Smoothing Effect, Positivity and Harnack Inequalities for Very Fast Evolution Equations

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Fast Diffusion Equation $m < 1$

Fast Diffusion Equation

$$\begin{cases} u_t = \dfrac{1}{m}\Delta(u^m)\,, & \text{in } Q = (0,T)\times\Omega \subseteq (0,+\infty)\times\mathbb{R}^d\,, \\ u(0,\cdot) = u_0\,, & u_0 \in \mathrm{L}^r_{loc}(\Omega) \quad \text{and } m < 1 \end{cases}$$

Finite Extinction Time (FET)

$$\exists\, T > 0 \ \text{ such that } \ u(t,x) = 0\,, \quad \forall\, x \in \Omega \ \text{ and } \ \forall\, t \ge T$$

- The homogeneous Dirichlet problem on bounded domains Ω always extinguish in finite time T when $0 \le m < 1$, and when $m < 0$, there is the effect of *immediate extinction*, $T = 0$.
- The Cauchy problem in $\mathbb{R}^d$ extinguish in finite time when the initial datum belongs to $\mathrm{L}^{r_c}(\mathbb{R}^d)$, with $r_c = d(1-m)/2$, for any $m < 1$, but $r_c > 1$ if $m > m_c$.
- Conservation of mass does not allow extinction in finite time, e.g. homogenous Neumann problem, large solutions.

Local Weak Solutions for Fast Diffusion Equation

$$u \in C\left(0,T;\mathrm{L}^2_{\mathrm{loc}}(\Omega)\right) \quad \text{and} \quad |u|^m \in \mathrm{L}^2_{loc}\left(0,T;W^{1,2}_{\mathrm{loc}}(\Omega)\right)$$

such that, for every open bounded subdomain $[t_1,t_2]\times K \subset (0,T]\times\Omega$, we have

$$\int_K u(t_2)\varphi(t_2)\,\mathrm{d}x - \int_K u(t_1)\varphi(t_1)\,\mathrm{d}x + \int_{t_1}^{t_2}\int_K \left(u\varphi_t + \nabla u^m\cdot\nabla\varphi\right)\,\mathrm{d}x\,\mathrm{d}t = 0,$$

for any test function $\varphi \in W^{1,2}_{loc}\left(0,T;L^2(K)\right)\cap L^2_{loc}\left(0,T;W^{1,2}_0(K)\right)$.

- Comparison principle *does not hold* for local weak solution, no boundary data is specified
- Bounded local weak solutions are continuous in $Q=[0,T]\times\Omega$.
    - Hölder continuity: DiBenedetto et al. 1988, 1992
    - Our results do not depend neither on an explicit modulus of continuity nor on the Hölder continuity.
    - Indeed our Harnack inequalities imply Hölder continuity.
- Using continuity, guaranteed by our sharp local smoothing effects (upper bounds), we can prove a *local comparison argument* that allows us to get sharp lower bounds and as a consequence new forms of Harnack inequalities.

Local comparison argument

Restriction of any continuous local weak solution

$$
\text{(RDP)}\begin{cases} \partial_t u = \Delta u^m\,, & \text{in } (0,T)\times B_{R_0}\,,\\ u(0,\cdot) = u_0\chi_{B_{R_0}} & \text{in } B_{R_0}\,,\\ u(t,x) = u_{\rm loc}(t,x) & \text{in } (0,T)\times \partial B_{R_0} \end{cases}
$$

$u_{\rm loc}$ is the continuous local weak solution under consideration and $u \equiv u_{\rm loc}$

Minimal Dirichlet Problem

$$
\text{(mDP)}\begin{cases} \partial_t \underline{u} = \Delta \underline{u}^m\,, & \text{in } (0,T)\times B_{R_0}\,,\\ \underline{u}(0,\cdot) = u_0\chi_{B_R} & \text{in } B_{R_0}\,,\ 0<2R<R_0\,,\\ \underline{u}(t,x) = 0 & \text{for any } (t,x)\in(0,T)\times \partial B_{R_0} \end{cases}
$$

We can conclude that $\underline{u}(t,x)\le u(t,x)$ on $[0,T)\times B_{R_0}$.
Minimal Life Time: $T_m(\underline{u}(0))\le T(u_0)$.

Maximal Dirichlet Problem - Large Solutions

$$
\text{(MDP)}\begin{cases} \partial_t \overline{u} = \Delta \overline{u}^{\,m}\,, & \text{in } (0,T)\times B_{R_0}\,,\\ \overline{u}(0,\cdot) = u_0\chi_{B_{R_0}} & \text{in } B_{R_0}\,,\\ \overline{u}(t,x) = +\infty & \text{in } (0,T)\times \partial B_{R_0} \end{cases}
$$

We can conclude that $u(t,x)\le \overline{u}(t,x)$ on $[0,T)\times B_{R_0}$

Theorem. (Local Smoothing effects)

Let $r \geq 1$ if $m > m_c = (d-2)/d$ or $r > r_c = d(1-m)/2$ if $m \leq m_c$. Let u be a local weak solution to the FDE in the cylinder $(0,T) \times \Omega \subseteq (0,+\infty) \times \mathbb{R}^d$. Then there are positive constants $\mathcal{C}_1, \mathcal{C}_2$ such that for any $0 < R < \mathrm{dist}(x_0, \partial\Omega)$ we have

$$\sup_{x \in B_{R/2}} u(t,x) \leq \frac{\mathcal{C}_1}{t^{d\vartheta_r}} \left[\int_{B_R} |u_0(x)|^r \,\mathrm{d}x \right]^{2\vartheta_r} + \mathcal{C}_2 \left[\frac{t}{R^2} \right]^{\frac{1}{1-m}} .$$

where $\vartheta_r = 1/(2r - d(1-m)) = 1/2(r - r_c)$, and the constants $\mathcal{C}_i$ depend on m, d and r. We give explicit expression for $\mathcal{C}_i$.

- We recover the well known smoothing effect in $\mathbb{R}^d$ by letting $R \longrightarrow +\infty$
- This result holds for any local weak solution, thus providing (Hölder) continuity of any local weak solutions (c.f. DiBenedetto et al.)
- This result holds for large solutions $\Rightarrow$ existence and interior boundedness.
- Small improvements:
  - Supremum in $[\varepsilon, t] \times B_R$ in the r.h.s , for any $\varepsilon \in (0,t)$.
  - Radii: $R/2$ can be improved up to any $R_0 < R = \mathrm{dist}(x_0, \partial\Omega)$
  - The result extends to $m \leq 0$ and to more general operators of the form $u_t = \nabla \cdot \mathbf{a}(t,x,u,\nabla u)$.
- Similar results previously proved for $m_c < m < 1$, by Herrero - Pierre, DiBenedetto - Gianazza - Vespri, Daskalopulos - Kenig [...]

Theorem. (Asymptotic behaviour of the Large-FDE problem)

Let $r \geq 1$ if $m > m_c = (d-2)/d$ or $r > r_c = d(1-m)/2$ if $m \leq m_c$. Consider the large-FDE problem on a bounded domain $\Omega \subset \mathbb{R}^d$

$$\begin{cases} \partial_t \overline{u} = \Delta \overline{u}^{\,m}\,, & \text{in } (0,T)\times\Omega\,,\\ \overline{u}(0,\cdot) = u_0 & \text{in } \Omega\,,\\ \overline{u}(t,x) = +\infty & \text{in } (0,T)\times\partial\Omega \end{cases}$$

then there exists a continuous local weak solution $\overline{u}$ defined in the cylinder $[0,\infty)\times\Omega \subseteq (0,+\infty)\times\mathbb{R}^d$. Moreover there are positive constants $\mathcal{C}_1, \mathcal{C}_2$ such that

$$\mathcal{C}_2\left[\frac{t}{\mathrm{dist}(x,\partial\Omega)^2}\right]^{\frac{1}{1-m}} \leq \overline{u}(t,x) \leq \frac{\mathcal{C}_1}{t^{d\vartheta_r}}\left[\int_\Omega |\overline{u}_0(x)|^r\,\mathrm{d}x\right]^{2\vartheta_r} + \mathcal{C}_2\left[\frac{t}{\mathrm{dist}(x,\partial\Omega)^2}\right]^{\frac{1}{1-m}}.$$

where $\vartheta_r = 1/(2r - d(1-m)) = 1/2(r-r_c)$, and the constants $\mathcal{C}_i$ depend on m, d and r. We give explicit expression for $\mathcal{C}_i$.

- This is the the sharp asymptotic behaviour of the Large-FDE problem.
- The upper estimates (LSE) imply existence and interior boundedness for the Large-FDE problem.
- The result extends to $m \leq 0$ and to more general operators of the form $u_t = \nabla\cdot\mathbf{a}(t,x,u,\nabla u)$.

Theorem (Aronson-Caffarelli type Estimates)

Let $0 < m < 1$ and let u be a local weak solution to the FDE over $(0,T)\times\Omega$. Let x_0 be a point in Ω such that $B_{5R}\subset\Omega$. Then the following inequality holds for all $0<t<T$

$$R^{-d}\int_{B_R(x_0)} u_0(x)\,\mathrm{d}x \le C_1\,R^{-2/(1-m)}\,t^{\frac{1}{1-m}} + C_2\,T^{\frac{1}{1-m}}\,R^{-2}\,t^{-\frac{m}{1-m}}\,u^m(t,x_0).$$

with C_1 and C_2 given positive constants depending only on d.
If moreover $u_0\in L^{r_c}_{loc}(\Omega)$we have

$$R^{-d}\|u_0\|_{L^1(B_R(x_0))} \le C_1\,R^{-2/(1-m)}\,t^{\frac{1}{1-m}} + C_3\,\|u_0\|_{L^{r_c}(B_R(x_0))}R^{-2}\,t^{-\frac{m}{1-m}}\,u^m(t,x_0).$$

- These estimates are the analogous of the celebrated Aronson-Caffarelli estimates, valid for the slow diffusion case, $m>1$, namely

$$R^{-d}\int_{B_R(x_0)} u_0(x)\,\mathrm{d}x \le C_1\,R^{2/(m-1)}\,t^{-\frac{1}{m-1}} + C_2\,R^{-d}t^{d/2}u^{1+(d(m-1)/2)}(t,x_0).$$

- In the case $m>1$ the AC estimates define the so-called *waiting time*

$$t_c = c(m,d)\|u_0\|^{1-m}_{L^1(B_R(x_0))}R^{2+d(m-1)}.$$

namely a time that we have to wait in order that positivity takes place, in view of the slow diffusion.

Theorem (Local Positivity Estimates)

Let $0 < m < 1$ and let u be a local weak solution to the FDE over $(0,T)\times\Omega$. Let x_0 be a point in Ω such that $B_{6R}(x_0)\subset\Omega$, and let $2R < R_0 \le \mathrm{dist}(x_0,\partial\Omega)$. There exists a time $t_* \in (0,T]$ such that for all $t\in(0,t_*]\subseteq(0,T]$

$$u^m(t,x_0) \ge C_1'\, R^{2-d}\|u_0\|_{L^1(B_R)}T^{-\frac{1}{1-m}}\ t^{\frac{m}{1-m}}.$$

where $C_1' > 0$ depends only on d and *the critical time*

$$t_* \ := k_d\,(R_0-2R)^2\,\mathrm{Vol}\big(B_{R_0}\setminus B_R\big)^{m-1}\|u_0\|_{L^1(B_R)}^{1-m} \le T$$

where $k_d > 0$ depends only on d. If moreover $u_0\in L^r_{loc}(\Omega)$, $r\ge\max\{r_c,1\}$

$$u(t,x_0) \ge C_1'\left[\frac{R^{\frac{d}{r}}\|u_0\|_{L^1(B_R)}}{R^d\|u_0\|_{L^r(B_R)}}\right]^{\frac{1}{m}}\left[\frac{t}{R^2}\right]^{\frac{1}{1-m}}.$$

- The role of the critical time is different from the slow diffusion case
- Positivity without dependence on T or T_m for general initial data is false in general. We provide a counterexample and also DiBenedetto-Gianazza-Vespri.
- The assumption $u_0\in L^r_{loc}$, $r > r_c$ is necessary to avoid T_m in the estimates when $m < m_c$. Since $r_c > 1$ iff $m < m_c$, *upper estimates on the minimal life time T_m* in terms of L^1-norm, are not possible.

The Good Fast Diffusion Range $m_c < m < 1$

(i) Sharp upper and lower estimates for the extinction time for the Dirichlet problem on any ball B_R of the form:

$$c_1 \|u_0\|_{L^1(B_{R/3})}^{1-m} R^{2-d(1-m)} \le T \le c_2 \|u_0\|_{L^1(B_R)}^{1-m} R^{2-d(1-m)}.$$

(ii) In that range of m our lower estimates imply the lower Harnack inequalities of DiBenedetto et al., in the form

$$u(t, x_0) \ge c_{m,d} \left[\frac{t}{R^2}\right]^{\frac{1}{1-m}}$$

for any $x \in B_R$ and any $0 < t < t_* := k_d \,(R_0 - 2R)^2 \,\mathrm{Vol}\big(B_{R_0} \setminus B_R\big)^{m-1} \|u_0\|_{L^1(B_R)}^{1-m}$.

Our upper and lower estimates (+work) imply **Global Harnack Principles:**

- ON DOMAINS Ω. *E. DiBenedetto, Y. C. Kwong and V. Vespri (1992).*
  For any $\varepsilon \in (0, T)$ there exist constants c, C depending only upon d, m, $\|u_0\|_{1+m}$, $\mathrm{diam}(\Omega)$, $\partial\Omega$ and ε, such that for all $(t, x) \in (0, T) \times \Omega$, $t > \varepsilon$

$$c\ \mathrm{dist}(x, \partial\Omega)^{1/m}\ (T - t)^{1/(1-m)} \le u(t, x) \le C\ \mathrm{dist}(x, \partial\Omega)^{1/m}\ (T - t)^{1/(1-m)}$$

- ON THE WHOLE SPACE $\mathbb{R}^d$. *M. B. - J. L. Vázquez, (2006)*
  Let $u_0 \in L^1(\mathbb{R}^d)$, $u_0 \ge 0$ and $u_0(x)\, |x|^{2/(1-m)} \le A$, for any $|x| \ge R_0$. Then, for any $\varepsilon > 0$ there exist constants τ_1, τ_2, M_1 and M_2, such that

$$\mathcal{B}(t - \tau_1, x; M_1) \le u(t, x) \le \mathcal{B}(t + \tau_2, x; M_2), \qquad \forall (t, x) \in (\varepsilon, \infty) \times \mathbb{R}^d$$

where

$$\mathcal{B}(t, x; M) = \left[b_1 \left(M^{m-1} t\right)^{\frac{2}{2-d(1-m)}} + b_2 |x|^2\right]^{\frac{-1}{(1-m)}} t^{\frac{1}{1-m}}, \quad b_i = b_i(m, d).$$

Main Ingredients for Positivity

The Flux Lemma for Minimal Dirichlet Problem

If u is a positive smooth solution of the mDP in $(0,T]\times B_{R_0}$ with extinction time $T>0$. Then, the following estimate holds true

$$k_0\,(R_0-2R)^2\int_{B_{R_0}} u(s,x)\,dx \le \int_s^T\int_{A_0} u^m\,\mathrm{d}x\,\mathrm{d}t,$$

for any $0\le s\le T$, and any $0<2R<R_0$, and for a suitable constant $k_0=k_0(d)$.

Estimates for Extinction Time

$$t_* = k_0\,(R_0-2R)^2\left[\frac{\int_{B_R} u_0\,\mathrm{d}x}{\mathrm{Vol}\big(B_{R_0}\setminus B_R\big)}\right]^{1-m} \le T \le \frac{8\,[d(1-m)-2]\,\mathcal{S}_2^2}{(d-2)^2(1-m)}\|u_0\|_{r_c}^{1-m}$$

Local Aleksandrov Principle for Minimal Dirichlet Problem

For any $t>0$, $0<2R<R_0$ one has $u(t,x_0)\ge u(t,x_2)$ for any $t>0$ and for any $x_2\in B_{R_0}(x_0)\setminus B_{2R_0}(x_0)$. Hence,

$$u(t,x_0)\ge |B_{R_0}(x_0)\setminus B_{2R_0}(x_0)|^{-1}\int_{B_{R_0}(x_0)\setminus B_{2R_0}(x_0)} u(t,x)\,\mathrm{d}x$$

Local Smoothing Effect + Lower Estimates ⇒ Harnack Inequalities

Local Smoothing

$$\sup_{x\in B_{R/2}} u(t,x) \le \mathcal{C}_1 \frac{\|u_0\|^{2r\vartheta_r}_{\mathrm{L}^r(B_R)}}{t^{d\vartheta_r}} + \mathcal{C}_2 \left[\frac{t}{R^2}\right]^{\frac{1}{1-m}}$$

Local Positivity

$$\inf_{x\in B_R} u(t,x) \ge \mathcal{C}_3 \left[\frac{R^{\frac{d}{r}}\|u_0\|_{L^1(B_R)}}{R^d\|u_0\|_{L^r(B_R)}}\right]^{\frac{1}{m}} \times \left[\frac{t}{R^2}\right]^{\frac{1}{1-m}}$$

↘ ↘ ↘ ↙ ↙ ↙

Harnack Inequalities

$$\sup_{x\in B_{R/2}} u(t,x) \le \mathcal{C}_1 \frac{\|u_0\|^{2r\vartheta_r}_{\mathrm{L}^r(B_R)}}{t^{d\vartheta_r}} + \mathcal{C}_4 \left[\frac{R^{\frac{d}{r}}\|u_0\|_{L^r(B_R)}}{R^d\|u_0\|_{L^1(B_R)}}\right]^{\frac{1}{m}} \inf_{x\in B_R} u(t,x)$$

$$\inf_{x\in B_R(x_0)} u\big(t_0\pm\theta, x\big) \ge \mathcal{H}\, u\big(t_0,x_0\big)$$

where $\vartheta_q = 1/(2q-d(1-m))$ for any $q>0$. Moreover $\theta = \delta u\big(t_0,x_0\big)^{1-m}R^2$ and $\mathcal{H} = \mathcal{H}(u_0,t_0,t_*,R)$ are explicitly calculated.

The intriguing realm of Intrinsic Harnack Inequalities

- Classical forms of Harnack inequalities do not hold for the nonlinear diffusion
- The intrinsic geometry enters the game: for the good fast diffusion range we have the result of DiBenedetto and collaborators (1994, 2007).
  There exist positive constants $\overline{c}$ and $\overline{\delta}$ depending only on m, d, such that for all $(t_0, x_0) \in Q = (0,T) \times \Omega$ and all cylinders of the type

  $$I_{8R}(t_0,x_0) = \left(t_0 - \overline{c}\, u(t_0,x_0)^{1-m}(8R)^2,\, t_0 + \overline{c}\, u(t_0,x_0)^{1-m}(8R)^2\right) \times B_{8R}(x_0) \subset Q\,,$$

  we have

  $$\overline{c}\, u(t_0,x_0) \le \inf_{x \in B_R(x_0)} u(t,x)$$

  for all times $t_0 - \overline{\delta}\, u(t_0,x_0)^{1-m} R^2 < t < t_0 + \overline{\delta}\, u(t_0,x_0)^{1-m} R^2$. The constants $\overline{\delta}$ and $\overline{c}$ tend to zero as $m \to 1$ or as $m \to m_c$.
- In the linear case, i.e. $m \to 1$, only forward Harnack inequalities hold:

  $$\overline{c}\, u(t_0,x_0) \le \inf_{x \in B_R(x_0)} u(t_0 + \vartheta, x)$$

  for suitable times $\vartheta = \overline{\delta} R^2$, where $\overline{c}, \overline{\delta} > 0$ depends only on d.

Intrinsic Harnack Inequalities of Forward-Backward-Elliptic type

Let $0 < m < 1$ and consider a local nonnegative weak solution u of the FDE defined in a cylinder $Q = (0, T) \times \Omega$, taking initial data $u(0, x) = u_0(x)$ in $\mathrm{L}^r_{\mathrm{loc}}(\Omega)$, with $r = 1$ if $m_c < m < 1$ or $r > r_c$ if $0 < m \le m_c$. Also, let x_0 be a point in Ω and let $6R \le \mathrm{dist}(x_0, \partial\Omega)$. There exists constants h_1 , h_2 depending only on m, d, p, such that, for any $\varepsilon \in [0, 1]$ the following inequality holds

$$\inf_{x \in B_R(x_0)} u(t \pm \theta, x) \ge h_1\, \varepsilon^{\frac{2p\vartheta_p}{1-m}} \left[\frac{\|u(t_0)\|_{\mathrm{L}^1(B_R)} R^{\frac{d}{p}}}{\|u(t_0)\|_{\mathrm{L}^r(B_R)} R^d} \right]^{2p\vartheta_p + \frac{1}{m}} u(t, x_0)$$

for any

$$t_0 + \varepsilon t_*(t_0) < t \pm \theta < t_0 + t_*(t_0)\,, \qquad t_*(t_0) = h_2\, R^{2-d(1-m)} \|u(t_0)\|^{1-m}_{\mathrm{L}^1(B_R(x_0))}$$

.

- The above estimate is completely of local type, since it involves only local quantities.
- The fact that the intrinsic cylinder $[t_0 + \varepsilon t_*(t_0), t_0 + t_*(t_0)] \times B_{R_0}(x_0)$ is contained in $(0, T] \times \Omega$ is a consequence, not as an hypothesis.
- We have shown that the size of the intrinsic cylinders is proportional to a ratio of local L^p norms. Note that in the supercritical range it simplifies and only depends on the local L^1 norm.

- In the supercritical range $m_c < m < 1$, we can let $r = 1$ to get

$$\inf_{x\in B_R(x_0)} u(t\pm\theta,x) \ge h_1\,\varepsilon^{\frac{2p\vartheta_p}{1-m}} u(t,x_0)$$

for any $t_0+\varepsilon t_*(t_0) < t\pm\theta < t_0+t_*(t_0)$.
We recover the above mentioned results of DiBenedetto et al.
Joining the upper and lower estimates for the Cauchy Problem, we obtain the Global Harnack principle.

- In the subcritical range $0 < m \le m_c$, the Harnack estimates cannot have a universal constant independent of u_0, (counterexamples).

- The quantity ε represents an arbitrary small waiting time, that is needed in order for the regularization to take place and to allow quantitative intrinsic Harnack inequalities.

- Backward Harnack inequalities are a bit surprising, but they reflect a typical feature of the fast diffusion processes, that is the extinction phenomena, namely

$$\inf_{x\in B_R(x_0)} u(t-\theta,x) \ge h_1\,\varepsilon^{\frac{2p\vartheta_p}{1-m}} \left[\frac{\|u(t_0)\|_{\mathrm{L}^1(B_R)} R^{\frac{d}{p}}}{\|u(t_0)\|_{\mathrm{L}^p(B_R)} R^d}\right]^{2p\vartheta_p+\frac{1}{m}} u(t,x_0)$$

for any $t_0+\varepsilon t_*(t_0) < t-\theta < t_0+t_*(t_0)$ This inequality is compatible with the fact that the solution extinguish at some later time, remaining strictly positive before. This backward inequality is typical of singular equation and can not hold for the degenerate -porous media- case $m > 1$, neither for the linear heat equation case, $m = 1$.
The same remark applies for the Elliptic Harnack inequality, that is when $\theta = 0$.

We provide a form of Harnack inequalities of forward, backward and elliptic type, avoiding the intrinsic framework, and the waiting time $\varepsilon \in [0, 1]$.

An alternative form of Harnack Inequalities

Under the running assumptions, there exists positive constants $\mathcal{C}_1$, $\mathcal{C}_2$ and h_2 depending only on m, d and p such that

$$\sup_{x\in B_R} u(t,x) \le \frac{\mathcal{C}_1}{t^{d\vartheta_p}} \|u(t_0)\|_p^{2p\vartheta_p} + \mathcal{C}_2 \left[\frac{\|u(t_0)\|_{\mathrm{L}^p(B_R)} R^d}{\|u(t_0)\|_{\mathrm{L}^1(B_R)} R^{\frac{d}{p}}} \right]^{\frac{1}{m}} \inf_{x\in B_R} u(t\pm\vartheta, x)$$

for any

$$0 \le t_0 < t \pm \vartheta < t_0 + t_*(t_0) \le T\,, \qquad t_*(t_0) = h_2\, R^{2-d(1-m)} \|u(t_0)\|_{\mathrm{L}^1(B_R(x_0))}^{1-m}\,,$$

where $\vartheta_p = 1/(2p - d(1-m))$.

This form of Harnack inequalities implies Hölder continuity.

- THE SIZE OF INTRINSIC CYLINDERS. The new critical time t^*,

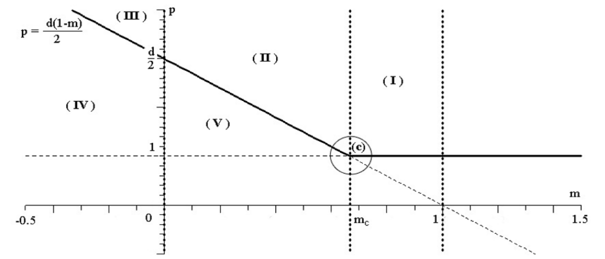
$$t_* := \frac{k_d}{2} \frac{(R_0 - 2R)^2}{\mathrm{Vol}\big(B_{R_0} \setminus B_R\big)^{1-m}} \left[\int_{B_R} u(t_0)\,\mathrm{d}x\right]^{1-m}$$

  gives a quantitative estimate on the maximum size of the intrinsic cylinders:

$$\Big(t_0 - \delta u(t_0,x_0)^{1-m} R^2, t_0 + \delta u(t_0,x_0)^{1-m} R^2\Big) \times B_R(x_0) \subseteq (t_0 - t_*, t_0 + t_*) \times B_R(x_0)$$

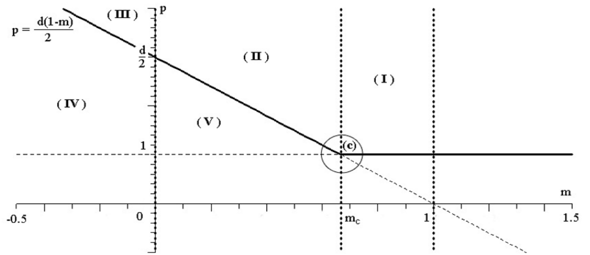
  in the supercritical fast diffusion range this time can be chosen a priori just in terms of the initial datum, but in the subcritical range its size changes with time; the diffusion is so fast that the initial information is lost after some time, which is represented by t^*.

- Never forget that a large class of solutions extinguish in finite time.



(I) *Good Fast Diffusion Range:* $m \in (m_c, 1)$ and $p \geq 1$. Local smoothing effect holds, also Reverse smoothing effect. Thus Intrinsic Harnack Inequalities of various types: Forward in time for small times; Elliptic, backward and forward Harnack Inequalities for intermediate times. For times close to the extinction time, in case extinction occurs, Elliptic Harnack inequalities hold up to extinction. The new results allow to recover the older, with a different proof.

(II) *Very Fast Diffusion Range:* $m \in (0, m_c)$ with local integrability exponent $p \geq p_c > 1$. Local smoothing effect and the Lower estimates hold. For any positive time we have the Aronson-Caffarelli estimates.For for small intrinsic cylinders, we have the Intrinsic Parabolic Harnack inequalities of Elliptic-Backward-Forward type. An open problem is to pass from local to global estimates in this very fast diffusion range.



(c) *Critical case:* $m = m_c$ and $p > p_c = 1$. The local upper and lower estimates of zone (II) apply, and also Harnack inequalities. *Work in progress:* how to pass from local to global estimates, lower bounds with super-exponential time decay.

(III) *Negative exponent range:* $m \le 0$ with $p > p_c$. The Local Smoothing effect still valid, the only known local upper bound. NO positivity result is known in this range, we cannot treat this case. Solutions of the homogeneous Dirichlet problem on bounded domains vanish instantaneously. Open problem: find positivity and a posteriori Harnack inequalities.

(IV) *Negative range:* $m \le 0$ with $p < p_c$. NO Smoothing Effect is true, u_0 not in L^p with $p > p_c$. Sols of the Cauchy problem with data in $L^p(\mathbb{R}^d)$ will vanish instantaneously and also sols of homogeneous Dirichlet problem. Open problem: positivity. NO Harnack inequalities since solution may not be neither locally bounded.

(V) *Very Fast Diffusion Range* $m \in (0, m_c)$ with small integrability exponent $p \in [1, p_c]$. NO smoothing effect (Backward effects), since initial data are not in L^p with $p > p_c$. Lower estimates are as in (II). NO Harnack inequalities since solution may not be neither locally bounded.

Short review of related works

- Bertsch et al. (1990): pressure equation $v = u^{1-m}$ they cover the whole fast diffusion range $0 < m < 1$ in terms of viscosity solutions.
- DiBenedetto, Kwong and Vespri (1991): Dirichlet problem on a bounded domain. Analiticity of positive solutions and Elliptic-Forward Intrinsic Harnack inequalities in the good range $m_c < m < 1$. Global Harnack Principle in the wider range $m_s < m < 1$.
- DiBenedetto and Kwong (1992): Intrinsic Harnack Inequalities of Forward type in the good fast diffusion range $m_c < m < 1$.
- The power $m_s = (d-2)/(d+2)$: Del Pino and Saez (2001), asymptotics of the evolutionary Yamabe problem, Elliptic Harnack inequalities.
- Bonforte, Vázquez (2006), in the good FDE range $m_c < m < 1$. Quantitative Elliptic and Forward Harnack Inequalities, and Global Harnack Principle for Cauchy problem on $\mathbb{R}^d$.
- DiBenedetto, Gianazza and Vespri, (2007). In the good FDE range $m_c < m < 1$: Harnack inequalities of Forward, Elliptic and Backward type for a a class of singular operartors of the form $u_t = \nabla \cdot \mathbf{a}(t, x, u, \nabla u)$.
- M. Bonforte, J. L. Vázquez, *Positivity, local smoothing, and Harnack inequalities for very fast diffusion equations*, Advances in Math. **223** (2010), 529–578.
- DiBenedetto, Gianazza and Vespri, (Preprint, 2009). *Harnack Type Estimates and Hölder Continuity for Non-Negative Solutions to Certain Sub-Critically Singular Parabolic Partial Differential Equations.* Ask Vincenzo details ;-)
- **Problem.** None of the above results considers the problem of finding suitable Harnack inequalities when the time approaches the finite extinction time. For the Dirichlet problem on domains this is has been done in a recent paper by B.-Grillo-Vázquez.

The Fast p-Laplacian $1 < p < 2$ (M.B, R. Iagar, J. L. Vázquez)

M. Bonforte, R. G. Iagar, J. L. Vázquez, *Local smoothing effects, positivity, and Harnack inequalities for the fast p -Laplacian equation*, To appear in Advances in Math. (2010).

Fast p-Laplacian Equation

$$\begin{cases} u_t = \Delta_p(u) = \nabla\cdot\left(|\nabla u|^{p-2}\nabla u\right)\,, & \text{in } Q=(0,T)\times\Omega\subseteq(0,+\infty)\times\mathbb{R}^d\,,\\ u(0,\cdot)=u_0\,, & u_0\in \mathrm{L}^r_{loc}(\Omega) \quad \text{and } 1<p<2 \end{cases}$$

Local Smoothing, Positivity, and Intrinsic Harnack Inequalities similar to the FDE.

Nice local energy inequality

Let u be a continuous local weak solution of the fast p-Laplacian equation over the domain $\Omega\subseteq\mathbb{R}^d$, with $1<p<2$, and let $0\le\varphi\in W^{2,2}_0(\Omega)$ be any admissible test function. Then the following inequality holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^p\varphi\,\mathrm{d}x \le -\frac{p}{d}\int_\Omega(\Delta_p u)^2\varphi\,\mathrm{d}x + \frac{p}{2}\int_\Omega |\nabla u|^{2(p-1)}\Delta\varphi\,\mathrm{d}x,$$

in the sense of distributions in $\mathcal{D}'(0,T)$. This inequality holds also in the case $p=1$.

The above inequality implies that continuous local weak solution are indeed strong solutions; we prove a bit more: $u_t\in\mathrm{L}^2_{\mathrm{loc}}\subset\mathrm{L}^1_{\mathrm{loc}}$. This result answers a question posed by P. Lindqvist about finding an estimate to prove that $u_t\in\mathrm{L}^2_{\mathrm{loc}}$.

THE END

Thank you very much!!