Hoja de problemas 3: Weak Solutions and Linear Elliptic Equations.

1. Let a, b, c be smooth functions, with a and c strictly positive. Let u be a solution to the boundary value problem
$$-au'' + bu' + cu = f \quad \text{en } I = (0,1), \qquad u(0) = u(1) = 0.$$
Show that u solves an equation of the form $-(\tilde{a}(x)u')' + \tilde{c}(x)u = \tilde{f}$: write the corresponding weak formulation and show that there exists a unique solution.

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2. Consider the boundary value problem
$$-u'' + ku' + u = f \quad \text{en } I = (0,1), \qquad u'(0) = u'(1) = 0.$$
Write the variational formulation and show that for k sufficiently small there is no unique solution. Find (at least) a value of k and (at least) a function $v \in H^1$, with $v \not\equiv 0$ such that $a(v,v) = 0$.

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3. Consider the problem
$$-u''(x) = f(x) \quad \text{en } I = (0,1), \qquad u'(0) - u(0) = 0,\ u'(1) + u(1) = 0.$$
   (a) Define a classical solution of the problem, when $f \in C([0,1])$.
   (b) Prove that classical solutions are weak i.e. they satisfy
$$u(0)v(0) + u(1)v(1) + \int_0^1 u'v' = \int_0^1 fv, \qquad \forall v \in H^1(I).$$
   Define a weak solution to the problem as a function $u \in H^1(I)$ satisfying the above equality.
   (c) Prove existence and uniqueness of weak solutions to the above problem.
   Hint: Prove and use the following Poincaré-type inequality
$$\int_0^1 u^2 \leq C\left((u(0))^2 + (u(1))^2 + \int_0^1 (u')^2\right) \qquad \forall u \in H^1(I).$$
   (d) Prove that $f \in C(\bar{I})$ implies $u \in C^2(\bar{I})$.
   (e) Show that any weak solution which is $C^2(\bar{I})$ is indeed a classical solution.

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4. Consider the boundary value problem
$$u''''(x) = f(x) \quad \text{in } I = (0,1), \qquad u(0) = u'(0) = u(1) = u'(1) = 0.$$
Here, u represents, for instance, deflection of a bar fixed at the extremals and under the influence of a transversal force of intensity f. Given $f \in C(\bar{I})$:
   (a) Define classical solutions.
   (b) Define weak solutions (the correct functional space is $H_0^2(I)$).
   (c) Show that every classical solution is a weak solution.
   (d) Prove that there exists a unique weak solution.
   (e) Prove that if $f \in C(\bar{I})$, then $u \in C^4(\bar{I})$.
   (f) Prove that if a weak solution is in $C^4(\bar{I})$, then it is a classical solution.

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5. Let $I = (0,1)$. Show that the functional $F : H^1(I) \mapsto \mathbb{R}$ defined by $F(u) = u(0)$ is linear and continuous. Show next that there exists a unique $v_0 \in H^1(I)$ such that

$$u(0) = \int_0^1 (u'v_0' + uv_0) \qquad \forall u \in H^1(I).$$

Show that v_0 is solution to a certain differential equation with suitable boundary conditions. Find an explicit expression for v_0.

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6. Find a function $u \in C^2([0,1/2])$ con $u(0) = u(1/2) = 0$ such that for any $v \in C^2([0,1/2])$ we have

$$\int_0^{1/2} (u'v' + (4u-1)v) = 0.$$

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7. Consider the boundary value problem $u'' = 2$, $u(1) = u(-1) = 0$, whose solution is given by $\bar{u}(x) = x^2 - 1$; write the variational formulation to conclude that for all $u \in C^2$ with $u(1) = u(-1) = 0$ we have

$$\frac{8}{3} + \int_{-1}^1 \left((u')^2 + 4u\right) \geq 0.$$

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8. (Hardy Inequality in dimension $N = 1$). Let $I = (0,1)$.

   (a) Given $u \in L^p(I)$, show that

   $$\left\| \frac{1}{x} \int_0^x u(t)\, dt \right\|_{L^p(I)} \leq \frac{p}{p-1} \|u\|_{L^p(I)}.$$

   Hint. Begin with $u \in C_c(I)$ by defining $\varphi(x) = \int_0^x u(t)\, dt$. Check that $|\varphi|^p \in C^1(\bar{I})$ and calculate the derivative. Finally, use the formula

   $$\int_0^1 |\varphi(x)|^p \frac{dx}{x^p} = \frac{1}{p-1} \int_0^1 |\varphi(x)|^p d\left(-\frac{1}{x^{p-1}}\right)$$

   and integrate by parts.

   (b) Let $u \in W^{1,p}(I)$, $1 < p < \infty$. Show that if $u(0) = 0$, then

   $$\left\| \frac{u(x)}{x} \right\|_{L^p(I)} \leq \frac{p}{p-1} \|u'\|_{L^p(I)}.$$

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9. (A problem with Hardy-type weights) Let $I = (0, 1)$ and $V = \{v \in H^1(I) : v(0) = 0\}$.

(a) Given $f \in L^2(I)$ such that $\frac{1}{x} f(x) \in L^2(I)$, show that there exists a unique $u \in V$ satisfying

$$\int_0^1 u'(x)v'(x)\,dx + \int_0^1 \frac{u(x)v(x)}{x^2}\,dx = \int_0^1 \frac{f(x)v(x)}{x^2}\,dx \qquad \forall v \in V. \tag{1}$$

(b) Write the minimization problem associated to (1)

(c) Here and in part (d) we will assume that $\frac{1}{x^2} f(x) \in L^2(I)$. Letting $v(x) = \frac{u(x)}{(x+\varepsilon)^2}$, $\varepsilon > 0$, show that

$$\int_0^1 \left| \frac{d}{dx} \left(\frac{u(x)}{x+\varepsilon} \right) \right|^2 dx \le \int_0^1 \frac{f(x)}{x^2} \frac{u(x)}{(x+\varepsilon)^2}\,dx.$$

(d) Prove that $\frac{u(x)}{x^2} \in L^2(I)$, $\frac{u(x)}{x} \in H^1(I)$ y $\frac{u'(x)}{x} \in L^2(I)$.

(e) As a consequence of part (d) show that $u \in H^2(I)$ and that

$$-u''(x) + \frac{u(x)}{x^2} = \frac{f(x)}{x^2} \quad \text{a.e. en } I, \qquad u(0) = u'(0) = 0, \quad u'(1) = 0. \tag{2}$$

(f) Viceversa, show that if $u \in H^2(I)$ with $\frac{u(x)}{x^2} \in L^2(I)$ satisfies equation (2), hence it satisfies (1).

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10. Let $I = (0, 1)$ and let us fix a constant $k > 0$.

(a) Given $f \in L^1(I)$, show that there is a unique $u \in H_0^1(I)$ such that

$$\int_I u'v' + k \int_I uv = \int_I fv \qquad \forall v \in H_0^1(I). \tag{3}$$

(b) Prove that $u \in W^{2,1}(I)$.

(c) Prove that

$$\|u\|_{L^1(I)} \le \frac{1}{k} \|f\|_{L^1(I)}.$$

Hint. Fix a function $\gamma \in C^1(\mathbb{R}, \mathbb{R})$ so that $\gamma'(t) \ge 0$ for all $t \in \mathbb{R}$, $\gamma(0) = 0$, $\gamma(t) = 1$ and all $t \ge 1$ and such that $\gamma(t) = -1$ for all $t \le -1$. Take $v = \gamma(n\,u)$ in (3) and let $n \to \infty$.

(d) Assume now $f \in L^p(I)$, $p \in (1, \infty)$. Show that there exists $\delta > 0$ independent of k and p such that

$$\|u\|_{L^p(I)} \le \frac{1}{k + \delta/pp'} \|f\|_{L^p(I)}.$$

Hint. If $p \in [2, \infty)$, take $v = \gamma(u)$ in (3), with $\gamma(t) = |t|^{p-1} \operatorname{sign} t$. If $p \in (1, 2)$, use duality.

(e) if $f \in L^\infty(I)$, show that

$$\|u\|_{L^\infty(I)} \le C_k \|f\|_{L^\infty(I)},$$

and find the best constant C_k. *Hint.* Find the explicit solution to (3) corresponding to $f \equiv 1$.

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11. Let $I = (0, 1)$.

(a) Prove that for any $\varepsilon > 0$ there exists a constant C_ε such that

$$|u(1)|^2 \leq \varepsilon \|u'\|^2_{L^2(I)} + C_\varepsilon \|u\|^2_{L^2(I)} \qquad \forall u \in H^1(I).$$

(b) Show that if the constant $k > 0$ is big enough, then for all $f \in L^2(I)$ there exists a unique $u \in H^2(I)$ satisfying

$$-u'' + ku = f \quad \text{en } I, \qquad u'(0) = 0, \quad u'(1) = u(1).$$

Write both the associated weak formulation and the minimization problem.

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12. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, let $h \in C^\infty(\partial\Omega)$ be such that $\int_{\partial\Omega} h = 0$.

(a) Define a reasonable concept of weak solution to the problem

$$\Delta u = 0 \quad \text{en } \Omega, \qquad \partial u/\partial n = h \quad \text{en } \partial\Omega.$$

(b) Prove that there exists a unique weak solution such that $\int_\Omega u = 0$ and check that the difference between two arbitrary weak solutions has to be constant in Ω.

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13. Let $\Omega \subset \mathbb{R}^N$ be a bounded connected domain with smooth boundary.

(a) Define weak solutions for the Poisson equation with Robin boundary conditions:

$$-\Delta u = f \quad \text{en } \Omega, \qquad u + \frac{\partial u}{\partial n} = 0 \quad \text{sobre } \partial\Omega,$$

Check that any classical solution to the problem is a weak solution, and that every weak solution which is also smooth enough, is a classical solution.

(b) Show existence and uniqueness of weak solutions to the problem, for any $f \in L^2(\Omega)$.
Hint. Use Friedrichs' Inequality.
