007. Pages 609–634.

# References

*References.* 1903: Boussinesq, ∼1930: Liebenzon.Muskat, ∼1950: Zeldovich, Barenblatt, 1958: Oleinik,...

Classical work after ∼1970 by Aronson, Benilan, Brezis, Crandall, Caffarelli, Friedman, Kamin, Kenig, Peletier, JLV, ... Recent by Carrillo, Toscani, MacCann, Markowich, Dolbeault, Lee, Daskalopoulos, ...

*Books. About the PME*


- J. L. Vázquez, "The Porous Medium Equation. Mathematical Theory", Oxford Univ. Press, 2007, xxii+624 pages.


*About estimates and scaling*


- J. L. Vázquez, "Smoothing and Decay Estimates for Nonlinear Parabolic Equations of Porous Medium Type", Oxford Univ. Press, 2006, 234 pages.


*About asymptotic behaviour.* *(Following Lyapunov and Boltzmann)*


- J. L. Vázquez. *Asymptotic behaviour for the Porous Medium Equation posed in the whole space.* Journal of Evolution Equations **3** (2003), 67–118.


*On Nonlinear Diffusion*


- J. L. Vázquez. *Perspectives in Nonlinear Diffusion. Between Analysis, Physics and Geometry.* Proceedings of the International Congress of Mathematicians, **ICM Madrid 2006**. Volume 1, (Marta Sanz-Solé et al. eds.), Eur. Math. Soc. Pub. House, 2007. Pages 609–634.

# References

*References.* 1903: Boussinesq, ∼1930: Liebenzon.Muskat, ∼1950: Zeldovich, Barenblatt, 1958: Oleinik,...

Classical work after ∼1970 by Aronson, Benilan, Brezis, Crandall, Caffarelli, Friedman, Kamin, Kenig, Peletier, JLV, ... Recent by Carrillo, Toscani, MacCann, Markowich, Dolbeault, Lee, Daskalopoulos, ...

*Books. About the PME*


- J. L. Vázquez, "The Porous Medium Equation. Mathematical Theory", Oxford Univ. Press, 2007, xxii+624 pages.


*About estimates and scaling*


- J. L. Vázquez, "Smoothing and Decay Estimates for Nonlinear Parabolic Equations of Porous Medium Type", Oxford Univ. Press, 2006, 234 pages.


*About asymptotic behaviour. (Following Lyapunov and Boltzmann)*


- J. L. Vázquez. *Asymptotic behaviour for the Porous Medium Equation posed in the whole space*. Journal of Evolution Equations **3** (2003), 67–118.


*On Nonlinear Diffusion*


- J. L. Vázquez. *Perspectives in Nonlinear Diffusion. Between Analysis, Physics and Geometry*. Proceedings of the International Congress of Mathematicians, **ICM Madrid 2006**. Volume 1, (Marta Sanz-Solé et al. eds.), Eur. Math. Soc. Pub. House, 2007. Pages 609–634.

# References

*References.* 1903: Boussinesq, ∼1930: Liebenzon.Muskat, ∼1950: Zeldovich, Barenblatt, 1958: Oleinik,...

Classical work after ∼1970 by Aronson, Benilan, Brezis, Crandall, Caffarelli, Friedman, Kamin, Kenig, Peletier, JLV, ... Recent by Carrillo, Toscani, MacCann, Markowich, Dolbeault, Lee, Daskalopoulos, ...

*Books. About the PME*


- J. L. Vázquez, "The Porous Medium Equation. Mathematical Theory", Oxford Univ. Press, 2007, xxii+624 pages.


*About estimates and scaling*


- J. L. Vázquez, "Smoothing and Decay Estimates for Nonlinear Parabolic Equations of Porous Medium Type", Oxford Univ. Press, 2006, 234 pages.


*About asymptotic behaviour. (Following Lyapunov and Boltzmann)*


- J. L. Vázquez. *Asymptotic behaviour for the Porous Medium Equation posed in the whole space*. Journal of Evolution Equations **3** (2003), 67–118.


*On Nonlinear Diffusion*


- J. L. Vázquez. *Perspectives in Nonlinear Diffusion. Between Analysis, Physics and Geometry*. Proceedings of the International Congress of Mathematicians, **ICM Madrid 2006**. Volume 1, (Marta Sanz-Solé et al. eds.), Eur. Math. Soc. Pub. House, 2007. Pages 609–634.

# References

*References.* 1903: Boussinesq, ∼1930: Liebenzon.Muskat, ∼1950: Zeldovich, Barenblatt, 1958: Oleinik,...

Classical work after ∼1970 by Aronson, Benilan, Brezis, Crandall, Caffarelli, Friedman, Kamin, Kenig, Peletier, JLV, ... Recent by Carrillo, Toscani, MacCann, Markowich, Dolbeault, Lee, Daskalopoulos, ...

*Books. About the PME*


- J. L. Vázquez, ”The Porous Medium Equation. Mathematical Theory”, Oxford Univ. Press, 2007, xxii+624 pages.


*About estimates and scaling*


- J. L. Vázquez, “Smoothing and Decay Estimates for Nonlinear Parabolic Equations of Porous Medium Type”, Oxford Univ. Press, 2006, 234 pages.


*About asymptotic behaviour. (Following Lyapunov and Boltzmann)*


- J. L. Vázquez. *Asymptotic behaviour for the Porous Medium Equation posed in the whole space*. Journal of Evolution Equations **3** (2003), 67–118.


*On Nonlinear Diffusion*

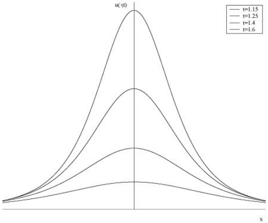

- J. L. Vázquez. *Perspectives in Nonlinear Diffusion. Between Analysis, Physics and Geometry*. Proceedings of the International Congress of Mathematicians, **ICM Madrid 2006**. Volume 1, (Marta Sanz-Solé et al. eds.), Eur. Math. Soc. Pub. House, 2007. Pages 609–634.

# Outline

# FDE Barenblatt profiles

- We have well-known explicit formulas for Self-smilar Barenblatt profiles with exponents less than one if $1 > m > (n-2)/n$:

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \frac{1}{(C + k\xi^2)^{1/(1-m)}}$$




The exponents are $\alpha = \frac{n}{2-n(1-m)}$ and $\beta = \frac{1}{2-n(1-m)} > 1/2$.

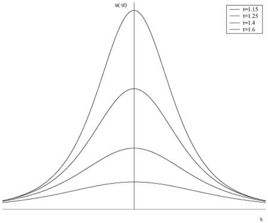
Solutions for $m > 1$ with fat tails (polynomial decay; anomalous distributions)

- Big problem: What happens for $m < (n-2)/n$?
- Main items: existence for very general data, non-existence for very fast diffusion, non-uniqueness for v.f.d., extinction, universal estimates, lack of standard Harnack.

# FDE Barenblatt profiles

- We have well-known explicit formulas for Self-smilar Barenblatt profiles with exponents less than one if $1 > m > (n-2)/n$:

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \frac{1}{(C + k\xi^2)^{1/(1-m)}}$$




The exponents are $\alpha = \frac{n}{2-n(1-m)}$ and $\beta = \frac{1}{2-n(1-m)} > 1/2$.

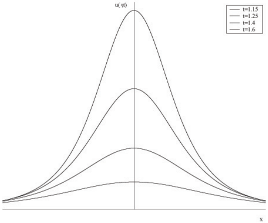
Solutions for $m > 1$ with fat tails (polynomial decay; anomalous distributions)

- Big problem: What happens for $m < (n-2)/n$?
- Main items: existence for very general data, non-existence for very fast diffusion, non-uniqueness for v.f.d., extinction, universal estimates, lack of standard Harnack.

# FDE Barenblatt profiles

- We have well-known explicit formulas for Self-smilar Barenblatt profiles with exponents less than one if $1 > m > (n-2)/n$:

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \frac{1}{(C + k\xi^2)^{1/(1-m)}}$$




The exponents are $\alpha = \frac{n}{2-n(1-m)}$ and $\beta = \frac{1}{2-n(1-m)} > 1/2$.

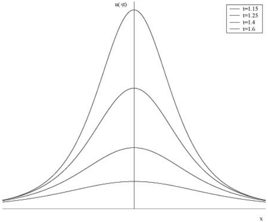
Solutions for $m > 1$ with fat tails (polynomial decay; anomalous distributions)

- Big problem: What happens for $m < (n-2)/n$?
- Main items: existence for very general data, non-existence for very fast diffusion, non-uniqueness for v.f.d., extinction, universal estimates, lack of standard Harnack.

# FDE Barenblatt profiles

- We have well-known explicit formulas for Self-smilar Barenblatt profiles with exponents less than one if $1 > m > (n-2)/n$:

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \frac{1}{(C + k\xi^2)^{1/(1-m)}}$$




The exponents are $\alpha = \frac{n}{2-n(1-m)}$ and $\beta = \frac{1}{2-n(1-m)} > 1/2$.

Solutions for $m > 1$ with fat tails (polynomial decay; anomalous distributions)

- Big problem: What happens for $m < (n-2)/n$?
- Main items: existence for very general data, non-existence for very fast diffusion, non-uniqueness for v.f.d., extinction, universal estimates, lack of standard Harnack.

# Applied Motivation

# Carleman model

Simple case of Diffusive limit of kinetic equations. Two types of particles in a one dimensional setting moving with speeds $c$ and $-c$.

Densities are $u$ and $v$ respectively. Dynamics is

$$(1) \qquad \begin{cases} \partial_t u + c\,\partial_x u = k(u,v)(v-u) \\ \partial_t v - c\,\partial_x v = k(u,v)(u-v), \end{cases}$$

for some interaction kernel $k(u,v) \ge 0$. Typical case $k = (u+v)^\alpha c^2$.

Write now the equations for $\rho = u + v$ and $j = c(u-v)$ and pass to the limit $c = 1/\varepsilon \to \infty$ and you will obtain to first order in powers or $\varepsilon = 1/c$:

$$(2) \qquad \frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{\rho^\alpha}\frac{\partial \rho}{\partial x}\right),$$

which is the FDE with $m = 1 - \alpha$, cf. Lions Toscani, 1997. The typical value $\alpha = 1$ gives $m = 0$, a surprising equation that we will find below! The rigorous investigation of the diffusion limit of more complicated particle/kinetic models is an active area of investigation.

# Carleman model

Simple case of Diffusive limit of kinetic equations. Two types of particles in a one dimensional setting moving with speeds $c$ and $-c$.
Densities are $u$ and $v$ respectively. Dynamics is

$$(1) \qquad \begin{cases} \partial_t u + c\,\partial_x u = k(u,v)(v-u) \\ \partial_t v - c\,\partial_x v = k(u,v)(u-v), \end{cases}$$

for some interaction kernel $k(u,v) \geq 0$. Typical case $k = (u+v)^\alpha c^2$.
Write now the equations for $\rho = u + v$ and $j = c(u-v)$ and pass to the limit $c = 1/\varepsilon \to \infty$ and you will obtain to first order in powers or $\varepsilon = 1/c$:

$$(2) \qquad \frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{\rho^\alpha}\frac{\partial \rho}{\partial x}\right),$$

which is the FDE with $m = 1-\alpha$, cf. Lions Toscani, 1997. The typical value $\alpha = 1$ gives $m = 0$, a surprising equation that we will find below! The rigorous investigation of the diffusion limit of more complicated particle/kinetic models is an active area of investigation.

# Carleman model

Simple case of Diffusive limit of kinetic equations. Two types of particles in a one dimensional setting moving with speeds $c$ and $-c$.
Densities are $u$ and $v$ respectively. Dynamics is

$$(1) \qquad \begin{cases} \partial_t u + c\,\partial_x u = k(u,v)(v-u) \\ \partial_t v - c\,\partial_x v = k(u,v)(u-v), \end{cases}$$

for some interaction kernel $k(u,v) \ge 0$. Typical case $k = (u+v)^\alpha c^2$.
Write now the equations for $\rho = u + v$ and $j = c(u-v)$ and pass to the limit $c = 1/\varepsilon \to \infty$ and you will obtain to first order in powers or $\varepsilon = 1/c$:

$$(2) \qquad \frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{\rho^\alpha}\frac{\partial \rho}{\partial x}\right),$$

which is the FDE with $m = 1 - \alpha$, cf. Lions Toscani, 1997. The typical value $\alpha = 1$ gives $m = 0$, a surprising equation that we will find below! The rigorous investigation of the diffusion limit of more complicated particle/kinetic models is an active area of investigation.

# Yamabe problem. Elliptic

Standard Yamabe problem . We have a Riemannian manifold $(M, g_0)$ in space dimension $n \geq 3$, Question: of finding another metric $g$ in the conformal class of $g_0$ having constant scalar curvature.
Write the conformal relation as

$$g = u^{4/(n-2)} g_0$$

locally on $M$ for some positive smooth function $u$. The conformal factor is $u^{4/(n-2)}$.
Denote by $R = R_g$ and $R_0$ the scalar curvatures of the metrics $g$, $g_0$ resp. Write $\Delta_0$ for the Laplace-Beltrami operator of $g_0$, we have the formula $R = -u^{-N} Lu$ on $M$, with $N = (n+2)/(n-2)$ and

$$Lu := \kappa \Delta_0 u - R_0 u, \quad \kappa = \frac{4(n-1)}{n-2}.$$

The Yamabe problem becomes then

$$(3) \qquad \Delta_0 u - \left(\frac{n-2}{4(n-1)}\right) R_0 u + \left(\frac{n-2}{4(n-1)}\right) R_g u^{(n+2)/(n-2)} = 0.$$

The equation should determine $u$ (hence, $g$) when $g_0$, $R_0$ and $R_g$ are known. In the standard case we take $M = \mathbb{R}^n$ and $g_0$ the standard metric, so that $\Delta_0$ is the standard Laplacian, $R_0 = 0$, we take $R_g = 1$ and then we get the well-known semilinear elliptic equation with critical exponent.

# Yamabe problem. Evolution

Evolution Yamabe flow is defined as an evolution equation for a family of metrics. Used as a tool to construct metrics of constant scalar curvature within a given conformal class. Wwe look for a one-parameter family $g_t(x) = g(x,t)$ of metrics solution of the evolution problem

$$\partial_t g = -R\, g, \quad g(0) = g_0 \quad \text{on } M. \tag{4}$$

It is easy to show that this is equivalent to the equation

$$\partial_t(u^N) = Lu, \quad u(0) = 1 \quad \text{on } \ M.$$

after rescaling the time variable. Let now $(M, g_0)$ be $\mathbb{R}^n$ with the standard flat metric, so that $R_0 = 0$. Put $u^N = v$, $m = 1/N = (n-2)/(n+2) \in (0,1)$. Then

$$\partial_t v = L v^m, \tag{5}$$

which is a fast diffusion equation with exponent $m_y \in (0,1)$ given by

$$m_y = \frac{n-2}{n+2}, \quad 1 - m_y = \frac{4}{n+2}.$$

If we now try separate variables solutions of the form $v(x,t) = (T-t)^\alpha f(x)$, then necessarily $\alpha = 1/(1-m_y) = (n+2)/4$, and $F = f^m$ satisfies the semilinear elliptic equation with critical exponent that models the stationary version:

$$\Delta F + \frac{n+2}{4} F^{\frac{n+2}{n-2}} = 0. \tag{6}$$

# Logarithmic Diffusion I

- Special case: the limit case $m = 0$ of the PME/FDE in two space dimensions

$$\boxed{\partial_t u = \operatorname{div}(u^{-1}\nabla u) = \Delta \log(u).}$$

- Application to Differential Geometry: it describes the evolution of a conformally flat metric $g$ given by $ds^2 = u\,dr^2$ by means of its Ricci curvature:

$$\frac{\partial}{\partial t} g_{ij} = -2\,\mathrm{Ric}_{ij} = -R\,g_{ij},$$

where Ric is the Ricci tensor and $R$ the scalar curvature.
This flow, proposed by R. Hamilton [1] is the equivalent of the Yamabe flow in two dimensions. Remark: what we usually call the mass of the solution (thinking in diffusion terms) becomes here the total area of the surface, $A = \iint u\,dx_1\,dx_2$.

- Work on existence, nonuniqueness, extinction, and asymptotics by several authors around 1995:
Daskalopoulos, Del Pino
Di Bendedetto, Diller
Esteban, Rodriguez, Vazquez

[1] RS Hamilton, The Ricci flow on surfaces. Mathematics and general relativity, 237–262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988.

# Logarithmic Diffusion I

- Special case: the limit case $m = 0$ of the PME/FDE in two space dimensions

$$\boxed{\partial_t u = \mathrm{div}\,(u^{-1}\nabla u) = \Delta \log(u).}$$

- Application to Differential Geometry: it describes the evolution of a conformally flat metric $g$ given by $ds^2 = u\,dr^2$ by means of its Ricci curvature:

$$\frac{\partial}{\partial t} g_{ij} = -2\,\mathrm{Ric}_{ij} = -R\,g_{ij},$$

  where Ric is the Ricci tensor and $R$ the scalar curvature.
  This flow, proposed by R. Hamilton [1] is the equivalent of the Yamabe flow in two dimensions. Remark: what we usually call the mass of the solution (thinking in diffusion terms) becomes here the total area of the surface, $A = \iint u\,dx_1\,dx_2$.

- Work on existence, nonuniqueness, extinction, and asymptotics by several authors around 1995:
  Daskalopoulos, Del Pino
  Di Bendedetto, Diller
  Esteban, Rodriguez, Vazquez

[1] RS Hamilton, The Ricci flow on surfaces. Mathematics and general relativity, 237–262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988.

# Logarithmic Diffusion I

- Special case: the limit case $m = 0$ of the PME/FDE in two space dimensions

$$\boxed{\partial_t u = \text{div}\,(u^{-1}\nabla u) = \Delta \log(u).}$$

- Application to Differential Geometry: it describes the evolution of a conformally flat metric $g$ given by $ds^2 = u\,dr^2$ by means of its Ricci curvature:

$$\frac{\partial}{\partial t} g_{ij} = -2\,\mathrm{Ric}_{ij} = -R\,g_{ij},$$

  where Ric is the Ricci tensor and $R$ the scalar curvature.
  This flow, proposed by R. Hamilton [1] is the equivalent of the Yamabe flow in two dimensions. Remark: what we usually call the mass of the solution (thinking in diffusion terms) becomes here the total area of the surface, $A = \iint u\,dx_1 dx_2$.

- Work on existence, nonuniqueness, extinction, and asymptotics by several authors around 1995:
  Daskalopoulos, Del Pino
  Di Bendedetto, Diller
  Esteban, Rodriguez, Vazquez

[1] RS Hamilton, The Ricci flow on surfaces. Mathematics and general relativity, 237–262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988.

# Logarithmic Diffusion I

- Special case: the limit case $m = 0$ of the PME/FDE in two space dimensions

$$\boxed{\partial_t u = \operatorname{div}(u^{-1}\nabla u) = \Delta \log(u).}$$

- Application to Differential Geometry: it describes the evolution of a conformally flat metric $g$ given by $ds^2 = u\,dr^2$ by means of its Ricci curvature:

$$\frac{\partial}{\partial t} g_{ij} = -2\,\mathrm{Ric}_{ij} = -R\,g_{ij},$$

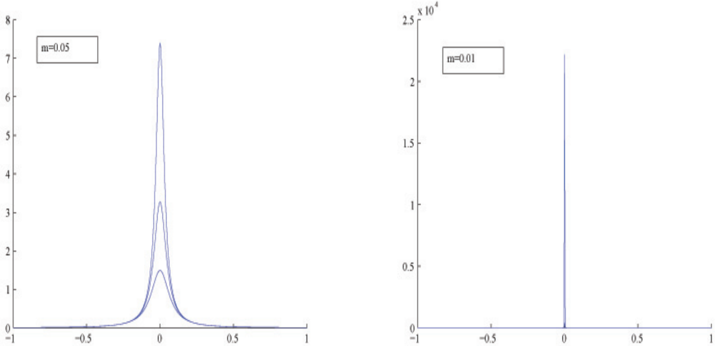
  where Ric is the Ricci tensor and $R$ the scalar curvature.
  This flow, proposed by R. Hamilton [1] is the equivalent of the Yamabe flow in two dimensions. Remark: what we usually call the mass of the solution (thinking in diffusion terms) becomes here the total area of the surface, $A = \iint u\,dx_1 dx_2$.

- Work on existence, nonuniqueness, extinction, and asymptotics by several authors around 1995:
  Daskalopoulos, Del Pino
  Di Bendedetto, Diller
  Esteban, Rodriguez, Vazquez

[1] RS Hamilton, The Ricci flow on surfaces. Mathematics and general relativity, 237–262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988.

# Pictures

About fast diffusion in the limit




- Evolution of the ZKB solutions; dimension $n = 2$.
  exponent near $m = 0$

# Functional Analysis Program

# Main facts

- Existence of an evolution semigroup.

$$u_0 \mapsto S_t(u_0) = u(t)$$

  A key issue is the choice of functional space.
  $X = L^1(\mathbb{R}^n)$ (Brezis, Benilan, Crandall, 1971)
  $Y = L^1_{loc}(\mathbb{R}^n)$ (Herrero, Pierre 1985)
  $M =$ Locally bounded measures (Pierre, 1987; Dalhberg - Kenig 1988)
  $B =$ (possibly locally unbounded) Borel mesaures (Chasseigne-Vazquez ARMA 2002)

- Positivity. Nonnegative data produce positive solutions.

- "Smoothing effect": In *many cases* $L^p \to L^q$ with $q > p$. Then solutions are $C^\infty$ smooth. In other cases, things go wrong (things=Functional Analysis)

- Theory for two signs is still poorly understood.
  Cf. Stefan Problem (Athanasopoulos, Caffarelli, Salsa)

# The good and bad range

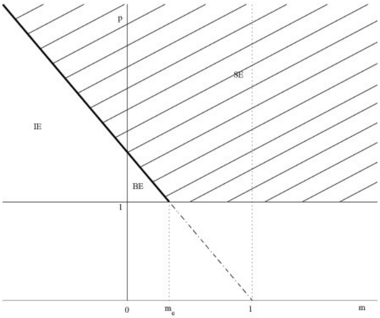



Figure 1. The $(m, p)$ diagram for the PME/FDE in dimensions $n \geq 3$.
SE: smoothing effect, BE: backwards effect, IE: instantaneous extinction
Critical line $p = n(1-m)/2$ (in boldface)

More exponents appear. One is $m = 0$. A third exponent $m = (n-2)/(n+2)$ (in dimensions $n \geq 3$), which is the inverse of the famous Sobolev exponent of the elliptic theory. The relation is clear by separation of variables. Exercise

# The good and bad range

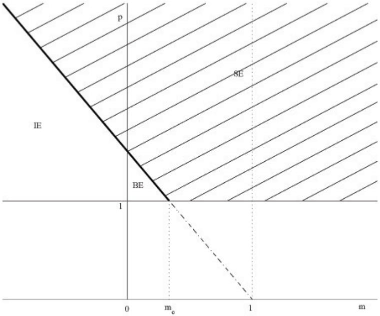



Figure 1. The $(m, p)$ diagram for the PME/FDE in dimensions $n \geq 3$.
SE: smoothing effect, BE: backwards effect, IE: instantaneous extinction
Critical line $p = n(1 - m)/2$ (in boldface)

More exponents appear. One is $m = 0$. A third exponent $m = (n-2)/(n+2)$ (in dimensions $n \geq 3$), which is the inverse of the famous Sobolev exponent of the elliptic theory. The relation is clear by separation of variables. Exercise

# The good and bad range

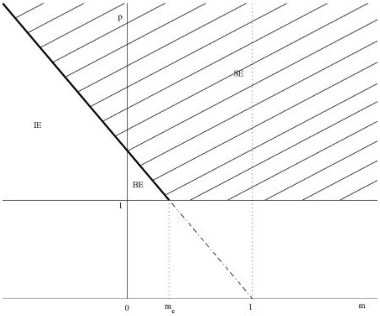



Figure 1. The $(m, p)$ diagram for the PME/FDE in dimensions $n \geq 3$.
SE: smoothing effect, BE: backwards effect, IE: instantaneous extinction
Critical line $p = n(1 - m)/2$ (in boldface)

More exponents appear. One is $m = 0$. A third exponent $m = (n-2)/(n+2)$ (in dimensions $n \geq 3$), which is the inverse of the famous Sobolev exponent of the elliptic theory. The relation is clear by separation of variables. Exercise

# The good and bad range II

- Smoothing effect means that data in $L^p$ imply that the weak solution is in $L^\infty$ for all $t > 0$. Over the "green" line the result is true even locally. Smoothing book, 2006, for $u_0 \in L^p$, Bonforte-Vazquez, preprint for $u_0 \in L^p_{loc}$. Here, $p = p_* = n(1-m)/2$.
- Backwards effect is a strange effect. Data in $L^p$ imply $u(t) \in L^1$ for all $t > 0$ Smoothing book, 2006.
- On the green line exactly there is extinction in finite time (not at the end value $p = 1$). The correct extinction space is a Marcinkiewicz space: $X = M^{p_*}(\mathbb{R}^n)$ Smoothing book, 2006.
- For $m < 0$ below the green line there are no solutions. Vazquez 92, Daskalopoulos del Pino 97.
- For $1 > m > m_c$ we can solve the initial value problem with any nonnegative measure, even a Borel Measure because there are very good local estimates Chasseigne-Vazquez 03. For $m < m_c$ Dirac masses cannot be initial data Brezis-Friedman 83, Pierre 87.
- Harnack holds for $m > m_c$ [2], but is very difficult for $m < m_c$ work just finished [3].

[2] M Bonforte- JL Vazquez, *Global positivity estimates and Harnack inequalities for the fast diffusion equation.* J. Funct. Anal. 240 (2006), no. 2, 399–428

[3] M Bonforte- JL Vazquez, *Positivity, local smoothing, and Harnack inequalities for very fast diffusion equations,* Preprint March 2008

# The good and bad range II

- Smoothing effect means that data in $L^p$ imply that the weak solution is in $L^\infty$ for all $t > 0$. Over the "green" line the result is true even locally. Smoothing book, 2006, for $u_0 \in L^p$, Bonforte-Vazquez, preprint for $u_0 \in L^p_{loc}$. Here, $p = p_* = n(1-m)/2$.
- Backwards effect is a strange effect. Data in $L^p$ imply $u(t) \in L^1$ for all $t > 0$ Smoothing book, 2006.
- On the green line exactly there is extinction in finite time (not at the end value $p = 1$). The correct extinction space is a Marcinkiewicz space: $X = M^{p_*}(\mathbb{R}^n)$ Smoothing book, 2006.
- For $m < 0$ below the green line there are no solutions. Vazquez 92, Daskalopoulos del Pino 97.
- For $1 > m > m_c$ we can solve the initial value problem with any nonnegative measure, even a Borel Measure because there are very good local estimates Chasseigne-Vazquez 03. For $m < m_c$ Dirac masses cannot be initial data Brezis-Friedman 83, Pierre 87.
- Harnack holds for $m > m_c$ [2], but is very difficult for $m < m_c$ work just finished [3].

---

[2] M Bonforte- JL Vazquez, *Global positivity estimates and Harnack inequalities for the fast diffusion equation.* J. Funct. Anal. 240 (2006), no. 2, 399–428

[3] M Bonforte- JL Vazquez, *Positivity, local smoothing, and Harnack inequalities for very fast diffusion equations,* Preprint March 2008

# The good and bad range II

- Smoothing effect means that data in $L^p$ imply that the weak solution is in $L^\infty$ for all $t > 0$. Over the "green" line the result is true even locally. Smoothing book, 2006, for $u_0 \in L^p$, Bonforte-Vazquez, preprint for $u_0 \in L^p_{loc}$. Here, $p = p_* = n(1-m)/2$.
- Backwards effect is a strange effect. Data in $L^p$ imply $u(t) \in L^1$ for all $t > 0$ Smoothing book, 2006.
- On the green line exactly there is extinction in finite time (not at the end value $p = 1$). The correct extinction space is a Marcinkiewicz space: $X = M^{p_*}(\mathbb{R}^n)$ Smoothing book, 2006.
- For $m < 0$ below the green line there are no solutions. Vazquez 92, Daskalopoulos del Pino 97.
- For $1 > m > m_c$ we can solve the initial value problem with any nonnegative measure, even a Borel Measure because there are very good local estimates Chasseigne-Vazquez 03. For $m < m_c$ Dirac masses cannot be initial data Brezis-Friedman 83, Pierre 87.
- Harnack holds for $m > m_c$ [2], but is very difficult for $m < m_c$ work just finished [3].

[2] M Bonforte- JL Vazquez, *Global positivity estimates and Harnack inequalities for the fast diffusion equation.* J. Funct. Anal. 240 (2006), no. 2, 399–428

[3] M Bonforte- JL Vazquez, *Positivity, local smoothing, and Harnack inequalities for very fast diffusion equations,* Preprint March 2008

# The good and bad range II

- Smoothing effect means that data in $L^p$ imply that the weak solution is in $L^\infty$ for all $t > 0$. Over the "green" line the result is true even locally. Smoothing book, 2006, for $u_0 \in L^p$, Bonforte-Vazquez, preprint for $u_0 \in L^p_{loc}$. Here, $p = p_* = n(1-m)/2$.
- Backwards effect is a strange effect. Data in $L^p$ imply $u(t) \in L^1$ for all $t > 0$ Smoothing book, 2006.
- On the green line exactly there is extinction in finite time (not at the end value $p = 1$). The correct extinction space is a Marcinkiewicz space: $X = M^{p_*}(\mathbb{R}^n)$ Smoothing book, 2006.
- For $m < 0$ below the green line there are no solutions. Vazquez 92, Daskalopoulos del Pino 97.
- For $1 > m > m_c$ we can solve the initial value problem with any nonnegative measure, even a Borel Measure because there are very good local estimates Chasseigne-Vazquez 03. For $m < m_c$ Dirac masses cannot be initial data Brezis-Friedman 83, Pierre 87.
- Harnack holds for $m > m_c$ [2], but is very difficult for $m < m_c$ work just finished [3].

---

[2] M Bonforte- JL Vazquez, *Global positivity estimates and Harnack inequalities for the fast diffusion equation.* J. Funct. Anal. 240 (2006), no. 2, 399–428

[3] M Bonforte- JL Vazquez, *Positivity, local smoothing, and Harnack inequalities for very fast diffusion equations,* Preprint March 2008

# The good and bad range II

- Smoothing effect means that data in $L^p$ imply that the weak solution is in $L^\infty$ for all $t > 0$. Over the "green" line the result is true even locally. Smoothing book, 2006, for $u_0 \in L^p$, Bonforte-Vazquez, preprint for $u_0 \in L^p_{loc}$. Here, $p = p_* = n(1-m)/2$.
- Backwards effect is a strange effect. Data in $L^p$ imply $u(t) \in L^1$ for all $t > 0$ Smoothing book, 2006.
- On the green line exactly there is extinction in finite time (not at the end value $p = 1$). The correct extinction space is a Marcinkiewicz space: $X = M^{p_*}(\mathbb{R}^n)$ Smoothing book, 2006.
- For $m < 0$ below the green line there are no solutions. Vazquez 92, Daskalopoulos del Pino 97.
- For $1 > m > m_c$ we can solve the initial value problem with any nonnegative measure, even a Borel Measure because there are very good local estimates Chasseigne-Vazquez 03. For $m < m_c$ Dirac masses cannot be initial data Brezis-Friedman 83, Pierre 87.
- Harnack holds for $m > m_c$ [2], but is very difficult for $m < m_c$ work just finished [3].

[2] M Bonforte- JL Vazquez, *Global positivity estimates and Harnack inequalities for the fast diffusion equation.* J. Funct. Anal. 240 (2006), no. 2, 399–428

[3] M Bonforte- JL Vazquez, *Positivity, local smoothing, and Harnack inequalities for very fast diffusion equations,* Preprint March 2008

# The good and bad range II

- Smoothing effect means that data in $L^p$ imply that the weak solution is in $L^\infty$ for all $t > 0$. Over the "green" line the result is true even locally. Smoothing book, 2006, for $u_0 \in L^p$, Bonforte-Vazquez, preprint for $u_0 \in L^p_{loc}$. Here, $p = p_* = n(1-m)/2$.
- Backwards effect is a strange effect. Data in $L^p$ imply $u(t) \in L^1$ for all $t > 0$ Smoothing book, 2006.
- On the green line exactly there is extinction in finite time (not at the end value $p = 1$). The correct extinction space is a Marcinkiewicz space: $X = M^{p_*}(\mathbb{R}^n)$ Smoothing book, 2006.
- For $m < 0$ below the green line there are no solutions. Vazquez 92, Daskalopoulos del Pino 97.
- For $1 > m > m_c$ we can solve the initial value problem with any nonnegative measure, even a Borel Measure because there are very good local estimates Chasseigne-Vazquez 03. For $m < m_c$ Dirac masses cannot be initial data Brezis-Friedman 83, Pierre 87.
- Harnack holds for $m > m_c$ [2], but is very difficult for $m < m_c$ work just finished [3].

[2] M Bonforte- JL Vazquez, *Global positivity estimates and Harnack inequalities for the fast diffusion equation.* J. Funct. Anal. 240 (2006), no. 2, 399–428

[3] M Bonforte- JL Vazquez, *Positivity, local smoothing, and Harnack inequalities for very fast diffusion equations,* Preprint March 2008

# The good and bad range III

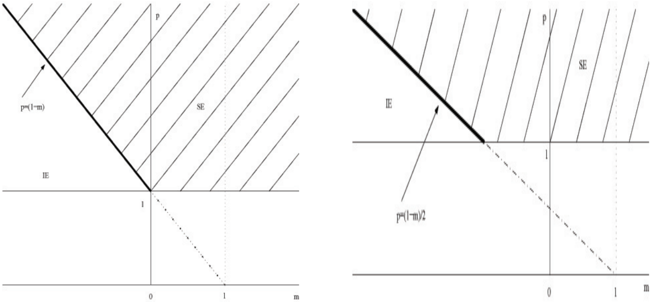



Figure 2. Left: $(m, p)$ diagram for the PME/FDE in dimension $n = 2$
Right: $(m, p)$ diagram for the PME/FDE in dimension $n = 1$

- There is existence and non-uniqueness if $n = 1$ and $-1 < m < 0$

# The question of intrinsic regularity

# The question of intrinsic regularity

# Universal Pointwise Estimates for Good Fast Diffusion

- CASE $m_c < m < 1$ This range has wonderful a priori estimates of local type. We assume that $u \ge 0$.
- If $u_0 \in L^1_{loc}(\mathbb{R}^n)$ then for all $t > 0$ we have $u(\cdot, t) \in L^\infty(\mathbb{R}^n)$, cf. Herrero-Pierre, 1985.
- There is a universal constant $C > 0$ such that if $v = u^{m-1}$

$$t|\Delta v| \le C, \quad t|\frac{v_t}{v}| \le C, \quad t\frac{|\nabla v|^2}{v} \le C. \tag{7}$$

Estimates for the PME were original of Aronson, Crandall and Benilan. [4] [5]

Note that $v$ satisfies the quadratic equation

$$v_t = v\Delta v - \gamma|\nabla v|^2, \qquad \gamma = 1/(1-m).$$

[4] DG Aronson, Ph. Bénilan, Regularité des solutions de l'équation des milieux poreux dans $R^N$. C. R. Acad. Sci. Paris Sér. A-B 288 (1979), no. 2, A103–A105.

[5] Ph. Bénilan, MG Crandall. Regularizing effects of homogeneous evolution equations. Contributions to analysis and geometry (Baltimore, Md., 1980), pp. 23–39, Johns Hopkins Univ. Press, Baltimore, Md., 1981.

# Universal Pointwise Estimates for Good Fast Diffusion

- CASE $m_c < m < 1$ This range has wonderful a priori estimates of local type. We assume that $u \geq 0$.
- If $u_0 \in L^1_{loc}(\mathbb{R}^n)$ then for all $t > 0$ we have $u(\cdot, t) \in L^\infty(\mathbb{R}^n)$, cf. Herrero-Pierre, 1985.
- There is a universal constant $C > 0$ such that if $v = u^{m-1}$

$$t|\Delta v| \leq C, \quad t|\frac{v_t}{v}| \leq C, \quad t\frac{|\nabla v|^2}{v} \leq C. \tag{7}$$

Estimates for the PME were original of Aronson, Crandall and Benilan. [4] [5]

Note that $v$ satisfies the quadratic equation

$$v_t = v\Delta v - \gamma|\nabla v|^2, \qquad \gamma = 1/(1-m).$$

[4] DG Aronson, Ph. Bénilan, Regularité des solutions de l'équation des milieux poreux dans $R^N$. C. R. Acad. Sci. Paris Sér. A-B 288 (1979), no. 2, A103–A105.

[5] Ph. Bénilan, MG Crandall. Regularizing effects of homogeneous evolution equations. Contributions to analysis and geometry (Baltimore, Md., 1980), pp. 23–39, Johns Hopkins Univ. Press, Baltimore, Md., 1981.

# Universal Estimates continued

- Universal estimates have been found in other problems.
- Some can can be found for the heat equation. They also work for the $p$-Laplacian equation (fast or slow) in similar exponent ranges[6]
- Similar estimates were discovered by Yau and Li [7] for flows on manifolds and they prove that they produce continuity.
- Hamilton for the Ricci flow. [8]
- For $m \leq m_c$ the first estimate from below fails and the second also from below and the third from above.

---

[6] JR Esteban, JL Vázquez. Régularité des solutions positives de l'équation parabolique $p$-laplacienne. [Regularity of nonnegative solutions of the $p$-Laplacian parabolic equation] C. R. Acad. Sci. Paris Sér. I Math. 310 (1990), no. 3, 105–110.

[7] Li, Peter; Yau, Shing-Tung, On the parabolic kernel of the Schrödinger operator. Acta Math. 156 (1986), no. 3-4, 153–201.

[8] RS Hamilton, The Harnack estimate for the Ricci flow. J. Differential Geom. 37 (1993), no. 1, 225–243.

# Universal Estimates continued

- Universal estimates have been found in other problems.
- Some can can be found for the heat equation. They also work for the *p*-Laplacian equation (fast or slow) in similar exponent ranges[6]
- Similar estimates were discovered by Yau and Li [7] for flows on manifolds and they prove that they produce continuity.
- Hamilton for the Ricci flow. [8]
- For $m \le m_c$ the first estimate from below fails and the second also from below and the third from above.

[6] JR Esteban, JL Vázquez. Régularité des solutions positives de l'équation parabolique *p*-laplacienne. [Regularity of nonnegative solutions of the *p*-Laplacian parabolic equation] C. R. Acad. Sci. Paris Sér. I Math. 310 (1990), no. 3, 105–110.

[7] Li, Peter; Yau, Shing-Tung, On the parabolic kernel of the Schrödinger operator. Acta Math. 156 (1986), no. 3-4, 153–201.

[8] RS Hamilton, The Harnack estimate for the Ricci flow. J. Differential Geom. 37 (1993), no. 1, 225–243.

# Universal Estimates continued

- Universal estimates have been found in other problems.
- Some can can be found for the heat equation. They also work for the $p$-Laplacian equation (fast or slow) in similar exponent ranges[6]
- Similar estimates were discovered by Yau and Li [7] for flows on manifolds and they prove that they produce continuity.
- Hamilton for the Ricci flow. [8]
- For $m \le m_c$ the first estimate from below fails and the second also from below and the third from above.

---

[6] JR Esteban, JL Vázquez. Régularité des solutions positives de l'équation parabolique $p$-laplacienne. [Regularity of nonnegative solutions of the $p$-Laplacian parabolic equation] C. R. Acad. Sci. Paris Sér. I Math. 310 (1990), no. 3, 105–110.

[7] Li, Peter; Yau, Shing-Tung, On the parabolic kernel of the Schrödinger operator. Acta Math. 156 (1986), no. 3-4, 153–201.

[8] RS Hamilton, The Harnack estimate for the Ricci flow. J. Differential Geom. 37 (1993), no. 1, 225–243.

# The Aronson Caffarelli Estimate for PME

- CASE $m > 1$ Aronson-Caffarelli's result [9] is a positivity estimate for the PME, $m > 1$, valid for all nonnegative weak solutions defined in the whole space.

  We take a point $x_0$ and a ball $B_R(x_0)$ and try to see how positive is the solution at time $t0$ if there is a "mass" $M_R(x_0) = \int_{B_R(x_0)} u_0(x)\,dx$ at $t = 0$. It says

$$\frac{M_R(x_0)}{R^d} \le C_1\, R^{2/(m-1)}\, t^{-\frac{1}{m-1}} + C_2\, R^{-d} t^{d/2} u^{\lambda/2}(t, x_0). \tag{8}$$

  with $\lambda = 2 + d(m-1)$. $C_1$ and $C_2$ given positive constants depending only on $m$ and $d$. Looking at the three terms we discover that there is a time $t_*$ where the second is already less than the first one. We can calculate this intrinsic positivity time as $t_* = C(m, d)R^\lambda / M^{m-1}$.

- For $t > t_*$ the third one is positive, hence $u(x_0, t) > 0$. Hence, for all large $t$ we have $u = O(t^{d/\lambda})$. OK!

- We go on to prove that $u \in C^\alpha$ for some $\alpha > 0$. There is no way you can get positivity for small times because of finite propagation (free boundaries).

[9] Aronson, D. G.; Caffarelli, L. A. The initial trace of a solution of the porous medium equation. Trans. Amer. Math. Soc. 280 (1983), no. 1, 351–366.

# The Aronson Caffarelli Estimate for PME

- CASE $m > 1$ Aronson-Caffarelli's result [9] is a positivity estimate for the PME, $m > 1$, valid for all nonnegative weak solutions defined in the whole space.

  We take a point $x_0$ and a ball $B_R(x_0)$ and try to see how positive is the solution at time $t0$ if there is a "mass" $M_R(x_0) = \int_{B_R(x_0)} u_0(x)\,dx$ at $t = 0$. It says

$$\frac{M_R(x_0)}{R^d} \le C_1\, R^{2/(m-1)}\, t^{-\frac{1}{m-1}} + C_2\, R^{-d} t^{d/2} u^{\lambda/2}(t, x_0). \tag{8}$$

  with $\lambda = 2 + d(m-1)$. $C_1$ and $C_2$ given positive constants depending only on $m$ and $d$. Looking at the three terms we discover that there is a time $t_*$ where the second is already less than the first one. We can calculate this intrinsic positivity time as $t_* = C(m, d)R^\lambda / M^{m-1}$.

- For $t > t_*$ the third one is positive, hence $u(x_0, t) > 0$. Hence, for all large $t$ we have $u = O(t^{d/\lambda})$. OK!

- We go on to prove that $u \in C^\alpha$ for some $\alpha > 0$. There is no way you can get positivity for small times because of finite propagation (free boundaries).

[9] Aronson, D. G.; Caffarelli, L. A. The initial trace of a solution of the porous medium equation. Trans. Amer. Math. Soc. 280 (1983), no. 1, 351–366.

# The Aronson Caffarelli Estimate for PME

- CASE $m > 1$ Aronson-Caffarelli's result [9] is a positivity estimate for the PME, $m > 1$, valid for all nonnegative weak solutions defined in the whole space.

  We take a point $x_0$ and a ball $B_R(x_0)$ and try to see how positive is the solution at time $t0$ if there is a "mass" $M_R(x_0) = \int_{B_R(x_0)} u_0(x)\,dx$ at $t = 0$. It says

$$\frac{M_R(x_0)}{R^d} \le C_1\, R^{2/(m-1)}\, t^{-\frac{1}{m-1}} + C_2\, R^{-d} t^{d/2} u^{\lambda/2}(t, x_0). \tag{8}$$

  with $\lambda = 2 + d(m-1)$. $C_1$ and $C_2$ given positive constants depending only on $m$ and $d$. Looking at the three terms we discover that there is a time $t_*$ where the second is already less than the first one. We can calculate this intrinsic positivity time as $t_* = C(m,d)R^\lambda/M^{m-1}$.

- For $t > t_*$ the third one is positive, hence $u(x_0, t) > 0$. Hence, for all large $t$ we have $u = O(t^{d/\lambda})$. OK!

- We go on to prove that $u \in C^\alpha$ for some $\alpha > 0$. There is no way you can get positivity for small times because of finite propagation (free boundaries).

[9] Aronson, D. G.; Caffarelli, L. A. The initial trace of a solution of the porous medium equation. Trans. Amer. Math. Soc. 280 (1983), no. 1, 351–366.

# AC Type Estimate for Good Fast Diffusion

- In the paper Bonforte- Vazquez, Global positivity estimates and Harnack inequalities for the fast diffusion equation. J. Funct. Anal., 2006

  we take the approach to regularity through positivity inspired by the work of Di Benedetto and collaborators for PME, FDE and PLE using intrinsic versions of Harnack.
- We study local solutions of the FDE in the good exponent range $m_c < m < 1$. The change in the sign of the exponent $m-1$ implies that we get good lower estimates for $0 < t \le t_*$ if the ideas of AC can be made to work. Moreover, we can continue these estimates for $t \ge t_*$ thanks to the fortunate circumstance that we have further differential inequalities, like $\partial_t u \ge -Cu/t$ in the case of the Cauchy problem. We get a continuation of the lower bounds with optimal decay rates in time. The final form is

$$u(t,x) \ge M_R(x_0)\, H(t/t_c), \quad M_R(x_0) = R^{-d} \int_{B_R(x_0)} u_0\, dx. \tag{9}$$

- The critical time is defined as before; the function $H(\eta)$ is defined as $K\eta^{1/(1-m)}$ for $\eta \le 1$ while $H(\eta) = K\eta^{-d\vartheta}$ for $\eta \ge 1$, with $K = K(m,d)$. Note that for $0 < t < t_c$ the lower bound means

$$u(t,x_0) \ge k(m,d)(t/R^2)^{1/(1-m)}$$

  which is independent of the initial mass.

# AC Type Estimate for Good Fast Diffusion

- In the paper Bonforte- Vazquez, Global positivity estimates and Harnack inequalities for the fast diffusion equation. J. Funct. Anal., 2006

  we take the approach to regularity through positivity inspired by the work of Di Benedetto and collaborators for PME, FDE and PLE using intrinsic versions of Harnack.
- We study local solutions of the FDE in the good exponent range $m_c < m < 1$. The change in the sign of the exponent $m-1$ implies that we get good lower estimates for $0 < t \le t_*$ if the ideas of AC can be made to work. Moreover, we can continue these estimates for $t \ge t_*$ thanks to the fortunate circumstance that we have further differential inequalities, like $\partial_t u \ge -Cu/t$ in the case of the Cauchy problem. We get a continuation of the lower bounds with optimal decay rates in time. The final form is

$$u(t,x) \ge M_R(x_0)\, H(t/t_c), \quad M_R(x_0) = R^{-d} \int_{B_R(x_0)} u_0\, dx. \tag{9}$$

- The critical time is defined as before; the function $H(\eta)$ is defined as $K\eta^{1/(1-m)}$ for $\eta \le 1$ while $H(\eta) = K\eta^{-d\vartheta}$ for $\eta \ge 1$, with $K = K(m,d)$. Note that for $0 < t < t_c$ the lower bound means

$$u(t,x_0) \ge k(m,d)(t/R^2)^{1/(1-m)}$$

  which is independent of the initial mass.

# The AC Estimate for Bad Fast Diffusion

We know that for $m, m_c$ all kinds of functional disasters may happen. In particular, extinction in finite holds for all integrable data (and some more) so that positivity for long times must be excluded. Let $u$ be a local solution with extinction time $> 0$. We prove this result in M Bonforte- JL Vazquez, *Positivity, local smoothing, and Harnack inequalities for very fast diffusion equations,* Preprint.

**Theorem**

*Let $0 < m < 1$ and let $u$ be the solution to the FDE under the above assumptions. Let $x_0$ be a point in $\Omega$ and let $d(x_0, \partial\Omega) \ge 5R$. Then the following inequality holds for all $0 < t < T$*

$$(10) \qquad R^{-d} \int_{B_R(x_0)} u_0(x)\,dx \le C_1\, R^{-2/(1-m)}\, t^{\frac{1}{1-m}} + C_2\, T^{\frac{1}{1-m}} R^{-2}\, t^{-\frac{m}{1-m}}\, u^m(t, x_0).$$

*with $C_1$ and $C_2$ given positive constants depending only on $d$. This implies that there exists a time $t_*$ such that for all $t \in (0, t_*]$*

$$(11) \qquad u^m(t, x_0) \ge C_1'\, R^{2-d} \|u_0(x)\|_{L^1(B_R)} T^{-\frac{1}{1-m}}\, t^{\frac{m}{1-m}}.$$

*where $C_1' > 0$ depends only on $d$; $t_*$ depends on $R$ and $\|u_0(x)\|_{L^1(B_R)}$ but not on $T$.*

# The local boundedness result for Fast Diffusion

- The main result of this part is the local upper bound that applies for the same type of solution and initial data, under different restrictions on $p$. Here is the precise formulation.

  We take $d \geq 3$. recall that $m_c = (d-2)/d$, that $p_c = d(1-m)/2$.

**Theorem**

*Let $p \geq 1$ if $m > m_c$ or $p > p_c$ if $m \leq m_c$. Then there are positive constants $C_1, \mathcal{C}_2$ such that for any $0 < R_1 < R_0$ we have*

$$\sup_{x \in B_{R_1}} u(t,x) \leq \frac{C_1}{t^{d\vartheta_p}} \left[ \int_{B_{R_0}} |u_0(x)|^p \, dx \right]^{2\vartheta_p} + C_2 \left[ \frac{t}{R_0^2} \right]^{\frac{1}{1-m}} . \tag{12}$$

- We recall that $\vartheta_p = 1/(2p - d(1-m)) = 1/2(p - p_c)$. The constants $C_i$ depend on $m$, $d$ and $p$, $R_1$ and $R_0$ and blow up when $R_1/R_0 \to 1$; an explicit formula for $C_i$ can be found.

# The local boundedness result for Fast Diffusion

- The main result of this part is the local upper bound that applies for the same type of solution and initial data, under different restrictions on $p$. Here is the precise formulation.

  We take $d \geq 3$. recall that $m_c = (d-2)/d$, that $p_c = d(1-m)/2$.

**Theorem**

*Let $p \geq 1$ if $m > m_c$ or $p > p_c$ if $m \leq m_c$. Then there are positive constants $C_1, \mathcal{C}_2$ such that for any $0 < R_1 < R_0$ we have*

$$\sup_{x \in B_{R_1}} u(t,x) \leq \frac{C_1}{t^{d\vartheta_p}} \left[ \int_{B_{R_0}} |u_0(x)|^p \, dx \right]^{2\vartheta_p} + C_2 \left[ \frac{t}{R_0^2} \right]^{\frac{1}{1-m}} . \tag{12}$$

- We recall that $\vartheta_p = 1/(2p - d(1-m)) = 1/2(p-p_c)$. The constants $C_i$ depend on $m, d$ and $p$, $R_1$ and $R_0$ and blow up when $R_1/R_0 \to 1$; an explicit formula for $C_i$ can be found.

# Local Boundedness II

- The proof consists in two steps: (1) The norm $\|u(\cdot,t)\|_{L^p_{loc}}$ grows with time in a controlled way in terms of its value at $t=0$, if $p\geq 1$, $p>1-m$. This uses Herrero-Pierre's approach.

  (2) Solutions in $L^p_{x,t}$ locally in space/time are in fact bounded in a smaller cylinder if $p>p_c$. This uses Moser iteration.

- Local Boundedness implies existence of Large Solucions having boundary data $u=+\infty$. Such solutions form the Maximal Semigroup. A reference is E Chasseigne, JL Vazquez, Theory of extended solutions for fast-diffusion equations in optimal classes of data. Radiation from singularities. Arch. Ration. Mech. Anal. 164 (2002), no. 2, 133–187.

- Reference for more general PME and Fast Diffusion $u_t=\Delta\Phi(u)$ in good fast diffusion: P. Daskalopoulos, C. Kenig. Degenerate diffusions. Initial value problems and local regularity theory. EMS Tracts in Mathematics, 1. European Mathematical Society (EMS), Zürich, 2007.

# Local Boundedness II

- The proof consists in two steps: (1) The norm $\|u(\cdot,t)\|_{L^p_{loc}}$ grows with time in a controlled way in terms of its value at $t = 0$, if $p \geq 1$, $p > 1 - m$. This uses Herrero-Pierre's approach.

  (2) Solutions in $L^p_{x,t}$ locally in space/time are in fact bounded in a smaller cylinder if $p > p_c$. This uses Moser iteration.

- Local Boundedness implies existence of Large Solucions having boundary data $u = +\infty$. Such solutions form the Maximal Semigroup. A reference is E Chasseigne, JL Vazquez, Theory of extended solutions for fast-diffusion equations in optimal classes of data. Radiation from singularities. Arch. Ration. Mech. Anal. 164 (2002), no. 2, 133–187.

- Reference for more general PME and Fast Diffusion $u_t = \Delta\Phi(u)$ in good fast diffusion: P. Daskalopoulos, C. Kenig. Degenerate diffusions. Initial value problems and local regularity theory. EMS Tracts in Mathematics, 1. European Mathematical Society (EMS), Zürich, 2007.

# Local Boundedness II

- The proof consists in two steps: (1) The norm $\|u(\cdot,t)\|_{L^p_{loc}}$ grows with time in a controlled way in terms of its value at $t=0$, if $p \geq 1$, $p > 1-m$. This uses Herrero-Pierre's approach.

  (2) Solutions in $L^p_{x,t}$ locally in space/time are in fact bounded in a smaller cylinder if $p > p_c$. This uses Moser iteration.

- Local Boundedness implies existence of Large Solucions having boundary data $u = +\infty$. Such solutions form the Maximal Semigroup. A reference is E Chasseigne, JL Vazquez, Theory of extended solutions for fast-diffusion equations in optimal classes of data. Radiation from singularities. Arch. Ration. Mech. Anal. 164 (2002), no. 2, 133–187.

- Reference for more general PME and Fast Diffusion $u_t = \Delta\Phi(u)$ in good fast diffusion: P. Daskalopoulos, C. Kenig. Degenerate diffusions. Initial value problems and local regularity theory. EMS Tracts in Mathematics, 1. European Mathematical Society (EMS), Zürich, 2007.

# Flows on manifolds

Paper: "Local Aronson-Bénilan estimates and entropy formulae for porous medium and fast diffusion equations on manifolds", *Peng Lu, Lei Ni, Juan-Luis Vazquez and Cedric Villani*, JMPA, to appear online (2008)

- Main result: *Let $u$ be a positive smooth solution to PME, $m > 1$, on a cylinder $Q := B(\Omega, R) \times [0, T]$. Let $v$ be the pressure and let $v_{\max}^{R,T} := \max_{B(\Omega,R)\times[0,T]} v$.*
  **(1)** *Assume that Ricci curvature $Ric \ge 0$ on $B(\Omega, R)$. Then, for any $\alpha > 1$ we have*

$$\frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \le a\alpha^2 \left( \frac{1}{t} + \frac{v_{\max}^{R,T}}{R^2} (C_1 + C_2(\alpha)) \right) \tag{13}$$

  *on $Q' := B(\Omega, R/2) \times [0, T]$. Here, $a := \frac{n(m-1)}{n(m-1)+2} = (m-1)\kappa$, and the positive constants $C_1$ and $C_2(\alpha)$ depend also on $m$ and $n$.*
  **(2)** *Assume that $Ric \ge -(n-1)K^2$ on $B(\Omega, R)$ for some $K \ge 0$. Then, for any $\alpha > 1$, we have that on $Q'$,*

$$\frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \le a\alpha^2 \left( \frac{1}{t} + C_3(\alpha) K^2 v_{\max}^{R,T} \right) + a\alpha^2 \frac{v_{\max}^{R,T}}{R^2} \left( C_2(\alpha) + C_1'(KR) \right). \tag{14}$$

  *Here, $a$ and $C_2(\alpha)$ are as before and the positive constants $C_3(\alpha)$ and $C_1'(KR)$ depend also on $m$ and $n$. Acceptable values of the constants are:*

$$C_1 := 40(m-1)(n+2), \qquad C_2(\alpha) := \frac{200 a\alpha^2 m^2}{\alpha - 1}$$

$$C_3(\alpha) := \frac{(m-1)(n-1)}{\alpha - 1}, \qquad C_1'(KR) := 40(m-1)[3 + (n-1)(1 + KR)].$$

# Flows on manifolds

Paper: "Local Aronson-Bénilan estimates and entropy formulae for porous medium and fast diffusion equations on manifolds", *Peng Lu, Lei Ni, Juan-Luis Vazquez and Cedric Villani*, JMPA, to appear online (2008)

- Main result: *Let $u$ be a positive smooth solution to PME, $m > 1$, on a cylinder $Q := B(\Omega, R) \times [0, T]$. Let $v$ be the pressure and let $v_{\max}^{R,T} := \max_{B(\Omega,R)\times[0,T]} v$.*
  **(1)** *Assume that Ricci curvature $Ric \ge 0$ on $B(\Omega, R)$. Then, for any $\alpha > 1$ we have*

$$\text{(13)} \qquad \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \le a\alpha^2 \left( \frac{1}{t} + \frac{v_{\max}^{R,T}}{R^2} \left( C_1 + C_2(\alpha) \right) \right)$$

  *on $Q' := B(\Omega, R/2) \times [0, T]$. Here, $a := \frac{n(m-1)}{n(m-1)+2} = (m-1)\kappa$, and the positive constants $C_1$ and $C_2(\alpha)$ depend also on $m$ and $n$.*
  **(2)** *Assume that $Ric \ge -(n-1)K^2$ on $B(\Omega, R)$ for some $K \ge 0$. Then, for any $\alpha > 1$, we have that on $Q'$,*

$$\text{(14)} \quad \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \le a\alpha^2 \left( \frac{1}{t} + C_3(\alpha) K^2 v_{\max}^{R,T} \right) + a\alpha^2 \frac{v_{\max}^{R,T}}{R^2} \left( C_2(\alpha) + C_1'(KR) \right).$$

  *Here, $a$ and $C_2(\alpha)$ are as before and the positive constants $C_3(\alpha)$ and $C_1'(KR)$ depend also on $m$ and $n$. Acceptable values of the constants are:*

$$C_1 := 40(m-1)(n+2), \qquad C_2(\alpha) := \frac{200 a\alpha^2 m^2}{\alpha - 1}$$

$$C_3(\alpha) := \frac{(m-1)(n-1)}{\alpha - 1}, \qquad C_1'(KR) := 40(m-1)[3 + (n-1)(1 + KR)].$$

End

Thank you

♠ ♠ ♠

*Gracias, Merci*

# End

Thank you

♠ ♠ ♠

*Gracias, Merci*

End

Thank you

♠ ♠ ♠

*Gracias, Merci*

End

Thank you

♠ ♠ ♠

*Gracias, Merci*