Nonlinear and Nonlocal Diffusions. Smoothing effects, Green functions and functional inequalities

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Nonlocal Equations: Analysis and Numerics
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The Cauchy Problem for Nonlinear Nonlocal Diffusion Equations in $\mathbb{R}^N$

$$\begin{cases} \partial_t u = \mathcal{L}[F(u)]\,, & \text{in } (0,+\infty)\times\mathbb{R}^N \\ u(0,\cdot) = u_0\,, & \text{in } \mathbb{R}^N \end{cases}$$

where:

- The linear operator is allowed to be local, nonlocal, ...
  Think $\mathcal{L} = -\Delta$ or $\mathcal{L} = (-\Delta)^s$, but we shall see many other examples
- The most studied nonlinearity is $F(u) = |u|^{m-1}u\,,$ with $m > 1$.
  We deal with Degenerate diffusion of Porous Medium type.
  More general classes of "degenerate" nonlinearities F are allowed.

The Question

Are solution bounded?

The Precise Question

Under which conditions on $\mathcal{L}$ and F,
$u_0 \in \mathrm{L}^p$ generates a bounded solution?

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Our goal is investigate the relations between

- Smoothing Effects, i.e. $L^p - L^\infty$ estimates (Ultracontractivity, linear eq.)
- Functional inequalities of Gagliardo-Nirenberg-Sobolev type (GNS)

$$\|f\|_{2^*} \le \mathcal{S}\|\nabla f\|_2 \qquad \text{or} \qquad \|f\|_p \le \mathcal{S}\|\mathcal{L}^{\frac{1}{2}} f\|^\theta \|f\|_q^{1-\theta}$$

- "Dual" functional inequalities of Hardy-Littlewood-Sobolev type (HLS)

$$\|\mathcal{L}^{-\frac{1}{2}}\|_2 \le \mathcal{S}\,\|f\|_{(2^*)'}$$

Comparing two methods:

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The Linear case $m = 1$

(HE)
$$\begin{cases} \partial_t u = \Delta u & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}$$

Solutions satisfy the ultracontractive estimates (smoothing effects)

$$\|u(t)\|_\infty \le C \frac{\|u_0\|_1^\alpha}{t^\beta}$$

the powers α, β are fixed by space-time scalings (and mass cons.).
The representation formula makes it easy to prove smoothings

$$|u(x,t)| = \left| \int_{\mathbb{R}^N} u_0(y) H_\Delta(x-y,t)\,\mathrm{d}y \right| \le \overline{\kappa} \frac{\|u_0\|_1}{t^{N/2}}$$

just using the **on diagonal bounds** on $H_{-\Delta}$

$$0 \le H_\Delta(x-y,t) = \frac{e^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^{N/2}} \le \frac{\overline{\kappa}}{t^{N/2}}$$

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The Nash/GNS Inequality via Smoothing Effect.

$$\|f\|_2 \le \mathcal{S}\|\nabla f\|_2^{\theta}\|f\|_1^{1-\theta}$$

Derive the L^2-Norm:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} u(t)^2\,\mathrm{d}x = -2\int_{\mathbb{R}^N}|\nabla u(t)|^2\,\mathrm{d}x \ge -\int_{\mathbb{R}^N}|\nabla u_0|^2\,\mathrm{d}x$$

where the latter follows by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}|\nabla u(t)|^2\,\mathrm{d}x = 2\int_{\mathbb{R}^N}\nabla u\cdot\nabla\partial_t u\,\mathrm{d}x = -\int_{\mathbb{R}^N}(\Delta u)^2\,\mathrm{d}x \le 0$$

Integrating the diff. ineq. and using the smoothing effects we obtain

$$\|u_0\|_2^2 \le t\,\|\nabla u_0\|_2^2 + \|u(t)\|_2^2 \le t\,\|\nabla u_0\|_2^2 + \overline{\kappa}\frac{\|u_0\|_1^2}{t^{N/2}}$$

Optimizing in t gives the Nash inequality for $f = u_0$.

Smoothing Effects via Nash/GNS inequalities

- Nash proved that the smoothing are implied by “his” inequality, using a nice duality trick, exploiting the symmetry of the heat semigroup.
- Moser showed that the symmetry of the semigroup is not needed, if one uses his celebrated iteration.
- **Nash/GNS inequalities and smoothing effects for the HE are equivalent!**

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Integrating the diff. ineq. and using the smoothing effects we obtain

$$\|u_0\|_2^2 \le t\,\|\nabla u_0\|_2^2 + \|u(t)\|_2^2 \le t\,\|\nabla u_0\|_2^2 + \overline{\kappa}\frac{\|u_0\|_1^2}{t^{N/2}}$$

Optimizing in t gives the Nash inequality for $f = u_0$.

Smoothing Effects via Nash/GNS inequalities

- Nash proved that the smoothing are implied by "his" inequality, using a nice duality trick, exploiting the symmetry of the heat semigroup.
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The Nonlinear Case

$$\boxed{\partial_t u = -\mathcal{L} u^m}$$

Nonlinear Nonlocal diffusion.

Nonlinear case ($m > 1$). Introduction

(GPME)
$$\begin{cases} \partial_t u + \mathcal{L}[u^m] = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}$$

Cons:

- We do not have a representation formula.
- It is harder to find the correct functional set-up.

Pros:

- We still have scaling (always time-scaling).
- Some estimates are true in the nonlinear, but not true in the linear.

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Smoothing Effects VS GNS inequalities

Smoothing Effects

$$\|u(t)\|_\infty \le \overline{\kappa}\,\frac{\|u_0\|_p^{2sp\vartheta_p}}{t^{N\vartheta_p}} \qquad \text{with} \qquad \vartheta_p = \frac{1}{2sp - N(1-m)}$$

GNS inequalities

$$\|f\|_{\frac{2q}{q+m-1}} \le \mathcal{S}_{\mathcal{L}}^{\vartheta}\, \|\mathcal{L}^{\frac{1}{2}} u\|_2^{\vartheta} \|f\|_{\frac{2p}{q+m-1}}^{1-\vartheta}$$

The Nash/GNS Inequality via Smoothing Effect. Nonlinear Setting: $\partial_t u = \mathcal{L}u^m$

$$\|f\|_{1+\frac{1}{m}} \le \mathcal{S}\|\mathcal{L}^{\frac{1}{2}}f\|_2^{\theta}\|f\|_{\frac{1}{m}}^{1-\theta}$$

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Smoothing Effects via Nash/GNS inequalities

- We have proved that the smoothing effects are implied by GNS inequalities, using a nonlinear version of Moser iteration or a variant of Gross' method (MB, Grillo, Vazquez 2005-2010)
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Moser Iteration in the nonlinear setting I. Main Ingredients

GNS inequalities: there exists $2^* > 2$ such that the Sobolev-type ineq. holds:

$$\|f\|_{2^*}^2 \le \mathcal{S}_{\mathcal{L}}^2 \, \|\mathcal{L}^{\frac{1}{2}} u\|_2^2 = \mathcal{S}_{\mathcal{L}}^2 \int_\Omega f \mathcal{L} f \, \mathrm{d}x \,.$$

Interpolate to get a family of GNS inequalities: let $p > q > 0$ and

$$\frac{2q}{q+m-1} \le 2^*, \qquad \text{and} \qquad \frac{q+m-1}{2q} = \frac{\vartheta}{2^*} + (1-\vartheta)\frac{q+m-1}{2p} \,,$$

so that

(GNS) $$\|f\|_{\frac{2q}{q+m-1}} \le \|f\|_{2^*}^{\vartheta} \, \|f\|_{\frac{2p}{q+m-1}}^{1-\vartheta} \le \mathcal{S}_{\mathcal{L}}^{\vartheta} \, \|\mathcal{L}^{\frac{1}{2}} u\|_2^{\vartheta} \|f\|_{\frac{2p}{q+m-1}}^{1-\vartheta}$$

Stroock-Varopoulos inequality: that there exists a constant $c_{m,q} > 0$

$$\int_\Omega u^{q-1} \mathcal{L} u^m \, \mathrm{d}x \ge c_{m,q} \int_\Omega u^{\frac{q+m-1}{2}} \mathcal{L} u^{\frac{q+m-1}{2}} \, \mathrm{d}x = c_{m,q} \left\| \mathcal{L}^{1/2} u^{\frac{q+m-1}{2}} \right\|_2^2$$

Combining the two above inequalities, one gets

(M1) $$\int_\Omega u^{q-1} \mathcal{L} u^m \, \mathrm{d}x \ge c_{m,q} \left\| \mathcal{L}^{1/2} u^{\frac{q+m-1}{2}} \right\|_2^2 \ge c_{m,q} \mathcal{S}_{\mathcal{L}}^{-2} \frac{\|u\|_q^{\frac{q+m-1}{2\vartheta}}}{\|u\|_p^{\frac{1-\vartheta}{\vartheta} \frac{q+m-1}{2}}} \,.$$

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Moser Iteration in the nonlinear setting II. $L^p - L^q$ Smoothing Effects.
We shall prove first:

$$(1) \qquad \|u(t)\|_q \le \overline{\kappa}_{p,q} \frac{\|u(t_0)\|_p^{\frac{p\vartheta_p}{q\vartheta_q}}}{(t-t_0)^{\frac{N(q-p)}{q}\vartheta_p}}, \qquad \text{with} \quad \vartheta_r = \frac{1}{2sr + N(1-m)}$$

The proof is formally simple:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^q \,\mathrm{d}x = q\int_\Omega u^{q-1}\partial_t u \,\mathrm{d}x = -q\int_\Omega u^{q-1}\mathcal{L}u^m \,\mathrm{d}x$$

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Then integrate the differential inequality to get (1).

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$$(1) \qquad \|u(t)\|_q \le \overline{\kappa}_{p,q} \frac{\|u(t_0)\|_p^{\frac{p\vartheta_p}{q\vartheta_q}}}{(t-t_0)^{\frac{N(q-p)}{q}\vartheta_p}}, \qquad \text{with} \quad \vartheta_r = \frac{1}{2sr + N(1-m)}$$

The proof is formally simple:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^q \,\mathrm{d}x = q\int_\Omega u^{q-1}\partial_t u \,\mathrm{d}x = -q\int_\Omega u^{q-1}\mathcal{L}u^m \,\mathrm{d}x$$

$$(M1) \quad \le -\mathcal{S}_{\mathcal{L}}^{-2}\frac{4q(q-1)m}{(q+m-1)^2}\frac{\|u\|_q^{\frac{q+m-1}{2\vartheta}}}{\|u\|_p^{\frac{1-\vartheta}{\vartheta}\frac{q+m-1}{2}}}.$$

Then integrate the differential inequality to get (1).

Moser Iteration in the nonlinear setting III. $L^p - L^\infty$ Smoothing Effects.

Rewrite (1) for each $k \geq 1$ with $p_k = 2^k p$ and t_k such that $t_k - t_{k-1} = \frac{t-t_0}{2^k}$,

$$\|u(t_k)\|_{p_k} \leq I_k^{\frac{N(p_k - p_{k-1})}{p_k}\vartheta_{k-1}} \|u(t_{k-1})\|_{p_{k-1}}^{\frac{p_{k-1}\,\vartheta_{k-1}}{p_k\,\vartheta_k}} \qquad \text{with} \qquad I_k \sim \frac{p_k}{t_k - t_{k-1}} \sim 4^k$$

where $\vartheta_k := \vartheta_{p_k} = (2sp_k - N(1-m))^{-1}$. Then we iterate

$$\begin{aligned}
\|u(t_k)\|_{p_k} &\leq I_k^{\frac{N(p_k - p_{k-1})}{p_k}\vartheta_{k-1}} \|u(t_{k-1})\|_{p_{k-1}}^{\frac{p_{k-1}\,\vartheta_{k-1}}{p_k\,\vartheta_k}} \\
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&\leq \prod_{j=1}^{k} I_j^{\frac{N(p_j - p_{j-1})}{p_k}\frac{\vartheta_j\vartheta_{j-1}}{\vartheta_k}} \|u(t_0)\|_q^{\frac{q\,\vartheta_q}{p_k\vartheta_k}} \leq \left[\prod_{j=1}^{k}\left(4^j\frac{\overline{c}}{t-t_0}\right)^{\frac{N(\vartheta_{j-1}-\vartheta_j)}{2s}}\right]^{\frac{1}{p_k\vartheta_k}} \|u(t_0)\|_q^{\frac{q\,\vartheta_q}{p_k\vartheta_k}}.
\end{aligned}$$

Finally, letting $k \to \infty$

$$\|u(t)\|_\infty \leq \lim_{k\to\infty}\|u(t_k)\|_{p_k} \leq \overline{\kappa}\,\frac{\|u(t_0)\|_p^{2sp\,\vartheta_p}}{(t-t_0)^{N\vartheta_p}}.$$

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The Green Function Method

(GPME) $$\begin{cases} \partial_t u = -\mathcal{L}[u^m] & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}$$

We need:

(1a) Dual formulation of the problem

(1b) $\mathcal{L}^{-1}$ with kernel $\mathbb{G}_{\mathcal{L}}(x-y) = \int_0^\infty H_{\mathcal{L}}(x-y,t)\mathrm{d}t$.

(2a) Time scaling. $u_\Lambda(x,t) := \Lambda^{\frac{1}{m-1}} u(x,\Lambda t)$ solution when u is.

(2b) Comparison principle.

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We can formulate a "dual problem", using the inverse $\mathcal{L}^{-1}$ as follows

$$\partial_t U = -F(u)\,,$$

where

$$U(t,x) := \mathcal{L}^{-1}[u(t,\cdot)](x) = \int_{\mathbb{R}^N} \mathbb{G}(x,y)u(t,y)\,\mathrm{d}y\,.$$

In the case of bounded domains, this formulation encodes the lateral boundary conditions in the inverse operator $\mathcal{L}^{-1}$.

Remark. This formulation has been used before by Pierre, Vázquez [...] to prove (in the $\mathbb{R}^N$ case) uniqueness of the "fundamental solution", i.e. the solution corresponding to $u_0 = \delta_{x_0}$,

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Given the "dual problem" $\partial_t U = -F(u)$,

Weak Dual Solutions (WDS)

We say that a nonnegative measurable function u is a *Weak Dual Solution* WDS of (GPME) if:

- $u \in C([0,T]; L^1(\mathbb{R}^N))$ and $u^m \in L^1((0,T); L^1_{\mathrm{loc}}(\mathbb{R}^N))$.
- For all $0 < \tau_1 \le \tau_2 \le T$, and all $\psi \in C^1_c([\tau_1,\tau_2]; L^\infty_c(\mathbb{R}^N))$,

(2) $$\begin{aligned} &\int_{\tau_1}^{\tau_2}\int_{\mathbb{R}^N} \left(\mathcal{L}^{-1}[u]\partial_t\psi - u^m\psi\right) \mathrm{d}x\,\mathrm{d}t \\ &= \int_{\mathbb{R}^N} \mathcal{L}^{-1}[u(\cdot,\tau_2)](x)\psi(x,\tau_2)\,\mathrm{d}x - \int_{\mathbb{R}^N} \mathcal{L}^{-1}[u(\cdot,\tau_1)](x)\psi(x,\tau_1)\,\mathrm{d}x. \end{aligned}$$

- $u(\cdot,0) = u_0$ a.e. in $\mathbb{R}^N$.

- WDS are Very Weak or Distributional Solutions
- Weak, Weak Energy, Mild (Gradient Flow, semigroup), Strong solutions are WDS.

The Green Function Method (1) Dual formulation of the Problem.

(GPME) $$\begin{cases} \partial_t u = -\mathcal{L}[u^m] & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}$$

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Weak Dual Solutions (WDS)

We say that a nonnegative measurable function u is a *Weak Dual Solution* WDS of (GPME) if:

- $u \in C([0,T]; L^1(\mathbb{R}^N))$ and $u^m \in L^1((0,T); L^1_{\text{loc}}(\mathbb{R}^N))$.
- For all $0 < \tau_1 \le \tau_2 \le T$, and all $\psi \in C^1_c([\tau_1,\tau_2]; L^\infty_c(\mathbb{R}^N))$,

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- $u(\cdot,0) = u_0$ a.e. in $\mathbb{R}^N$.

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The Green Function Method (2) Time Monotonicity

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We need:

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(1b) $\mathcal{L}^{-1}$ with kernel $\mathbb{G}_{\mathcal{L}}(x-y) = \int_0^\infty H_{\mathcal{L}}(x-y,t)\mathrm{d}t$.

(2a) Time scaling. $u_\Lambda(x,t) := \Lambda^{\frac{1}{m-1}} u(x,\Lambda t)$ solution when u is.

(2b) Comparison principle.

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Benilan-Crandall Time Monotonicity Estimates

$$\partial_t u \geq -\frac{u}{(m-1)t} \qquad \text{(in the distributional sense)}$$

This is a "weak formulation" of the fact that

$$t \mapsto t^{\frac{1}{m-1}} u(t,x) \qquad \text{is nondecreasing in } t > 0 \text{ for a.e. } x \in \Omega\,.$$

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An "Almost" Representation Formula

Theorem. (First Pointwise Estimates) (M.B. and J. L. Vázquez)

Let $u \geq 0$ be a weak dual solution to Problem (CDP) with $u_0 \in \mathrm{L}^p(\Omega)$, $p > N/2s$. Then,

$$\int_{\mathbb{R}^N} u(t_1,x)\mathbb{G}(x,x_0)\,\mathrm{d}x \leq \int_{\mathbb{R}^N} u(t_0,x)\mathbb{G}(x,x_0)\,\mathrm{d}x\,, \qquad \text{for all } t_1 \geq t_0 \geq 0\,.$$

Moreover, for almost every $0 \leq t_0 \leq t_1$ and almost every $x_0 \in \Omega$, we have

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Sketch of the proof of the First Pointwise Estimates We would like to take as test function

$$\psi(t,x) = \psi_1(t)\psi_2(x) = \chi_{[t_0,t_1]}(t)\,\mathbb{G}(x_0,x)\,,$$

(This is NOT an admissible test in the Definition of WDS: approximation needed)
Idea: plug such test function in very weak formulation, use $\mathcal{L}\mathbb{G}(x_0,\cdot) = \delta_{x_0}$ to get

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This formula can be proven rigorously though careful approximation.
Next, we use the monotonicity estimates, $t \mapsto t^{\frac{1}{m-1}}\,u(t,x)$ non-decreasing

$$\left(\frac{t_0}{t}\right)^{\frac{1}{m-1}} u(t_0) \le u(t) \le u(t_1)\left(\frac{t_1}{t}\right)^{\frac{1}{m-1}}$$

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Absolute bounds when $\mathbb{G}(x_0,\cdot) \in \mathrm{L}^1$
In the case when $\mathbb{G}(x_0,\cdot)$ is globally integrable:

Theorem. (Absolute upper bounds)(M.B. & J. Endal & J. L. Vázquez)

Let u be a WDS, then there exists constants $\overline{\kappa} > 0$ depending only on N, s, m (but not on u_0 !!) , such that

$$\|u(t)\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \le \frac{\overline{\kappa}}{t^{\frac{1}{m-1}}}, \qquad \text{for all } t > 0 .$$

- This is a very strong regularization *independent* of the initial datum u_0.
- Time decay is sharp, but only for large times, say $t \ge 1$. For small times when $0 < t < 1$ a better time decay is obtained in the form of smoothing effects.

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Sketch of the proof of Absolute Bounds on bounded domains

Assume that we are on bounded domains

$$0 \le \mathbb{G}(x_0, x) \le \frac{c_\Omega}{|x - x_0|^{N-2s}} \quad \text{hence} \quad \sup_{x_0 \in \Omega} \mathbb{G}(x_0, \cdot) \in L^q(\Omega) \text{ with } q < \frac{N}{N-2s}$$

• STEP 1. *First upper estimates.* Recall the pointwise estimate:

$$\left(\frac{t_0}{t_1}\right)^{\frac{m}{m-1}} (t_1 - t_0)\, u^m(t_0, x_0) \le \int_\Omega u(t_0, x)\mathbb{G}(x, x_0)\,dx - \int_\Omega u(t_1, x)\mathbb{G}(x, x_0)\,dx\,.$$

for any $u \in L^p$, $p > N/2s$ all $0 \le t_0 \le t_1$ and all $x_0 \in \Omega$. Choose $t_1 = 2t_0$ to get

$$(*) \qquad \boxed{u^m(t_0, x_0) \le \frac{2^{\frac{m}{m-1}}}{t_0} \int_\Omega u(t_0, x)\mathbb{G}(x, x_0)\,dx\,.}$$

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Nonlinear case 2.0: Using Green function of $I+\mathcal{L}$ (MB & J. Endal)
Consider the operator $\mathcal{L} \mapsto I+\mathcal{L}$, i.e.,

$$\partial_t u + (I+\mathcal{L})[u^m] = 0 \qquad \Longleftrightarrow \qquad \partial_t u + \mathcal{L}[u^m] = -u^m.$$

x-independent supersolution:
$t \mapsto Y(t)$ solves $Y'(t) = -Y(t)^{1+(m-1)}$, hence

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Holds independently of the operator! But needs "good" nonlinearity.

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$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq Y(t) \leq \frac{c_m}{t^{\frac{1}{m-1}}}.$$

Holds independently of the operator! But needs "good" nonlinearity.

L. VÉRON. Effets régularisants de semi-groupes non linéaires dans des espaces de Banach. *Ann. Fac. Sci. Toulouse Math. (5)*, 1(2):171–200, 1979.

General Assumptions

$$(\mathrm{G}_1)\quad \begin{cases} \int_{B_R(x_0)} \mathbb{G}_{\mathcal{L}}^{x_0}(x)\,\mathrm{d}x \le K_1 R^{\alpha} & \text{for all } R>0 \text{ and some } \alpha\in(0,2],\\ \mathbb{G}_{\mathcal{L}}^{x_0}(x) \le K_2 R^{-(N-\alpha)} & \text{when } x\in\mathbb{R}^N\setminus B_R(x_0). \end{cases}$$

$$(\mathrm{G}_1')\quad \begin{cases} \int_{B_R(x_0)} \mathbb{G}_{\mathcal{L}}^{x_0}(x)\,\mathrm{d}x \le K_1 R^{\alpha} & \text{for all } R>0 \text{ and some } \alpha\in(0,2],\\ \mathbb{G}_{\mathcal{L}}^{x_0}(x) \le K_3 & \text{when } x\in\mathbb{R}^N\setminus B_R(x_0). \end{cases}$$

$$(\mathrm{G}_2)\quad \|\mathbb{G}_{\mathcal{L}}^{x_0}\|_{L^1(\mathbb{R}^N)} = \|\mathbb{G}_{\mathcal{L}}^{0}\|_{L^1(\mathbb{R}^N)} \le C_1 < \infty.$$

$$(\mathrm{G}_3)\quad \|\mathbb{G}_{I+\mathcal{L}}^{x_0}\|_{L^p(\mathbb{R}^N)} = \|\mathbb{G}_{I+\mathcal{L}}^{0}\|_{L^p(\mathbb{R}^N)} \le C_p < \infty \text{ for some } p\in(1,\infty).$$

Notation. We systematically identify $\alpha = 2s$

Theorem (L^1–L^∞-smoothing under (G_1) MB & J. Endal)

Let u be a weak dual solution of (GPME) with initial data u_0. If (G_1) hold with $\alpha \in (0,2)$ when $0 < R \le 1$ and with $\alpha = 2$ when $R > 1$, then:

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le \tilde{C}(m) \begin{cases} t^{-N\theta_\alpha} \|u_0\|_{L^1(\mathbb{R}^N)}^{\alpha\theta_\alpha} & \text{if } 0 < t \le \|u_0\|_{L^1(\mathbb{R}^N)}^{-(m-1)}, \\ t^{-N\theta_2} \|u_0\|_{L^1(\mathbb{R}^N)}^{2\theta_2} & \text{if } t > \|u_0\|_{L^1(\mathbb{R}^N)}^{-(m-1)}, \end{cases}$$

where $\theta_\alpha = (\alpha + N(m-1))^{-1}$ (defined for $\alpha \in (0,2]$)

If moreover we have scaling in x properties of $\mathcal{L}$, $\mathcal{L}[u(rx)] = r^\alpha \mathcal{L}[u](x)$ then we can prove the stronger smoothing

$$\|u(t)\|_\infty \le \overline{\kappa} \frac{\|u_0\|_1^{\alpha\vartheta_\alpha}}{t^{N\vartheta_\alpha}}$$

Theorem (L^1–L^∞-smoothing under (G_1), (G_1') MB & J. Endal)

Let u be a weak dual solution of (GPME) *with initial data u_0.*

- (a) *If* (G_1) *hold, then:*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C(m,\alpha,N)}{t^{N\theta_\alpha}} \|u_0\|_{L^1(\mathbb{R}^N)}^{\alpha\theta_\alpha} \qquad \textit{for all } t>0,$$

where $\theta_\alpha := (\alpha + N(m-1))^{-1}$

- (b) *If* (G_1') *hold, then:*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{\tilde{C}(m)}{t^{\frac{1}{m}}} \|u_0\|_{L^1(\mathbb{R}^N)}^{\frac{1}{m}} \qquad \textit{for all } t>0,$$

Theorem (L^1–L^∞-smoothing under (G_3) MB & J. Endal)

Let u be a weak dual solution of (GPME) *with initial data u_0. If* (G_3) *hold, then:*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq \begin{cases} c_m t^{-\frac{1}{m-1}} & \text{if } 0 < t \leq t_0, \\ c_2 \|u_0\|_{L^1(\mathbb{R}^N)} & \text{if } t > t_0, \end{cases}$$

where

$$t_0 := c_0 \|u_0\|_{L^1(\mathbb{R}^N)}^{-(m-1)}$$

We can also rewrite it as

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{c_m'}{t^{\frac{1}{m-1}}} + c_2' \|u_0\|_{L^1(\mathbb{R}^N)}$$

Nonlinear case ($m > 1$). Examples

$$\partial_t u = -\mathcal{L}[u^m]$$

- $\mathcal{L} = (-\Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (0,2]$ gives

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \lesssim \frac{\|u_0\|_{L^1(\mathbb{R}^N)}^{\alpha\theta}}{t^{-N\theta}} \qquad \text{where } \theta := (\alpha + N(m-1))^{-1}.$$

A. de Pablo, F. Quirós, A. Rodríguez, and J. L. Vázquez. A general fractional porous medium equation. *Comm. Pure Appl. Math.*, 65(9):1242–1284, 2012.

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Here, $\mathbb{G}_{\mathcal{L}} \equiv |x - x_0|^{-(N-\alpha)}$. The "standard" case (also coeff. are allowed).

- $\mathcal{L} = (\kappa^2 I - \Delta)^{\frac{\alpha}{2}} - \kappa^\alpha I$ with $\kappa > 0$ and $\alpha \in (0, 2)$ gives

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- $\mathcal{L} = \sum_{i=1}^N (-\partial^2_{x_i x_i})^{\frac{\alpha}{2}}$ with $\alpha \in (0, 2)$ gives

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Nonlinear does *not* imply linear

Consider

$$\partial_t u + \mathcal{L}[u^m] = 0,$$

with

$$\mathcal{L}[\psi](x) = \psi(x) - \int_{\mathbb{R}^N} \psi(z)J(x-z)\mathrm{d}z = (I - J*_x)[\psi](x)$$

where $J \geq 0$ such that $\|J\|_{L^1(\mathbb{R}^N)} = 1$ and $J \in L^p(\mathbb{R}^N)$.

- If $m = 1$, then

$$u(x,t) = u_0(x)e^{-t} + W(x,t),$$

where $W \geq 0$ is some smooth function. Hence, no smoothing.

F. Andreu-Vaillo, J. M. Mazón, J. D. Rossi, J. Toledo-Melero. *Nonlocal diffusion problems*. Mathematical Surveys and Monographs, volume 165. American Mathematical Society, Providence, RI; Real Sociedad Matemática Española, Madrid, 2010.

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Theorem (MB & J. Endal)

Assume $m > 1$ *and* $0 \le u_0 \in L^1(\mathbb{R}^N)$*. Then solutions* u *of* (GPME) *are bounded when* $t > 0$ *in the following cases:*

1. (Linear implies nonlinear) *The operator* $\mathcal{L}$ *is such that* $H_{\mathcal{L}}^{x_0}$ *satisfies*

$$\|H_{\mathcal{L}}^{x_0}(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le C(t) \qquad \text{and} \qquad \int_0^\infty e^{-t} C(t)^{\frac{p-1}{p}}\,\mathrm{d}t < \infty.$$

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On negatively curved manifolds...

Theorem (Smoothing effects on M (E. Berchio, MB, G. Grillo, M. Muratori))

Let u be the WDS to $u_t = -(-\Delta_M)^s u^m$, corresponding to any nonnegative initial datum $u_0 \in L^1(M)$. Then there exists $C = C(N, k, c, s, m) > 0$ such that

$$(3) \quad \|u(t)\|_{L^\infty(M)} \le C \left(\frac{\|u(t)\|_{L^1(M)}^{2s\vartheta_1}}{t^{N\vartheta_1}} \vee \|u_0\|_{L^1(M)} \right) \le C \left(\frac{\|u_0\|_{L^1(M)}^{2s\vartheta_1}}{t^{N\vartheta_1}} \vee \|u_0\|_{L^1(M)} \right) .$$

If, in addition, M is Cartan-Hadamard with negative curvature, then for some $C = C(N, s, m) > 0$ we have

$$(4) \quad \|u(t)\|_{L^\infty(M)} \le C \frac{\|u(t)\|_{L^1(M)}^{2s\vartheta_1}}{t^{N\vartheta_1}} \le C \frac{\|u_0\|_{L^1(M)}^{2s\vartheta_1}}{t^{N\vartheta_1}} ,$$

Furthermore, if M has negative sectional curvature, (and $u_0 \not\equiv 0$), then
(5)

$$\|u(t)\|_{L^\infty(M)} \le \frac{C}{t^{\frac{1}{m-1}}} \left[\log\left(t\, \|u_0\|_{L^1(M)}^{m-1} \right) \right]^{\frac{s}{m-1}} \qquad \forall t \ge e^{(N-1)(m-1)\sqrt{\mathsf{c}}}\, \|u_0\|_{L^1(M)}^{-(m-1)} ,$$

for another $C = C(N, s, \mathsf{c}, m) > 0$.

The End

Thank You!!!

Grazie Mille!!!

Dankeschön!!!

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- *See my web-page for slides and some videos:* `http://verso.mat.uam.es/~matteo.bonforte`