1. Let $\mathcal{M} = \left\{ u \in C([0,1]) : \int_0^{1/2} u - \int_{1/2}^1 u = 1 \right\}$ and $I : \mathcal{M} \mapsto \mathbb{R}$ given by $I(u) = \|u\|_\infty$.

   (i) Show that $\text{ínf}_{u\in\mathcal{M}} I(u) = 1$.

   (ii) Show that there is no function $u \in \mathcal{M}$ such that $I(u) = 1$.

   The problem is that this space is not reflexive.

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2. Let $\mathcal{M}$ the convex closed subset of $H^1([0,1])$ given by $\mathcal{M} = \{u \in H^1([0,1]) : u(0) = 1, u(1) = 0\}$. Consider the functional $I : \mathcal{M} \mapsto \mathbb{R}$ defined by $I(u) = \int_0^1 x|u'(x)|^2\, dx$.

   (i) Show that $\text{ínf}_{u\in\mathcal{M}} I(u) = 0$.

   (ii) Show that there does not exist any $u \in \mathcal{M}$ such that $I(u) = 0$.

   In this case the problem is that the functional is not coercive.

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3. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, let $\beta : \mathbb{R} \mapsto \mathbb{R}$ be a smooth function such that there exist a and b such that

$$0 < a \le \beta'(z) \le b \quad \text{for all } z \in \mathbb{R},$$

   and $f \in L^2(\Omega)$.

   (i) Define a concept of weak solution for the nonlinear problem

$$-\Delta u = f \quad \text{en } \Omega, \qquad \partial u/\partial n + \beta(u) = 0 \quad \text{in } \partial\Omega.$$

   (ii) Prove that there exists a weak solution (and is unique).

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4. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Given $u \in H^1(\Omega)$, we define the *surface of the graphic* of u by

$$F(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx.$$

   (i) Prove that the functional F is C^1 in $H^1(\Omega)$.

   (ii) Let $g \in H^1(\Omega)$, and $\mathcal{A} = \{u = g + v : v \in H_0^1(\Omega)\}$. Prove that a critical point of F in $\mathcal{A}$ is a weak solution to the *equation of minimal surfaces*:

$$\nabla \cdot \left(\frac{\nabla u}{(1 + |\nabla u|^2)^{1/2}} \right) = 0 \quad \text{en } \Omega, \qquad u = g \quad \text{en } \partial\Omega.$$

   The expression on the left of this equality is N-times the *mean curvature of the graphic* of u. Hence a minimal surface has zero mean curvature.

   (iii) Check wether or not the direct method of calculus of variations can be used to deduce existence of a minimizer of F in $\mathcal{A}$.

   (iv) Let $J(w) = \int_\Omega w\, dx$. assume that u is a smooth minimizer of F in $\mathcal{A} \cap \{w : J(w) = 1\}$. Show that the graphic of u is a minimal surface with constant mean curvature.

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5. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Consider the eigenvalue problem for the bi-harmonic opeartor with Dirichlet boundary conditions,

$$\Delta^2 u = \lambda u \quad \text{en } \Omega, \qquad u = \partial u / \partial n = 0 \quad \text{in } \partial\Omega.$$

Show that there exists a non-trivial weak solution (λ, u) to the problem when $\lambda > 0$.

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6. Let $f \in L^2(\Omega)$. Show that there exists a unique minimizer u of

$$J(w) = \int_\Omega \left(\frac{1}{2} |\nabla w|^2 - fw \right)$$

in $\mathcal{A} = \{ w \in H_0^1(\Omega) : |\nabla w| \leq 1 \text{ a.e. } \}$. Show that

$$\int_\Omega \nabla u \cdot \nabla (w - u) \geq \int_\Omega f(w - u) \qquad \text{for all } w \in \mathcal{A}.$$
