On Wigner and Bohmian Measures

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The Schrödinger and Wigner Pictures of Quantum Mechanics and Semiclassical Asymptotics

Erwin Schrödinger 1926:

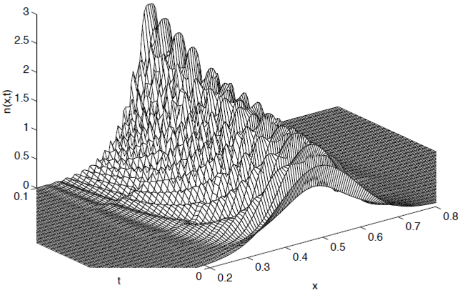
$$\left\{ \begin{array}{l} i\varepsilon u_t^\varepsilon = -\frac{\varepsilon^2}{2}\Delta u^\varepsilon + V(x)u^\varepsilon, \quad x \in \mathbb{R}^m, t \in \mathbb{R}, \\ u^\varepsilon(x, t=0) = u_I^\varepsilon(x), \quad x \in \mathbb{R}^m \end{array} \right\} \text{ IVP}$$

- $u^\varepsilon = u^\varepsilon(x,t)$: complex-valued wave-function
- $V = V(x)$: real-valued potential
- $0 < \varepsilon \ll 1$: semi-classical parameter

Observable densities are quadratic in the wave-function:

- $n^\varepsilon(x,t) := |u^\varepsilon(x,t)|^2$ position density
- $J^\varepsilon(x,t) := \varepsilon \mathrm{Im}(\overline{u^\varepsilon}(x,t)\nabla u^\varepsilon(x,t))$ current density vector
- $e^\varepsilon(x,t) := \frac{\varepsilon^2}{2}\left|\nabla u^\varepsilon(x,t)\right|^2 + V(x)|u^\varepsilon(x,t)|^2$ energy density

- High frequency limit $\varepsilon \to 0^+$ in the observables.
- <u>Problem:</u> $u^\varepsilon \xrightarrow{\varepsilon \to 0} u^0$ only weakly since the Schrödinger equation propagates oscillations of wave length ε in x and t.
  Weak convergence does not commute with (quadratically) nonlinear operations!



Assumption 1: $V = V(x)$ smooth, bounded from below

$$\Rightarrow \; H^{\varepsilon} := -\frac{\varepsilon^2}{2}\Delta + V(x) \quad \text{is essentially self-adjoint on } L^2(\mathbb{R}^m)$$

Assumption 2:

1. $\int_{\mathbb{R}^m} |u_I^{\varepsilon}(x)|^2 \, dx \leq C$ (uniformly bounded initial total mass)
2. $\int_{\mathbb{R}^m} \left(\frac{\varepsilon^2}{2} \left|\nabla u^{\varepsilon}(x,t)\right|^2 + V(x)|u^{\varepsilon}(x,t)|^2 \right) \, dx \leq C$ (uniformly bounded intitial total energy

- Stone's Theorem: iH^ε generates a strongly continuous group of unitary operators on $L^2(\mathbb{R}^m) \Rightarrow n^\varepsilon := |u^\varepsilon|^2$ satisfies

$$\int_{\mathbb{R}^m} n^\varepsilon(x,t)\,dx = \int_{\mathbb{R}^m} n^\varepsilon(x,t=0)\,dx \quad \forall t \in \mathbb{R}$$

- Also

$$E^\varepsilon(t) := \int_{\mathbb{R}^m} \left(\frac{\varepsilon^2}{2} \left|\nabla u^\varepsilon(x,t)\right|^2 + V(x)|u^\varepsilon(x,t)|^2 \right) \,dx \equiv E^\varepsilon(t=0)$$

$$n^\varepsilon \xrightarrow{\varepsilon\to 0} n^0 \text{ in } \mathcal{M}^+(\mathbb{R}^m_x) \text{ weak-}*\text{, identify } n^0!!$$

$$p^\varepsilon(\xi,t) := \frac{1}{\varepsilon^m} \left| \hat{u}^\varepsilon\left(\frac{\xi}{\varepsilon}, t\right)\right|^2 \quad \text{momentum density}$$

$$p^\varepsilon \xrightarrow{\varepsilon\to 0} p^0 \text{ in } \mathcal{M}^+(\mathbb{R}^m_\xi) \text{ weak-*, identify } p^0 \text{ !!!}$$

Pseudo-differential Framework

- General observables: $a(x, \varepsilon D)^w$ with $a = a(x, \xi)$ real

$$E_a(u^\varepsilon) = (a(x, \varepsilon D)^w u^\varepsilon, u^\varepsilon)_{L^2}$$ average value of the observable in the state u^ε

$$E_a(u^\varepsilon) \xrightarrow{\varepsilon \to 0^+} E_a^0 \quad \text{identify !!!}$$

- Wigner transform: $f \in \mathcal{S}'(\mathbb{R}^m)$, scale $\varepsilon > 0$

$$\mathcal{S}'(\mathbb{R}^m) \to \mathcal{S}'(\mathbb{R}^m_x \times \mathbb{R}^m_\xi) \text{ quadratic, continuous}$$

$$w^\varepsilon[f](x, \xi) := \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} f\left(x - \frac{\varepsilon}{2} v\right) \overline{f\left(x + \frac{\varepsilon}{2} v\right)} e^{iv\cdot\xi} \, dv$$

- Proposition: $E_a(f) = \int_{\mathbb{R}^m_x \times \mathbb{R}^m_\xi} a(x, \xi) w^\varepsilon[f](x, \xi) \, dx d\xi$
- Proposition: f in a bounded subset of $L^2(\mathbb{R}^m)$. Then $w^\varepsilon[f]$ is uniformly bounded in $\mathcal{S}'(\mathbb{R}^m_x \times \mathbb{R}^m_\xi)$.

Pseudo-differential Framework

- Take $f^\varepsilon \in L^2(\mathbb{R}^m)$ unif. as $\varepsilon \to 0$. By weak compactness, after extracting a subsequence ε_k, there exists $w^0 \in \mathcal{S}'(\mathbb{R}^m_x \times \mathbb{R}^m_\xi)$ such that

$$w^{\varepsilon_k}[f^{\varepsilon_k}] \xrightarrow{\varepsilon_k \to 0} w^0 \text{ in } \mathcal{S}'(\mathbb{R}^m_x \times \mathbb{R}^m_\xi)$$

- Proposition: $w^0 \in \mathcal{M}^+(\mathbb{R}^m_x \times \mathbb{R}^m_\xi)$ (Wigner measure of f^ε)
- Proposition: $f^\varepsilon \in L^2(\mathbb{R}^m)$ uniformly as $\varepsilon \to 0$ and $a \in \mathcal{S}$. Then

$$E_a(f^{\varepsilon_k}) \to \int_{\mathbb{R}^m_x \times \mathbb{R}^m_\xi} a(x,\xi)\,dw^0(x,\xi)$$

$$\left\{ \begin{array}{l} i\varepsilon u_t^\varepsilon = -\frac{\varepsilon^2}{2}\Delta u^\varepsilon + V(x)u^\varepsilon \\ u^\varepsilon(x, t=0) = u_I^\varepsilon(x) \end{array} \right\} \text{ denote } w^\varepsilon(x,\xi,t) := w^\varepsilon[u^\varepsilon(t)](x,\xi)$$

$$w^\varepsilon(t) \xrightarrow{\varepsilon \to 0^+} w^0(t) \in \mathcal{M}^+(\mathbb{R}_x^m \times \mathbb{R}_\xi^m),$$

$$E_a(u^\varepsilon(t)) \to \int a(x,\xi) w^0(x,\xi,t)\, dx d\xi$$

local in t uniform convergence! identify w^0 !!!

- Proposition: w^0 satisfies the transport (Liouville) equation:

$$\left\{ \begin{array}{l} w_t^0 + \xi \cdot \nabla_x w^0 - \nabla_x V(x) \cdot \nabla_\xi w^0 = 0,\ (x,\xi) \in \mathbb{R}_x^m \times \mathbb{R}_\xi^m,\ t \in \mathbb{R} \\ w^0(t=0) = \lim_{\varepsilon \to 0} w^\varepsilon[u_I^\varepsilon] =: w_I^0 \end{array} \right\}$$

Characteristics: $\dot{x} = \xi,\ \dot{\xi} = -\nabla_x V(x)$

Newton's ODEs

Also:

1. $n^\varepsilon \rightharpoonup n^0 = \int_{\mathbb{R}^m} w^0(x, d\xi, t)$
2. $J^\varepsilon \rightharpoonup J^0 = \int_{\mathbb{R}^m} \xi w^0(x, d\xi, t) \ll n^0$

If additionally, the measures $|u_I^\varepsilon(x)|^2 =: n_I^\varepsilon(x)$ form a tight sequence as $\varepsilon \to 0$, then

3. $\int_{\mathbb{R}^m} n^\varepsilon(x, t)\, dx \to \iint_{\mathbb{R}^{2m}} w^0(dx, d\xi, t) \equiv \iint_{\mathbb{R}^{2m}} w_I^0(dx, d\xi)$

Phase space analysis of the high-frequency limit of the linear Schrödinger equation:

- A. Shnirelman, Uspekhi Mat. Nauk., 1974
- P. Gerard, Seminaire Ecole Polytechnique, 1991
- P.L. Lions and T. Paul, Rivista Math. Iberoam. 1993
- P. Gérard, P.A.M., N. Mauser and F. Poupaud, CP.A.M. 1997

based on work by E. Wigner, 1932

Bohmian Mechanics

$$\text{Velocity field } v^\varepsilon(x,t) := \frac{J^\varepsilon(x,t)}{\rho^\varepsilon(x,t)} = \varepsilon \mathrm{Im}\left(\frac{\nabla u^\varepsilon(x,t)}{u^\varepsilon(x,t)}\right)$$

$$\left\{ \begin{array}{c} \dot{X}^\varepsilon(t,x) = v^\varepsilon(X^\varepsilon(t,x),t) \\ X^\varepsilon(t=0,x) = x \end{array} \right\} \text{Eulerian viewpoint of Bohmian Mechanics}$$

Thm (Berndl et al, 1995): $X_t^\varepsilon : x \to X^\varepsilon(t,x)$ exists globally in time for almost all x, relative to the measure $n_I^\varepsilon dx$ and $X_t^\varepsilon \in \mathcal{C}^1$ on its maximal domain.

Note: generally, $v^\varepsilon(\cdot,t)$ is not Lipschitz in x, not $W^{1,1}$ and not even BV !!

Moreover: $n^\varepsilon(t) = X_t^\varepsilon \# n_I^\varepsilon \Leftrightarrow \forall \sigma \in \mathcal{C}^0(\mathbb{R}^m)$:

$$\int_{\mathbb{R}^m} \sigma(x) n^\varepsilon(x,t)\, dx = \int_{\mathbb{R}^m} \sigma(X^\varepsilon(t,x)) n_I^\varepsilon(x)\, dx$$

n^ε is a weak solution of the continuity equation

$$n_t^\varepsilon + \operatorname{div}(\underbrace{n^\varepsilon v^\varepsilon}_{J^\varepsilon}) = 0$$

Lagrangian Viewpoint of Bohmian Mechanics:

$$\dot{X}^\varepsilon = v^\varepsilon(X^\varepsilon, t) \;\Big|\; \frac{d}{dt} \Rightarrow \dot{X}^\varepsilon = P^\varepsilon,\; \dot{P}^\varepsilon = v_t^\varepsilon + (v^\varepsilon \cdot \nabla)\, v^\varepsilon$$

The Madelung equations:

$$n_t^\varepsilon + \operatorname{div}(n^\varepsilon v^\varepsilon) = 0$$
$$v_t^\varepsilon + (v^\varepsilon \cdot \nabla) v^\varepsilon + \nabla V = \frac{\varepsilon^2}{2} \nabla \left(\frac{\Delta \sqrt{n^\varepsilon}}{\sqrt{n^\varepsilon}} \right)$$

leads to:

$$\begin{cases} \dot{X}^\varepsilon = P^\varepsilon, & X^\varepsilon(t=0) = x \\ \dot{P}^\varepsilon = -\nabla V(X^\varepsilon) - \nabla V_B^\varepsilon(X^\varepsilon, t), & P^\varepsilon(t=0) = v_I^\varepsilon(x) \end{cases}$$

with the Bohm potential

$$V_B^\varepsilon(x,t) := -\frac{\varepsilon}{2}^2 \frac{\Delta\sqrt{n^\varepsilon}}{\sqrt{n^\varepsilon}}$$

formally $O(\varepsilon^2)$, but highly oscillatory as $\varepsilon \to 0$ and singular in nodes of $u^\varepsilon(x,t)$!!!

Lagrangian Bohmian equations $\Leftrightarrow$ formal $O(\varepsilon^2)$-perturbation of Newton's ODEs

Def.(Bohmian Measures): $\varepsilon > 0$ scale, $u^\varepsilon \in H^1(\mathbb{R}^m)$,

$$n^\varepsilon := |u^\varepsilon|^2, \; J^\varepsilon = \varepsilon \mathrm{Im}(\overline{u^\varepsilon} \nabla u^\varepsilon)$$

$$\text{Then } \beta^\varepsilon(x,\xi) := n^\varepsilon(x)\delta\left(\xi - \frac{J^\varepsilon(x)}{n^\varepsilon(x)}\right) \in \mathcal{M}^+(\mathbb{R}^m_x \times \mathbb{R}^m_\xi)$$

Note: $\displaystyle\int_{\mathbb{R}^m} \beta^\varepsilon(x,d\xi) = n^\varepsilon(x), \; \int_{\mathbb{R}^m} \xi \beta^\varepsilon(x,d\xi) = J^\varepsilon(x)$

but:
$$\begin{aligned}
\iint_{\mathbb{R}^{2m}} \frac{|\xi|}{2}^2 \beta^\varepsilon(dx,d\xi) &= \int_{\mathbb{R}^m} \frac{|J^\varepsilon(x)|^2}{n^\varepsilon(x)} \, dx \\
&\neq \int_{\mathbb{R}^m} \frac{\varepsilon}{2}^2 |\nabla u^\varepsilon(x)|^2 \, dx \\
&= \frac{\varepsilon}{2}^2 \int_{\mathbb{R}^m} \left|\nabla \sqrt{n^\varepsilon}\right|^2 \, dx + \int_{\mathbb{R}^m} \frac{|J^\varepsilon(x)|}{n^\varepsilon(x)} \, dx
\end{aligned}$$

Now let $u^\varepsilon = u^\varepsilon(x,t)$ be the solution of the Schrödinger IVP. Set

$$\beta^\varepsilon(x,\xi,t) := n^\varepsilon(x,t)\delta\left(\xi - \frac{J^\varepsilon(x,t)}{\rho^\varepsilon(x,t)}\right)$$
$$\beta_I^\varepsilon(x,\xi) = \beta^\varepsilon(x,\xi,t=0)$$

Prop.: $\beta^\varepsilon(t)$ is the push-forward of β_I^ε with respect to the Lagrangian Bohmian (phase space) flow $(X^\varepsilon(t,\cdot), P^\varepsilon(t,\cdot))$ which is β_I^ε-a.e. well defined.

Formally:

$$\begin{cases} \frac{\partial}{\partial t}\beta^\varepsilon + \xi\cdot\nabla_x\beta^\varepsilon - \nabla\left(V+V_B\right)\cdot\nabla_\xi\beta^\varepsilon = 0, n^\varepsilon = \int_{\mathbb{R}^m}\beta^\varepsilon\, d\xi \\ \beta^\varepsilon(t=0) = \beta_I^\varepsilon \end{cases}$$

β^ε encodes the full quantum mechanical information of n^ε, J^ε and evolves through the Bohmian Lagrangian flow

Study $\lim_{\varepsilon\to 0} \beta^\varepsilon$!

Prop:

- $u^\varepsilon \in L^2(\mathbb{R}^m)$ uniform as $\varepsilon \to 0$. Then, up to extraction of subsequences, $\beta^\varepsilon \to \beta^0$ in $\mathcal{M}^+(\mathbb{R}^m_x \times \mathbb{R}^m_\xi)$ w-$*$
- If u^ε has uniform bounded kinetic energy, then

$$n^0(x) = \int_{\mathbb{R}^m} \beta^0(x, d\xi), \quad J^0(x) = \int_{\mathbb{R}^m} \xi \beta^0(x, d\xi)$$

if, additionally, n^ε is tight then

$$\int_{\mathbb{R}^m} n^\varepsilon \, dx \to \int_{\mathbb{R}^m} n^0 \, dx = \iint_{\mathbb{R}^{2m}} \beta^0(dx, d\xi)$$

Compare β^0 to the Wigner measure

$$w^0 := \lim_{\varepsilon \to 0} w^\varepsilon[u^\varepsilon] \; !$$

Young measures: Sequence $f^\varepsilon : \mathbb{R}^m \to \mathbb{R}^n$ measureable. Then there exists a map $\mu_x \equiv \mu(x) : \mathbb{R}^m \to \mathcal{M}^+(\mathbb{R}^n)$ called Young measure associated to f^ε such that

- $x \to \langle \mu(x), g \rangle$ is measurable for all $g \in \mathcal{C}_0(\mathbb{R}^n)$
- (after selection of a subsequence)

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^m} \sigma(x, f^\varepsilon(x)) \, dx = \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} \sigma(x, \lambda) \, d\mu_x(\lambda) dx$$

for all testfunctions σ in $\mathcal{C}_0(\mathbb{R}^m \times \mathbb{R}^n)$.

Connection Bohmian measure $\leftrightarrow$ Young measure

$$\langle \beta^\varepsilon, \varphi \rangle = \int_{\mathbb{R}^m} n^\varepsilon(x) \varphi\left(x, \frac{J^\varepsilon(x)}{n^\varepsilon(x)}\right) \, dx \to \langle \beta^0, \varphi \rangle$$

Prop. Let $u^\varepsilon \in L^2(\mathbb{R}^m)$ uniformly with uniformly bounded kinetic energy. Let μ_x be the Young measure of the sequence $(n^\varepsilon, J^\varepsilon)$. Then

- $\beta^0(x, \xi) \geq \int_0^\infty r^{d+1} \mu_x(r, r\xi) \, dr$
- If $n^\varepsilon \rightharpoonup n^0$ in $L^1(\mathbb{R}^m)$-weakly, then "$=$" holds above and μ_x is a probability measure on $\mathbb{R}^{m+1}$.

Prop. (mono kinetic): If $n^\varepsilon \to n^0$ in $L^1(\mathbb{R}^m)$ strongly and $J^\varepsilon \to \tilde{J}$ in measure then β^0 is mono-kinetic:

$$\beta^0(x,\xi) = n^0(x)\delta\left(\xi - \frac{\tilde{J}(x)}{n^0(x)}\right)$$

$$\text{and} \qquad \tilde{J}(x) = J^0(x).$$

"$\Leftarrow$": NOT TRUE !

Prop.: Let $\gamma_{t,x}$ be the Young measure associated to the Bohmian Lagrangian flow $(X^\varepsilon(t,x), P^\varepsilon(t,x))$. Then, if n_I^ε converges strongly in $L^1(\mathbb{R}^m)$ to n_I^0, we have:

$$\beta^0(t,y,\xi) = \int_{\mathbb{R}^m} \gamma_{t,x}(y,\xi) n_I^0(x)\, dx$$

β^0 can be obtained from $\gamma_{t,x}$ but not the other way around !

$$u^\varepsilon(x) = \sqrt{n^\varepsilon(x)}\exp\left(\frac{i}{\varepsilon}S^\varepsilon(x)\right)$$

Thm 1: u^ε uniformly bounded in $L^2(\mathbb{R}^m)$ with uniform bounded kinetic energy. If

(a) $n^\varepsilon \to n^0$ in $L^1(\mathbb{R}^m)$-strongly

(b) "$\nabla S^\varepsilon \to \nabla S^0$ uniformly"

then $\beta^0(x,\xi) = w^0(x,\xi) = n^0(x)\delta(\xi - \nabla S^0(x))$.

Thm 2: Let u^ε be as in Thm 1. If

(a) $\varepsilon\nabla\sqrt{n^\varepsilon} \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^m)$

(b) $\varepsilon\sup_x\left|\frac{\partial^2 S^\varepsilon}{\partial x_i \partial x_j}\right| \to 0 \ \ \forall i,j$

then $w^0 \equiv \beta^0$.

$$w^0 \equiv \beta^0 \overset{\text{almost}}{\Rightarrow} \text{(a),(b) !!}$$

Case Studies

1. Oscillatory functions: $u^\varepsilon(x) = f(x) g\left(\frac{x}{\varepsilon}\right), \ f \in \mathcal{C}_0^\infty(\mathbb{R}^m)$, $g \in \mathcal{C}^\infty(\mathbb{R}^d)$ periodic on some lattice L

   Prop.: $\beta^0 = w^0$ iff g carries only a single oscillation, i.e. if $g(y) = ce^{-iy\cdot l^*}$, $c \in \mathbb{C}, \ l^* \in L^*$.

2. Concentrating functions: $u^\varepsilon(x) = \varepsilon^{-\frac{m}{2}} f\left(\frac{x - x_0}{\varepsilon}\right), \ f \in \mathcal{C}^\infty(\mathbb{R}^m)$
   note: $|u^\varepsilon(x)|^2 = n^\varepsilon(x) \rightharpoonup \delta(x - x_0)$

   Prop.: $\beta^0 = w^0$ iff $f \equiv 0$.

3. Semi-classical wave packets (coherent states)

$$u^\varepsilon(x) = \varepsilon^{-\frac{m}{4}} f\left(\frac{x - x_0}{\sqrt{\varepsilon}}\right) e^{i\frac{\xi_0 \cdot x}{\varepsilon}}, \; x_0, \xi_0 \in \mathbb{R}^m$$

$$\Rightarrow \; w^0(x,\xi) = \beta^0(x,\xi) = \int_{\mathbb{R}^m} |f(x)|^2 \, dx \delta(x - x_0)\delta(\xi - \xi_0).$$

Note: coherent states concentrate on the scale $\sqrt{\varepsilon}$ not on the scale ε !

4. Eigenfunctions $H^\varepsilon = -\frac{\varepsilon}{2}^2 \Delta + \frac{1}{2}|x|^2$

spectral problem: $H^\varepsilon u_l^\varepsilon = \lambda_l^\varepsilon u_l^\varepsilon, \; \|u_l^\varepsilon\|_{L^2}^2 = 1, \; \lambda_l^\varepsilon \in \mathbb{R}, \; u_l^\varepsilon$ real.

Let ε_l be a sequence, $\varepsilon_l \to 0$ and $\lambda_l^{\varepsilon_l} \to \Lambda \in \mathbb{R}$, $(u_l^{\varepsilon_l})^2 \rightharpoonup n^0$

$$w^0(x,\xi) = \delta(|x|^2 + |\xi|^2 - \Lambda)$$

$$\beta^0(x,\xi) = n^0(x)\delta(\xi)$$

WKB-Approximation

$$u_I^\varepsilon(x) := \sqrt{n_I(x)}\exp\left(\frac{i}{\varepsilon}S_I(x)\right),\ n_I,\ S_I \text{ real valued, smooth}$$

$$\left\{\begin{array}{l} S_t^0 + \frac{1}{2}|\nabla S^0|^2 + V(x) = 0 \\ S^0(t=0) = S_I \end{array}\right\} \text{ Hamilton-Jacobi equation}$$

$$\left\{\begin{array}{l} n_t^0 + \operatorname{div}(\nabla S^0 n^0) = 0 \\ n^0(t=0) = n_I \end{array}\right\} \text{ Continuity equation}$$

Then: $u^\varepsilon(x,t) \sim \sqrt{n^0(x,t)}\,\exp\left(\frac{i}{\varepsilon}S^0(x,t)\right)$ before caustic onset time of the HJ-solution S^0 (J. Keller, P. Lax, 1950-1960)

Prop.: $\beta^0(x,\xi,t) = w^0(x,\xi,t) = n^0(x,t)\delta(\xi - \nabla S^0(x,t))$ before caustic onset time.

After caustic onset: $u^\varepsilon(x,t) \approx$ superposition of WKB-states.

Consider:

$u^\varepsilon(x) = \sqrt{n_1(x)}\exp\left(\frac{i}{\varepsilon}S_1(x,t)\right) + \sqrt{n_2(x)}\exp\left(\frac{i}{\varepsilon}S_2(x,t)\right), \ \nabla S_1 \neq \nabla S_2$

Then: $w^0(x,\xi) = n_1(x)\delta(\xi - \nabla S_1(x)) + n_2(x)\delta(\xi - \nabla S_2(x))$

and

$$\beta^0(x,\xi) = \int_0^{2\pi} m(x,\theta)\delta(\xi - \Phi(x,\Theta))\, d\Theta$$

$$m(x,\Theta) = n_1(x) + n_2(x) + 2\sqrt{n_1(x)}\sqrt{n_2(x)}\cos(\Theta)$$

$$\Phi(x,\Theta) = \frac{1}{m(x,\Theta)}(n_1(x)\nabla S_1(x) + n_2(x)\nabla S_2(x) + \sqrt{n_1(x)}\sqrt{n_2(x)}(\nabla S_1(x) + \nabla S_2(x))\cos\Theta)$$

Conjecture: after caustic onset: $\beta^0 \neq w^0$