Nonlinear and Nonlocal Degenerate Diffusions on Bounded Domains

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References:


[BFV1] M. B., A. FIGALLI, J. L. VÁZQUEZ, Sharp boundary estimates and higher regularity for nonlocal porous medium-type equations in bounded domains. 
*To Appear in Analysis & PDE*. https://arxiv.org/abs/1610.09881


A talk more focussed on the first three papers is available online: 
Outline of the talk

- **Introduction**
  - The Parabolic problem
  - Assumptions on the (inverse) operator
  - Boundary behaviour Linear Elliptic problem
  - Some important examples

- **Semilinear Elliptic Equations**
  - Sharp boundary behaviour for Semilinear Elliptic equations
  - Parabolic solutions by separation of variables

- **Back to the Parabolic problem**
  - (More) Assumptions on the operator
  - Basic theory: existence, uniqueness and boundedness
  - Elliptic VS Parabolic: Asymptotic Behaviour as $t \to \infty$

- **Sharp Boundary Behaviour**
  - Upper Boundary Estimates
  - Infinite Speed of Propagation
  - Lower Boundary Estimates
  - Harnack-type Inequalities
  - Numerics

- **Regularity Estimates**
Homogeneous Dirichlet Problem for Fractional Nonlinear Degenerate Diffusion Equations

\[
\begin{aligned}
\text{(HDP)} \quad \begin{cases}
    u_t + \mathcal{L} F(u) = 0, & \text{in } (0, +\infty) \times \Omega \\
    u(0, x) = u_0(x), & \text{in } \Omega \\
    u(t, x) = 0, & \text{on the lateral boundary.}
\end{cases}
\end{aligned}
\]

where:

- $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N \geq 1$.
- The linear operator $\mathcal{L}$ will be:
  - sub-Markovian operator
  - densely defined in $L^1(\Omega)$.

A wide class of linear operators fall in this class: all fractional Laplacians on domains.

- The most studied nonlinearity is $F(u) = |u|^{m-1}u$, with $m > 1$.

We deal with Degenerate diffusion of Porous Medium type. More general classes of “degenerate” nonlinearities $F$ are allowed.

- The homogeneous boundary condition is posed on the lateral boundary, which may take different forms, depending on the particular choice of the operator $\mathcal{L}$. 
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- The homogeneous boundary condition is posed on the lateral boundary, which may take different forms, depending on the particular choice of the operator $\mathcal{L}$. 
Assumptions on the inverse of $\mathcal{L}$

The linear operator $\mathcal{L} : \text{dom}(A) \subseteq L^1(\Omega) \to L^1(\Omega)$ is assumed to be densely defined and *sub-Markovian*, more precisely satisfying (A1) and (A2) below:

(A1) $\mathcal{L}$ is $m$-accretive on $L^1(\Omega)$,
(A2) If $0 \leq f \leq 1$ then $0 \leq e^{-t\mathcal{L}}f \leq 1$.

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We will assume that the operator $\mathcal{L}$ has an inverse $\mathcal{L}^{-1} : L^1(\Omega) \to L^1(\Omega)$ with a kernel $\mathcal{G}$ - the Green function - such that

$$\mathcal{L}^{-1}f(x) = \int_{\Omega} \mathcal{G}(x, y) f(y) \, dy,$$

and that satisfies (one of) the following estimates for some $\gamma, s \in (0, 1]$

(K1) \[0 \leq \mathcal{G}(x, y) \leq \frac{c_{1,\Omega}}{|x - y|^{N-2s}}\]

Assumption (K1) implies that $\mathcal{L}^{-1}$ is compact on $L^2(\Omega)$ and has discrete spectrum.

(K2) \[c_{0,\Omega}\delta^\gamma(x) \delta^\gamma(y) \leq \mathcal{G}(x, y) \leq \frac{c_{1,\Omega}}{|x - y|^{N-2s}} \left( \frac{\delta^\gamma(x)}{|x - y|^\gamma} \wedge 1 \right) \left( \frac{\delta^\gamma(y)}{|x - y|^\gamma} \wedge 1 \right)\]

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Boundary behaviour for Elliptic equations
We always assume that $\mathcal{L}$ satisfies (A1), (A2) and zero Dirichlet boundary conditions.

**The Linear Problem** $\mathcal{L}v = f$ with $f \in L^{q'}(\Omega)$

Let $G$ be the kernel of $\mathcal{L}^{-1}$, and assume (K2) and that $0 \leq f \in L^{q'}$ with $q' > N/2s$. Then $q = \frac{q'}{q'-1} \in \left(0, \frac{N}{N-2s}\right)$ and the (weak dual) solution $v \geq 0$ satisfies $\forall x \in \Omega$

\[
\|f\|_{L^{q'}} \leq v(x) \leq \left\|f\right\|_{L^{q'}} \begin{cases}
\delta(x)^\gamma, & 0 < q \in \left(0, \frac{N}{N-2s+\gamma}\right), \\
\delta(x)^\gamma \left(1 + \left|\log \delta(x)\right|\right)^{\frac{1}{q}}, & q = \frac{N}{N-2s+\gamma}, \\
\delta(x)^{\frac{N-q(N-2s)}{q}}, & q \in \left(\frac{N}{N-2s+\gamma}, \frac{N}{N-2s}\right).
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**The Eigenvalue Problem** $\mathcal{L}\Phi_k = \lambda_k \Phi_k$

Assumption (K1) implies that $\mathcal{L}^{-1}$ is compact in $L^2(\Omega)$.
Hence the operator $\mathcal{L}$ has a discrete spectrum $(\lambda_k, \Phi_k)$ and $\Phi_k \in L^\infty(\Omega)$.
If we assume moreover that $\mathcal{L}^{-1}$ satisfies (K2) we have that

$\Phi_1 \approx \text{dist}(\cdot, \partial\Omega)^\gamma = \delta^\gamma$ and $\left|\Phi_k\right| \lesssim \text{dist}(\cdot, \partial\Omega)^\gamma = \delta^\gamma$
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Some remarks about boundary behaviour for Elliptic equations

Assuming (K2), that we recall here: \[ \text{[recall } \text{dist}(\cdot, \partial\Omega) \gamma = \delta \gamma] \]

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Consider for simplicity \( \mathcal{L}v = f \in L^\infty(\Omega) \geq 0 \), hence \( q = 1 \). Then we have:

\[ \|f\|_{L_1^{\delta \gamma}} \delta(x)^\gamma \lesssim v(x) \lesssim \|f\|_{L^\infty} \begin{cases} \delta(x)^\gamma, & 2s > \gamma, \\ \delta(x)^\gamma \left( 1 + |\log \delta(x)| \right), & 2s = \gamma, \\ \delta(x)^{2s}, & 2s < \gamma. \end{cases} \]

The boundary behaviour may change depending on the relation between \( 2s \) and \( \gamma \).

On the other hand, for eigenfunctions we always have

\[ \Phi_1 \asymp \text{dist}(\cdot, \partial\Omega) \gamma = \delta \gamma \quad \text{and} \quad |\Phi_k| \lesssim \text{dist}(\cdot, \partial\Omega) \gamma = \delta \gamma \]

This reveals a deep and strong difference in the boundary behaviour, typical of the different definitions of Fractional Laplacians on domains.

Many “nonlocal” results by Cabré, Caffarelli, Capella, Davila, Dupaigne, Grubb, Kassmann, Ros-Oton, Serra, Silvestre, Sire, Stinga, Torrea [...]
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Reminder about the fractional Laplacian operator on $\mathbb{R}^N$

We have several equivalent definitions for $(-\Delta_{\mathbb{R}^N})^s$:

1. **By means of Fourier Transform**, 
   \[
   \left((-\Delta_{\mathbb{R}^N})^s f\right)(\xi) = |\xi|^{2s} \hat{f}(\xi).
   \]
   This formula can be used for positive and negative values of $s$.

2. **By means of an Hypersingular Kernel**: 
   if $0 < s < 1$, we can use the representation
   \[
   (-\Delta_{\mathbb{R}^N})^s g(x) = c_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{g(x) - g(z)}{|x - z|^{N+2s}} \, dz,
   \]
   where $c_{N,s} > 0$ is a normalization constant.

3. **Spectral definition**, in terms of the heat semigroup associated to the standard Laplacian operator:
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   (-\Delta_{\mathbb{R}^N})^s g(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_{\mathbb{R}^N}} g(x) - g(x)\right) \frac{dt}{t^{1+s}}.
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Examples of operators $\mathcal{L}$

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The Spectral Fractional Laplacian operator (SFL)

\[
(-\Delta_{\Omega})^s g(x) = \sum_{j=1}^{\infty} \lambda_j^s \hat{g}_j \phi_j(x) = \frac{1}{\Gamma(-s)} \int_0^{\infty} \left( e^{t\Delta_{\Omega}} g(x) - g(x) \right) \frac{dt}{t^{1+s}}.
\]

- \(\Delta_{\Omega}\) is the classical Dirichlet Laplacian on the domain \(\Omega\)
- **Eigenvalues:** \(0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \leq \lambda_{j+1} \leq \ldots\) and \(\lambda_j \asymp j^{2/N}\).
- **Eigenfunctions:** \(\phi_j\) are the eigenfunctions of the classical Laplacian \(\Delta_{\Omega}\):

\[
\phi_1 \asymp \text{dist}(\cdot, \partial \Omega) \quad \text{and} \quad |\phi_j| \lesssim \text{dist}(\cdot, \partial \Omega),
\]

and \(\phi_j\) are as smooth as \(\partial \Omega\) allows: \(\partial \Omega \in C^k \Rightarrow \phi_j \in C^\infty(\Omega) \cap C^k(\overline{\Omega})\)

\[
\hat{g}_j = \int_{\Omega} g(x) \phi_j(x) \, dx,
\]

with \(\|\phi_j\|_{L^2(\Omega)} = 1\).

The Green function of SFL satisfies a stronger assumption than (K2) or (K3), i.e.

\[
(K4) \quad G(x, y) \asymp \frac{1}{|x - y|^{N-2s}} \left( \frac{\delta^\gamma(x)}{|x - y|^\gamma} \wedge 1 \right) \left( \frac{\delta^\gamma(y)}{|x - y|^\gamma} \wedge 1 \right), \quad \text{with} \quad \gamma = 1
\]

Lateral boundary conditions for the SFL

\[
u(t, x) = 0, \quad \text{in} \ (0, \infty) \times \partial \Omega.
\]
The Spectral Fractional Laplacian operator (SFL)

\[ (-\Delta_{\Omega})^s g(x) = \sum_{j=1}^{\infty} \lambda_j^s \hat{g}_j \phi_j(x) = \frac{1}{\Gamma(-s)} \int_{0}^{\infty} \left( e^{t\Delta_{\Omega}} g(x) - g(x) \right) \frac{dt}{t^{1+s}}. \]

- \( \Delta_{\Omega} \) is the classical Dirichlet Laplacian on the domain \( \Omega \)
- **Eigenvalues:** \( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \leq \lambda_{j+1} \leq \ldots \) and \( \lambda_j \asymp j^{2/N} \).
- **Eigenfunctions:** \( \phi_j \) are the eigenfunctions of the classical Laplacian \( \Delta_{\Omega} \):
  \[
  \phi_1 \asymp \text{dist}(\cdot, \partial \Omega) \quad \text{and} \quad |\phi_j| \lesssim \text{dist}(\cdot, \partial \Omega),
  \]
and \( \phi_j \) are as smooth as \( \partial \Omega \) allows:
  \[
  \partial \Omega \in C^k \Rightarrow \phi_j \in C^\infty(\Omega) \cap C^k(\overline{\Omega})
  \]
  \[
  \hat{g}_j = \int_{\Omega} g(x) \phi_j(x) \, dx, \quad \text{with} \quad \|\phi_j\|_{L^2(\Omega)} = 1.
  \]

The Green function of SFL satisfies a stronger assumption than (K2) or (K3), i.e.

(K4) \[ \mathcal{G}(x, y) \asymp \frac{1}{|x - y|^{N-2s}} \left( \frac{\delta^\gamma(x)}{|x - y|^{\gamma}} \wedge 1 \right) \left( \frac{\delta^\gamma(y)}{|x - y|^{\gamma}} \wedge 1 \right), \]

with \( \gamma = 1 \).

**Lateral boundary conditions for the SFL**

\[ u(t, x) = 0, \quad \text{in} \ (0, \infty) \times \partial \Omega. \]
Definition via the hypersingular kernel in $\mathbb{R}^N$, “restricted” to functions that are zero outside $\Omega$.

**The (Restricted) Fractional Laplacian operator (RFL)**

$$( -\Delta |_{\Omega} )^s g(x) = c_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{g(x) - g(z)}{|x - z|^{N+2s}} \, dz , \quad \text{with supp}(g) \subseteq \overline{\Omega}.$$  

where $s \in (0, 1)$ and $c_{N,s} > 0$ is a normalization constant.

- $( -\Delta |_{\Omega} )^s$ is a self-adjoint operator on $L^2(\Omega)$ with a discrete spectrum:
  - **EIGENVALUES**: $0 < \bar{\lambda}_1 \leq \bar{\lambda}_2 \leq \ldots \leq \bar{\lambda}_j \leq \bar{\lambda}_{j+1} \leq \ldots$ and $\bar{\lambda}_j \asymp j^{2s/N}$.
  - Eigenvalues of the RFL are smaller than the ones of SFL: $\bar{\lambda}_j \leq \lambda_j^s$ for all $j \in \mathbb{N}$.
  - **EIGENFUNCTIONS**: $\phi_j \in C^s(\overline{\Omega}) \cap C^\infty(\Omega)$ (J. Serra - X. Ros-Oton), and $\phi_1 \asymp \text{dist}(\cdot, \partial\Omega)^s$ and $|\phi_j| \lesssim \text{dist}(\cdot, \partial\Omega)^s$.

The Green function of RFL satisfies a stronger assumption than (K2) or (K3), i.e.

$$(K4) \quad G(x, y) \asymp \frac{1}{|x - y|^{N-2s}} \left( \frac{\delta^\gamma(x)}{|x - y|^{\gamma}} \wedge 1 \right) \left( \frac{\delta^\gamma(y)}{|x - y|^{\gamma}} \wedge 1 \right) , \quad \text{with } \gamma = s$$

**Lateral boundary conditions for the RFL**

$$u(t, x) = 0 , \quad \text{in } (0, \infty) \times (\mathbb{R}^N \setminus \Omega) .$$

Definition via the hypersingular kernel in $\mathbb{R}^N$, “restricted” to functions that are zero outside $\Omega$.

**The (Restricted) Fractional Laplacian operator (RFL)**

$$
(-\Delta|_{\Omega})^s g(x) = c_{N,s}, \text{ P.V. } \int_{\mathbb{R}^N} \frac{g(x) - g(z)}{|x - z|^{N+2s}} \, dz,
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Examples of operators $\mathcal{L}$

Introduced in 2003 by Bogdan, Burdzy and Chen.

**Censored (Regional) Fractional Laplacians (CFL)**

$$\mathcal{L}f(x) = \text{P.V.} \int_{\Omega} \frac{f(x) - f(y)}{|x - y|^{N + 2s}} \, dy, \quad \text{with} \quad \frac{1}{2} < s < 1,$$

- It is a self-adjoint operator on $L^2(\Omega)$ with a discrete spectrum $(\lambda_j, \phi_j)$
- **EIGENFUNCTIONS:** $\overline{\phi}_j \in C^{s-1/2}(\Omega) \cap C^{2s+\alpha}(\Omega)$ (MB, A.Figalli, J. L. Vázquez)

$$\phi_1 \asymp \text{dist}(\cdot, \partial \Omega)^{s-\frac{1}{2}} \quad \text{and} \quad |\phi_j| \lesssim \text{dist}(\cdot, \partial \Omega)^{s-\frac{1}{2}},$$

The Green function $G(x, y)$ satisfies $(K4)$ (Chen, Kim and Song (2010))

$$G(x, y) \asymp \frac{1}{|x - y|^{N-2s}} \left( \frac{\delta^\gamma(x)}{|x - y|^{\gamma}} \wedge 1 \right) \left( \frac{\delta^\gamma(y)}{|x - y|^{\gamma}} \wedge 1 \right), \quad \text{with} \quad \gamma = s - \frac{1}{2}.$$  

**Remarks.**

- This is a third model of Dirichlet fractional Laplacian **not equivalent** to SFL nor to RFL.
- Roughly speaking, $s \in (0, 1/2]$ corresponds to Neumann boundary conditions.
- We can allow “coefficients”, i.e. replace $K(x, y) \asymp a(x, y)|x - y|^{N-2s}$ where $a(x, y)$ is a measurable, symmetric function bounded between two positive constants, and $|a(x, y) - a(x, x)| \chi_{|x-y|<1} \lesssim |x - y|^{\sigma}$, with $0 < s < \sigma \leq 1$.
Censored (Regional) Fractional Laplacians (CFL)

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\mathcal{L}f(x) = \text{P.V.} \int_{\Omega} \frac{f(x) - f(y)}{|x - y|^{N+2s}} \, dy, \quad \text{with} \quad \frac{1}{2} < s < 1,
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The Green function $\mathcal{G}(x, y)$ satisfies $(K4)$ (Chen, Kim and Song (2010))

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\[ a(x, y) \leq a(x, x) \chi_{|x-y|<1} \lesssim |x - y|^{\frac{\sigma}{2}}, \quad \text{with} \quad \sigma \leq 1 \].
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\begin{tcolorbox}[colback=orange!7!white]
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\[ \int_{\Omega} \frac{f(x) - f(y)}{|x - y|^{N+2s}} \, dy, \quad \text{with} \quad \frac{1}{2} < s < 1, \]
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Spectral powers of uniformly elliptic operators. Consider a linear operator $A$ in divergence form, with uniformly elliptic bounded measurable coefficients:

$$A = \sum_{i,j=1}^{N} \partial_i (a_{ij} \partial_j) , \quad s\text{-power of } A \text{ is: } \mathcal{L}f(x) := A^s f(x) := \sum_{k=1}^{\infty} \lambda_k^s \hat{f_k} \phi_k(x)$$

$\mathcal{L} = A^s$ satisfies (K3) estimates with $\gamma = 1$

$$(\text{K3}) \quad c_{0,\Omega} \phi_1(x) \phi_1(y) \leq \mathcal{G}(x, y) \leq \frac{c_{1,\Omega}}{|x - y|^{N-2s}} \left( \frac{\phi_1(x)}{|x - y|} \wedge 1 \right) \left( \frac{\phi_1(y)}{|x - y|} \wedge 1 \right)$$

[General class of intrinsically ultra-contractive operators, Davies and Simon JFA 1984].

Fractional operators with “rough” kernels. Integral operators of Levy-type

$$\mathcal{L}f(x) = \text{P.V.} \int_{\mathbb{R}^N} (f(x + y) - f(y)) \frac{a(x, y)}{|x - y|^{N+2s}} \, dy.$$

where $K$ is measurable, symmetric, bounded between two positive constants, and

$$|a(x, y) - a(x, x)| \chi_{|x - y|<1} \leq c|x - y|^\sigma , \quad \text{with } 0 < s < \sigma \leq 1 ,$$

for some positive $c > 0$. We can allow even more general kernels.

The Green function satisfies a stronger assumption than (K2) or (K3), i.e.

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Sums of two Restricted Fractional Laplacians. Operators of the form

$$\mathcal{L} = (\Delta|_{\Omega})^s + (\Delta|_{\Omega})^{\sigma}, \quad \text{with} \ 0 < \sigma < s \leq 1,$$

where $(\Delta|_{\Omega})^s$ is the RFL. Satisfy $(K4)$ with $\gamma = s$.

Sum of the Laplacian and operators with general kernels. In the case

$$\mathcal{L} = a\Delta + A_s, \quad \text{with} \ 0 < s < 1 \quad \text{and} \quad a \geq 0,$$

where

$$A_sf(x) = \text{P.V.} \int_{\mathbb{R}^N} (f(x+y) - f(y) - \nabla f(x) \cdot y \chi_{|y| \leq 1}) \chi_{|y| \leq 1} \, d\nu(y),$$

the measure $\nu$ on $\mathbb{R}^N \setminus \{0\}$ is invariant under rotations around origin and satisfies

$$\int_{\mathbb{R}^N} 1 \lor |x|^2 \, d\nu(y) < \infty,$$

together with other assumptions.

Relativistic stable processes. In the case

$$\mathcal{L} = c - \left(c^{1/s} - \Delta\right)^s, \quad \text{with} \ c > 0, \ \text{and} \ 0 < s \leq 1.$$

The Green function $G(x, y)$ of $\mathcal{L}$ satisfies assumption $(K4)$ with $\gamma = s$.

Many other interesting examples. Schrödinger equations for non-symmetric diffusions, Gradient perturbation of RFL...

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$$\mathcal{L} = c - \left(c^{1/s} - \Delta\right)^s,$$

with $c > 0$, and $0 < s \leq 1$.

The Green function $G(x, y)$ of $\mathcal{L}$ satisfies assumption (K4) with $\gamma = s$.

Many other interesting examples. Schrödinger equations for non-symmetric diffusions, Gradient perturbation of RFL...

Semilinear Elliptic Equations

- Sharp boundary behaviour for Semilinear Elliptic equations
- Parabolic solutions by separation of variables
Sharp boundary behaviour for Elliptic Equations

We always assume that $\mathcal{L}$ satisfies (A1), (A2) and zero Dirichlet boundary conditions.

The Semilinear Dirichlet Problem $\mathcal{L}v = f(v) \sim v^p$ with $0 < p < 1$

Assume moreover that $\mathcal{L}^{-1}$ satisfies (K2). Let $u \geq 0$ be a (weak dual) solution to the Dirichlet Problem, where $f$ is a nonnegative increasing function with $f(0) = 0$ such that $F = f^{-1}$ is convex and $F(a) \sim a^{1/p}$ when $0 \leq a \leq 1$, for some $0 < p < 1$. Then, the following sharp absolute bounds hold true for all $x \in \Omega$

$$v(x) \sim \begin{cases} 
\Phi_1^\sigma(x) & \text{when } 2s \neq \gamma(1 - p) \\
\Phi_1(x) (1 + |\log \Phi_1(x)|)^{\frac{1}{1-p}} & \text{when } 2s = \gamma(1 - p), \text{ assuming (K4)} 
\end{cases}$$

where

$$\sigma := 1 \wedge \frac{2s}{\gamma(1 - p)}$$

and

$$\Phi_1 \sim \text{dist}(\cdot, \partial \Omega)^\gamma = \delta^\gamma$$

When $2s = \gamma(1 - p)$, if (K4) does not hold, then the upper bound still holds, but the lower bound holds in a non-sharp form without the extra logarithmic term.

Remarks.

- When $2s < \gamma(1 - p)$, the new power $\sigma$ becomes less than 1.
- Somehow $\sigma$ interpolates between the two extremal cases: $p = 0$ i.e. $\mathcal{L}v = 1$ and $p = 1$, i.e. $\mathcal{L}v = \lambda v$. 
Examples.

- For the RFL ($\gamma = s$) and CFL ($\gamma = s - 1/2$) we always have $\sigma = 1$ and $2s \neq \gamma(1 - p)$, hence
  \[v(x) \asymp \Phi_1(x) \asymp \text{dist}(\cdot, \partial \Omega)^\gamma = \delta^\gamma\]

- For the SFL we have $\gamma = 1$ hence we have three possibilities:
  \[
v(x) \asymp \begin{cases} 
  \text{dist}(x, \partial \Omega) & \text{when } s > \frac{1-p}{2} \\
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  \end{cases}\]

Regularity. Under some mild assumptions on $L$ and $f \in C^\beta(\mathbb{R})$ for some $\beta > 0$, with $0 \leq f(a) \leq c_p a^p$ when $0 \leq a \leq 1$ for some $0 < p \leq 1$.

- Solutions are Hölder continuous in the interior, and (when the operator allows it) are classical in the interior, namely $C^{2s+\beta}(\Omega)$.
- Assuming moreover that $L^{-1}$ satisfies (K2), solutions are Hölder continuous up to the boundary:
  \[\|u\|_{C^\eta(\Omega)} \leq C \quad \forall \eta \in (0, \gamma) \cap (0, 2s).\]
  (When $2s \geq \gamma$ the exponent is sharp. When $2s < \gamma$ actually we can reach any $\eta < \gamma$)
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Change of notations from Elliptic to Parabolic In order to make the elliptic results “compatible” with the parabolic, we will perform the change of notations

\[
m = \frac{1}{p} > 1 \quad \text{and} \quad v = S^m \quad \text{or} \quad v^p = S.
\]

The elliptic equation transforms: (we deal only with pure powers for simplicity)

\[
\mathcal{L}v = f(v) = v^p \quad \text{becomes} \quad \mathcal{L}S^m = \mathcal{L}F(S) = S.
\]

Parabolic solutions by separation of variables. We have the following solution for the Dirichlet problem for the equation \( u_t + \mathcal{L} u^m = 0 \)

\[
U_T(t, x) = \frac{S(x)}{(T + t)^{\frac{1}{m-1}}}
\]

where \( \mathcal{L}S^m = S \), and the initial datum is \( U_T(0, x) = T^{-1/(m-1)} S(x) \).

When \( T = 0 \) we have the so-called Friendly Giant, corresponding to the biggest possible initial datum (useful in the asymptotic study as \( t \to \infty \)).

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U(t, x) = \frac{S(x)}{t^{\frac{1}{m-1}}} \quad \text{with} \quad U(0, x) = +\infty.
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Back to the Parabolic problem

- (More) Assumptions on the operator
- Basic theory: existence, uniqueness and boundedness
- Elliptic VS Parabolic: Asymptotic Behaviour as $t \to \infty$

For the rest of the talk we deal with the special case:

$$F(u) = u^m := |u|^{m-1}u, \quad m > 1$$
Recall that the linear operator $\mathcal{L} : \text{dom}(A) \subseteq L^1(\Omega) \rightarrow L^1(\Omega)$ is assumed to be densely defined and \textit{sub-Markovian}, and we have already explained the assumptions (K1) and (K2) on the inverse.

**Assumptions on the kernel.**

- Whenever $\mathcal{L}$ is defined in terms of a kernel $K(x, y)$ via the formula

$$\mathcal{L}f(x) = P.V. \int_{\mathbb{R}^N} (f(x) - f(y)) K(x, y) \, dy,$$

assumption (L1) states that there exists $\kappa_{\Omega} > 0$ such that

$$(L1) \quad \inf_{x, y \in \Omega} K(x, y) \geq \kappa_{\Omega} > 0.$$  

- Whenever $\mathcal{L}$ is defined in terms of a kernel $K(x, y)$ and a zero order term:

$$\mathcal{L}f(x) = P.V. \int_{\mathbb{R}^N} (f(x) - f(y)) K(x, y) \, dy + B(x)f(x),$$

assumptions (L2) states that there exists $\kappa_{\Omega} > 0$ and $\gamma \in (0, 1]$

$$(L2) \quad K(x, y) \geq \kappa_{\Omega} \text{dist}(x, \partial \Omega)^\gamma \text{dist}(y, \partial \Omega)^\gamma, \quad \text{and} \quad B(x) \geq 0,$$
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About the kernels of spectral nonlocal operators. Most of the examples of nonlocal operators, but the SFL, admit a representation with a kernel. A natural question is: does the SFL admit such a representation?

Let $A$ be a uniformly elliptic linear operator. Define the $s^{th}$ power of $A$:

$$
\mathcal{L}^s g(x) = A^s g(x) = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{tA} g(x) - g(x)) \frac{dt}{t^{1+s}}
$$

Then it admits a representation with a Kernel plus zero order term:

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A^s g(x) = P.V. \int_{\mathbb{R}^N} (g(x) - g(y)) K(x, y) \, dy + \kappa(x) g(x).
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where $K \geq 0$ is compactly supported in $\Omega \times \Omega$ with

$$
K(x, y) \simeq \frac{1}{|x - y|^{N+2s}} \left( \Phi_1(x) \wedge 1 \right) \left( \Phi_1(y) \wedge 1 \right) \quad \text{and} \quad \kappa(x) \simeq \frac{1}{\text{dist}(x, \partial \Omega)^{2s}}.
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Basic theory: existence, uniqueness and boundedness (in one page)

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\begin{align*}
\frac{\partial t}{\partial t} u &= -L u^m, & \text{in } (0, +\infty) \times \Omega \\
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\end{align*}
\]  

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  \[
  \|u(t, \cdot)\|_{L^\infty(\Omega)} \leq \bar{k} t^{-\frac{1}{m-1}},
  \]

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  \[
  \|u(t)\|_{L^\infty(\Omega)} \leq \frac{\bar{k}}{t^{N+\frac{\rho}{\gamma}}\|u(t)\|_{L^1_{\Phi_1}(\Omega)}} \leq \frac{\bar{k}}{t^{N+\frac{\rho}{\gamma}}} \|u_0\|_{L^1_{\Phi_1}(\Omega)}.
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Elliptic VS Parabolic: Asymptotic Behaviour as $t \to \infty$

**Theorem. (Asymptotic behaviour)** (M.B., A. Figalli, Y. Sire, J. L. Vázquez)

Assume that $\mathcal{L}$ satisfies (A1), (A2), and (K2), and let $S$ be the solution to $\mathcal{L}S^m = S$. Let $u$ be any weak dual solution to the Cauchy-Dirichlet problem. Then, unless $u \equiv 0$,

$$\left\| t^{m-1} u(t, \cdot) - S \right\|_{L^\infty(\Omega)} \xrightarrow{t \to \infty} 0.$$ 

This result, gives a clear suggestion of what the boundary behaviour of parabolic solutions should be,

$$u(t, x) \asymp \mathcal{U}(t, x) = \frac{S(x)}{t^{m-1}}$$

at least for large times, as it happens in the local case $s = 1$. Hence the boundary behaviour shall be dictated by the behaviour of the solution to the elliptic equation. We shall see that this is not always the case.
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$$
\left\| t^{m-1} u(t, \cdot) - S \right\|_{L^\infty(\Omega)} \xrightarrow{t \to \infty} 0.
$$

This result, gives a clear suggestion of what the boundary behaviour of parabolic solutions should be,

$$
u(t, x) \asymp \mathcal{U}(t, x) = \frac{S(x)}{t^{\frac{1}{m-1}}},$$

at least for large times, as it happens in the local case $s = 1$. Hence the boundary behaviour shall be dictated by the behaviour of the solution to the elliptic equation. We shall see that this is not always the case.
Sharp Boundary Behaviour

- Upper Boundary Estimates
- Infinite Speed of Propagation
- Lower Boundary Estimates
- Harnack-type Inequalities
- Numerics
**Theorem. (Upper boundary behaviour)** (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K2) hold. Let $u \geq 0$ be a weak dual solution to the (CDP). Let $\sigma \in (0, 1]$ be

$$\sigma = \frac{2sm}{\gamma(m - 1)} \land 1$$

Then, there exists a computable constant $\kappa > 0$, depending only on $N, s, m,$ and $\Omega$, (but not on $u_0$) such that for all $t \geq 0$ and all $x \in \Omega$:

$$u(t, x) \leq \frac{\kappa}{t^{m-1}} \left\{ \begin{array}{ll}
\Phi_1(x)^{\sigma/m} & \text{if } \gamma \neq 2sm/(m - 1), \\
\Phi_1(x)^{\frac{1}{m}} \left(1 + |\log \Phi_1(x)|\right)^{\frac{1}{m-1}} & \text{if } \gamma = 2sm/(m - 1).
\end{array} \right.$$  

- **When $\sigma = 1$ and $\gamma \neq 2sm/(m - 1)$** we have sharp boundary estimates: we will show lower bounds with matching powers.

- **When $\sigma < 1$** the estimates are not sharp in all cases:
  - The solution by separation of variables $U(t, x) = S(x)t^{-1/(m-1)}$ (asymptotic behaviour) behaves like $\Phi_1^{\sigma/m} t^{-1/(m-1)}$.
  - We will show that for small data, the boundary behaviour is different.
  - In examples, $\sigma < 1$ only happens for SFL-type, where $\gamma = 1$, and $s$ can be small, $0 < s < 1/2 - 1/(2m)$. 
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Infinite Speed of Propagation

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Universal Lower Bounds
Theorem. (Universal lower bounds)  
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Let $\mathcal{L}$ satisfy (A1), (A2) and (L2). Let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in L^1_{\Phi_1}(\Omega)$. Then there exists a constant $\kappa_0 > 0$, so that the following inequality holds:

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for all $t > 0$ and all $x \in \Omega$.

Here $t^* = \kappa_* \|u_0\|_{L^1_{\Phi_1}(\Omega)}^{-(m-1)}$ and $\kappa_0, \kappa_*$ depend only on $N, s, \gamma, m, c_0$, and $\Omega$.

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- The assumption on the kernel $K$ of $\mathcal{L}$ holds for all examples and represent somehow the “worst case scenario” for lower estimates:

  $$\mathcal{L}f(x) = \text{P.V.} \int_{\mathbb{R}^N} (f(x) - f(y)) K(x, y) \, dy + B(x)f(x), \quad \text{with} \quad \left\{ \begin{array}{l} K(x, y) \gtrsim \delta^\gamma(x) \delta^\gamma(y), \\ B(x) \geq 0, \end{array} \right.$$  

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Infinite speed of propagation.

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- As a consequence, of the above universal bounds for all times, we have proven that all nonnegative solutions have **infinite speed of propagation**.
  - No free boundaries when \( s < 1 \), contrary to the “local” case \( s = 1 \), cf. Barenblatt, Aronson, Caffarelli, Vázquez, Wolansky [...]
  - Qualitative version of infinite speed of propagation for the Cauchy problem on \( \mathbb{R}^N \), by De Pablo, Quíros, Rodríguez, Vázquez [Adv. Math. 2011, CPAM 2012]
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Sharp Lower boundary estimates
Sharp lower boundary estimates I: the non-spectral case.

Let $\sigma = \frac{2sm}{\gamma(m-1)} \wedge 1$. Let $\mathcal{L}$ satisfy (A1) and (A2), and assume moreover that

$$\mathcal{L}f(x) = \int_{\mathbb{R}^N} (f(x) - f(y)) K(x, y) \, dy, \quad \text{with} \quad \inf_{x,y \in \Omega} K(x, y) \geq \kappa_{\Omega} > 0.$$  \hspace{1cm} (1)

Assume moreover that $\mathcal{L}$ has a first eigenfunction $\Phi_1 \asymp \text{dist}(x, \partial \Omega)^{\gamma}$ and that

- either $\sigma = 1$;
- or $\sigma < 1$, $K(x, y) \leq c_1 |x - y|^{-(N+2s)}$ for a.e. $x, y \in \mathbb{R}^N$, and $\Phi_1 \in C^{\gamma}(\overline{\Omega})$.

**Theorem. (Sharp lower bounds for all times) (M.B., A. Figalli and J. L. Vázquez)**

Under the above assumptions, let $u \geq 0$ be a weak dual solution to the (CDP) with $u_0 \in L^1_{\Phi_1}(\Omega)$. Then there exists a constant $\kappa_1 > 0$ such that

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- The boundary behavior is sharp for all times in view of the upper bounds.
- Within examples, this applies to RFL and CFL type, but not to SFL-type.
- For RFL, this result was obtained first by MB, A. Figalli and X. Ros-Oton.
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Sharp absolute lower estimates for large times: the case $\sigma = 1$.
When $\sigma = 1$ we can establish a quantitative lower bound near the boundary that matches the separate-variables behavior for large times.

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Let $(A1)$, $(A2)$, and $(K2)$ hold, and let $\sigma = 1$ and $2sm \neq \gamma(m - 1)$. Let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in L^1_\Phi(\Omega)$. There exists a constant $\kappa_2 > 0$ such that

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for all $t \geq t_*$ and a.e. $x \in \Omega$.

Here, $t_* = \kappa_* \|u_0\|_{L^1_{\Phi_1}(\Omega)}^{-(m - 1)}$, and the constants $\kappa_*$, $\kappa_2$ depend only on $N, s, \gamma, m,$ and $\Omega$.

- It holds for $s = 1$, the local case, where there is finite speed of propagation.
- When $s = 1$, $t_*$ is the time that the solution needs to be positive everywhere.
- When $\mathcal{L} = -\Delta$, proven by Aronson-Peletier (’81) and Vázquez (’04)
- Our method applies when $\mathcal{L}$ is an elliptic operator with $C^1$ coefficients (new result).
- In the limit case $2sm = \gamma(m - 1)$, we have $\sigma = 1$, but the estimates are not sharp, as we show below.
**Sharp absolute lower estimates for large times: the case $\sigma = 1$.**

When $\sigma = 1$ we can establish a quantitative lower bound near the boundary that matches the separate-variables behavior for large times.

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Here, $t_* = \kappa_* \|u_0\|_{L^1_{\Phi_1}(\Omega)}^{(m-1)}$, and the constants $\kappa_*$, $\kappa_2$ depend only on $N, s, \gamma, m$, and $\Omega$.

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- When $s = 1$, $t_*$ is the time that the solution needs to be positive everywhere.
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Sharp absolute lower estimates for large times: the case $\sigma = 1$.

When $\sigma = 1$ we can establish a quantitative lower bound near the boundary that matches the separate-variables behavior for large times.

**Theorem. (Sharp lower bounds for large times)** (M.B., A. Figalli and J. L. Vázquez)

Let $(A1)$, $(A2)$, and $(K2)$ hold, and let $\sigma = 1$ and $2sm \neq \gamma(m-1)$. Let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in L^1_{\Phi_1}(\Omega)$. There exists a constant $\kappa_2 > 0$ such that

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**Positivity for large times II: the case $\sigma < 1$.**

The intriguing case $\sigma < 1$ is where new and unexpected phenomena appear. Recall that

$$\sigma = \frac{2sm}{\gamma(m-1)} < 1 \quad \text{i.e.} \quad 0 < s < \frac{\gamma}{2} - \frac{\gamma}{2m}.$$  

**Solutions by separation of variables: the standard boundary behaviour?**

Let $S$ be a solution to the Elliptic Dirichlet problem for $\mathcal{L}S^m = c_m S$. We can define

$$\mathcal{U}(t,x) = S(x)t^{-\frac{1}{m-1}} \quad \text{where} \quad S \simeq \Phi^{\sigma/m}_1,$$

which is a solution to the (CDP), which behaves like $\Phi^{\sigma/m}_1$ at the boundary.

By comparison, we see that the same lower behaviour is shared ‘big’ solutions:

$$u_0 \geq \epsilon_0 S \quad \text{implies} \quad u(t) \geq \frac{S}{(\epsilon_0^{1-m} + t)^{1/(m-1)}}$$

This behaviour seems to be sharp: we have shown matching upper bounds, and also $S$ represents the large time asymptotic behaviour:

$$\lim_{t \to \infty} \left\| t^{\frac{1}{m-1}} u(t) - S \right\|_{L^\infty} = 0 \quad \text{for all } 0 \leq u_0 \in L^1_{\Phi_1}(\Omega).$$

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**Different boundary behaviour when** $\sigma < 1$. The next result shows that, in general, we cannot hope to prove that $u(t)$ is larger than $\Phi_1^{1/m}$, but always smaller than $\Phi_1^{\sigma/m}$.

**Proposition. (Counterexample I)**

(M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K2) hold, and $u \geq 0$ be a weak dual solution to the (CDP). Then, there exists a constant $\hat{\kappa}$, depending only $N, s, \gamma, m$, and $\Omega$, such that

$$0 \leq u_0 \leq c_0\Phi_1 \text{ implies } u(t, x) \leq c_0\hat{\kappa}\frac{\Phi_1^{1/m}(x)}{t^{1/m}} \quad \forall t > 0 \text{ and a.e. } x \in \Omega.$$

In particular, if $\sigma < 1$, then

$$\lim_{x \to \partial \Omega} \frac{u(t, x)}{\Phi_1(x)\sigma/m} = 0 \quad \text{for any } t > 0.$$  

When $\sigma = 1$ and $2sm = \gamma(m - 1)$, then

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**Idea:** The proposition above could make one wonder whether or not the sharp general lower bound could be actually given by $\Phi_1^{1/m}$, as in the case $\sigma = 1$.

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We next show that assuming (K4), the bound $u(t) \gtrsim \Phi_1^{1/m} t^{-1/(m-1)}$ is false for $\sigma < 1$.

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Let (A1), (A2), and (K4) hold, and let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to a nonnegative initial datum $u_0 \leq c_0 \Phi_1$ for some $c_0 > 0$. If there exist constants $\kappa, T, \alpha > 0$ such that

$$u(T, x) \geq \kappa \Phi_1^\alpha(x) \quad \text{for a.e. } x \in \Omega,$$

then $\alpha \geq 1 - \frac{2s}{\gamma}$.

In particular, when $\sigma < 1$, we have $\alpha > \frac{1}{m} > \frac{\sigma}{m}$.

Under mild assumptions on the operator (for example SFL-type), we can prove:

$$0 \leq u_0 \leq A \Phi_1^{1-\frac{2s}{\gamma}} \quad \Rightarrow \quad u(t) \leq [A^{1-m} - \tilde{C}t]^{-(m-1)} \Phi_1^{1-\frac{2s}{\gamma}}$$

for small times $t \in [0, T_A]$, where $T_A := 1/(\tilde{C}A^{m-1})$, for some $\tilde{C} > 0$. Recall that we have a universal lower bound (under minimal assumptions on $K$)

$$u(t, x) \geq \kappa_0 \left(1 \wedge \frac{t}{t^*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{m-1}}$$

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[Outline of the talk]

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Semilinear Elliptic Equations

Back to the Parabolic problem

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Regularity Estimates

Sharp Lower boundary estimates
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Harnack-type Inequalities

- Global Harnack Principle I. The non-spectral case.
- Consequences of GHP.
- Global Harnack Principle II. The remaining cases.
Global Harnack Principle I. The non-spectral case.

Recall that

\[ \Phi_1 \asymp \text{dist}(\cdot, \partial \Omega) \gamma, \quad \sigma = 1 \wedge \frac{2sm}{\gamma(m-1)}, \quad t_* = \kappa_* \|u_0\|^{-{(m-1)}}_{L^1_{\Phi_1}(\Omega)} \]

**Theorem.** (Global Harnack Principle I. The non-spectral case.) (MB & AF & JLV)

Let (A1), (A2), (L1) and (K2). Let \( u \geq 0 \) be a weak dual solution to the (CDP). Also, when \( \sigma < 1 \), assume that \( K(x, y) \leq c_1 |x - y|^{-(N+2s)} \) for a.e. \( x, y \in \mathbb{R}^N \) and that \( \Phi_1 \in C^\gamma(\Omega) \).

Then, there exist constants \( \kappa, \kappa > 0 \), so that the following inequality holds:

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for all \( t > 0 \) and all \( x \in \Omega \).

The constants \( \kappa, \kappa \) depend only on \( N, s, \gamma, m, c_1, \kappa_\Omega, \Omega, \) and \( \|\Phi_1\|_{C^\gamma(\Omega)} \).

- For large times \( t \geq t_* \), the estimates are independent on the initial datum.
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**Corollary. (Local Harnack Inequalities of Elliptic/Backward Type)**

Assume that the (GHP-I) holds for a weak dual solution $u$ to the (CDP). Then there exists a constant $\hat{H}$ depending only on $N, s, \gamma, m, c_1, \Omega$, s. t. for all $t > 0$ and $h \geq 0$

$$\sup_{x \in B_R(x_0)} u(t, x) \leq \hat{H} \left[ \left(1 + \frac{h}{t}\right) \left(1 \wedge \frac{t}{t^*}\right)^{-m} \right]^{\frac{1}{m-1}} \inf_{x \in B_R(x_0)} u(t + h, x).$$

When $s = 1$, backward Harnack inequalities are typical of Fast Diffusion eq. ($m < 1$, possible extinction in finite time), and they do not happen when $m > 1$ (finite speed of propagation).

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Assume that a GHP with matching powers hold. Set $U(t, x) := t^{-\frac{1}{m-1}} S(x)$. Then there exists $c_0 > 0$ such that, for all $t \geq t_0 := c_0 \|u_0\|_{L^1_{\Phi_1}(\Omega)}^{-(m-1)}$, we have

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This asymptotic result is sharp: check by considering $u(t, x) = U(t + 1, x)$. For the classical case $\mathcal{L} = \Delta$, we recover the results of Aronson-Peletier and Vazquez with a different proof.
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- For large times, we can prove as before Local Harnack inequalities of Elliptic/Backward type.
- Also in this case the Sharp Asymptotic behaviour follows from GHP with matching powers.
- For small times we can not find matching powers for a global Harnack inequality (except for special data) and such result is actually false for \( s = 1 \) (finite speed of propagation).
- Backward Harnack inequalities for the linear heat equation \( s = 1 \) and \( m = 1 \), by Fabes, Garofalo, Salsa [Ill. J. Math, 1986] and also Safonov, Yuan [Ann. of Math, 1999]
- For \( s = 1 \), Intrinsic (Forward) Harnack inequalities by DiBenedetto [ARMA, 1988], Daskalopoulos and Kenig [EMS Book, 2007], cf. also DiBenedetto, Gianazza, Vespri [LNM, 2011].
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- For small times we can not find matching powers for a global Harnack inequality (except for special data) and such result is actually false for \( s = 1 \) (finite speed of propagation).
- Backward Harnack inequalities for the linear heat equation \( s = 1 \) and \( m = 1 \), by Fabes, Garofalo, Salsa [Ill. J. Math, 1986] and also Safonov, Yuan [Ann. of Math, 1999]
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Global Harnack Principles II. The remaining cases.

**Theorem. (Global Harnack Principle II)**

(M.B., A. Figallì and J. L. Vázquez)

Let (A1), (A2), and (K2) hold, and let \( u \geq 0 \) be a weak dual solution to the (CDP) corresponding to \( u_0 \in L^1_{\Phi_1}(\Omega) \). Assume that:
- either \( \sigma = 1 \) and \( 2sm \neq \gamma(m - 1) \);
- or \( \sigma < 1 \), \( u_0 \geq \kappa_0 \Phi_1^{\sigma/m} \) for some \( \kappa_0 > 0 \), and (K4) holds.

Then there exist constants \( \kappa, \bar{\kappa} > 0 \) such that the following inequality holds:

\[
\kappa \frac{\Phi_1(x)^{\sigma/m}}{t^{m-1}} \leq u(t,x) \leq \bar{\kappa} \frac{\Phi_1(x_0)^{\sigma/m}}{t^{m-1}}
\]

for all \( t \geq t^* \) and all \( x \in \Omega \).

The constants \( \kappa, \bar{\kappa} \) depend only on \( N, s, \gamma, m, \kappa_0, \kappa_\Omega \), and \( \Omega \).

- For large times, we can prove as before Local Harnack inequalities of Elliptic/Backward type.
- Also in this case the Sharp Asymptotic behaviour follows from GHP with matching powers.
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$$\frac{\Phi_1(x)^{\sigma/m}}{t^{m-1}} \leq u(t, x) \leq \frac{\Phi_1(x_0)^{\sigma/m}}{t^{m-1}}$$

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Hence, in the remaining cases, we have only the following general result.

**Theorem. (Global Harnack Principle III)** (M.B., A. Figalli and J. L. Vázquez)

Let $\mathcal{L}$ satisfy (A1),(A2), (L2) and (K2). Let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in L^1_{\Phi_1}(\Omega)$.

Then, there exist constants $\kappa, \overline{\kappa} > 0$, so that the following inequality holds:

$$\frac{\kappa}{\overline{\kappa}} \left(1 \wedge \frac{t}{t_*}\right)^{m-1} \frac{\phi_1(x)}{t^{1/m-1}} \leq u(t, x) \leq \frac{\overline{\kappa}}{\kappa} \frac{\phi_1(x_0)^{\sigma/m}}{t^{1/m-1}}$$

for all $t > 0$ and all $x \in \Omega$.

- This is sufficient to ensure interior regularity, under ‘minimal’ assumptions.
- This bound holds for all times and for a large class of operators.
- This is not sufficient to ensure $C^\alpha_x$ boundary regularity.
Hence, in the remaining cases, we have only the following general result.

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$$
\frac{\kappa}{\left(1 \wedge \frac{t}{t_*}\right)^{\frac{m-1}{m}} \Phi_1(x)} \leq u(t, x) \leq \frac{\bar{\kappa}}{t^{\frac{1}{m-1}}} \frac{\Phi_1(x_0)^{\sigma/m}}{t^{\frac{1}{m-1}}}
$$

for all $t > 0$ and all $x \in \Omega$.

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Numerical Simulations*

Graphics and videos: courtesy of F. Del Teso (NTNU, Trondheim, Norway)
Numerical simulation for the SFL with parameters $m = 2$ and $s = 1/2$, hence $\sigma = 1$.

Left: the initial condition $u_0 \leq C_0 \Phi_1$

Right: solid line represents $\Phi_1^{1/m}$

the dotted lines represent $t^{m-1} u(t)$ at time at $t = 1$ and $t = 5$

While $u(t)$ appears to behave as $\Phi_1 \asymp \text{dist}(\cdot, \partial \Omega)$ for very short times
already at $t = 5$ it exhibits the matching boundary behavior $t^{m-1} u(t) \asymp \Phi_1^{1/m}$
**Compare** $\sigma = 1$ VS $\sigma < 1$: same $u_0 \leq C_0 \Phi_1$, solutions with different parameters

**Left:** $t^{\frac{1}{m-1}}u(t)$ at time $t = 30$ and $t = 150$; $m = 4$, $s = 3/4$, $\sigma = 1$.

**Matching:** $u(t)$ behaves like $\Phi_1 \asymp \text{dist}(\cdot, \partial \Omega)$ for quite some time, and only around $t = 150$ it exhibits the matching boundary behavior $u(t) \asymp \Phi_1^{1/m}$

**Right:** $t^{\frac{1}{m-1}}u(t)$ at time $t = 150$ and $t = 600$; $m = 4$, $s = 1/5$, $\sigma = 8/15 < 1$.

**Non-matching:** $u(t) \asymp \Phi_1$ even after long time.

**Idea:** maybe when $\sigma < 1$ and $u_0 \lesssim \Phi_1$, we have $u(t) \asymp \Phi_1$ for all times...

**Not True:** there are cases when $u(t) \gg \Phi_1^{1-2s}$ for large times...
**Non-matching when** $\sigma < 1$: same data $u_0$, with $m = 2$ and $s = 1/10$, $\sigma = 2/5 < 1$

In both pictures, the solid line represents $\Phi_1^{1-2s}$ (anomalous behaviour)

**Left:** $t^{m-1} u(t)$ at time $t = 4$ and $t = 25.$

$u(t) \asymp \Phi_1$ for short times $t = 4$, then $u(t) \sim \Phi_1^{1-2s}$ for intermediate times $t = 25$

**Right:** $t^{m-1} u(t)$ at time $t = 40$ and $t = 150.$ $u(t) \gg \Phi_1^{1-2s}$ for large times.

**Both non-matching** always different behaviour from the asymptotic profile $\Phi_1^{\sigma/m}$.

In this case we show that if $u_0(x) \leq C_0 \Phi_1(x)$ then for all $t > 0$

$$u(t, x) \leq C_1 \left[ \frac{\Phi_1(x)}{t} \right]^{1/m}$$

and

$$\lim_{x \to \partial\Omega} \frac{u(t, x)}{\Phi_1(x)^{\sigma/m}} = 0 \quad \text{for any } t > 0.$$
Regularity Estimates

- Interior Regularity
- Hölder continuity up to the boundary
- Higher interior regularity for RFL
The regularity results, require the validity of a Global Harnack Principle. 

**(R)** The operator $\mathcal{L}$ satisfies (A1) and (A2), and $\mathcal{L}^{-1}$ satisfies (K2). Moreover, we consider

$$\mathcal{L}f(x) = P.V. \int_{\mathbb{R}^N} (f(x) - f(y)) K(x, y) \, dy + B(x)f(x),$$

with

$$K(x, y) \asymp |x - y|^{-(N+2s)} \text{ in } B_{2r}(x_0) \subset \Omega, \quad K(x, y) \lesssim |x - y|^{-(N+2s)} \text{ in } \mathbb{R}^N \setminus B_{2r}(x_0).$$

As a consequence, for any ball $B_{2r}(x_0) \subset \subset \Omega$ and $0 < t_0 < T_1$, there exist $\delta, M > 0$ such that

$$0 < \delta \leq u(t, x) \quad \text{for a.e. } (t, x) \in (T_0, T_1) \times B_{2r}(x_0),$$

$$0 \leq u(t, x) \leq M \quad \text{for a.e. } (t, x) \in (T_0, T_1) \times \Omega.$$

The constants in the regularity estimates will depend on the solution only through $\delta, M$.

---

**Theorem. (Interior Regularity)**

(M.B., A. Figalli and J. L. Vázquez)

Assume (R) and let $u$ be a nonnegative bounded weak dual solution to problem (CDP).

1. Then $u$ is **Hölder continuous in the interior**. More precisely, there exists $\alpha > 0$ such that, for all $0 < T_0 < T_2 < T_1$,

$$\|u\|_{C^{\alpha/2s, \alpha}_{t,x}((T_2, T_1) \times B_{r}(x_0))} \leq C.$$

2. Assume in addition $|K(x, y) - K(x', y)| \leq c|x - x'|^{\beta} |y|^{-(N+2s)}$ for some $\beta \in (0, 1 \wedge 2s)$ such that $\beta + 2s \notin \mathbb{N}$. Then $u$ is a **classical solution in the interior**. More precisely, for all $0 < T_0 < T_2 < T_1$,

$$\|u\|_{C^{1+\beta/2s, 2s+\beta}_{t,x}((T_2, T_1) \times B_{r}(x_0))} \leq C.$$
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The regularity results require the validity of a Global Harnack Principle. 

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As a consequence, for any ball \( B_{2r}(x_0) \subset \subset \Omega \) and \( 0 < t_0 < T_1 \), there exist \( \delta, M > 0 \) such that

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\[
\|u\|_{C^{1+\beta/2s, 2s+\beta}_{t,x}((T_2, T_1) \times B_{2r}(x_0))} \leq C.
\]
Hölder continuity up to the boundary

**Theorem. (Hölder continuity up to the boundary)** (M.B., A. Figalli and J. L. Vázquez)

Assume (R), hypothesis 2 of the interior regularity and in addition that $2s > \gamma$. Then \( u \) is **Hölder continuous up to the boundary**.

More precisely, for all \( 0 < T_0 < T_2 < T_1 \) there exists a constant \( C > 0 \) such that

\[
\|u\|_{C^{\frac{\gamma}{m\vartheta}, \frac{\gamma}{m}}_{t,x}((T_2, T_1) \times \Omega)} \leq C \quad \text{with} \quad \vartheta := 2s - \gamma \left(1 - \frac{1}{m}\right).
\]

- Since \( u(t, x) \approx \Phi_1(x)^{1/m} \approx \text{dist}(x, \partial \Omega)^{\gamma/m} \), the spacial Hölder exponent is sharp, while the Hölder exponent in time is the natural one by scaling. (\( 2s > \gamma \) implies \( \sigma = 1 \))
- Previous regularity results: (I apologize if I forgot someone)
  - **\( C^{\alpha} \) regularity:**
    Athanasopoulos and Caffarelli [Adv. Math, 2010], (RFL domains)
    De Pablo, Quirós, Rodriguez, Vázquez [CPAM 2012] (RFL on \( \mathbb{R}^N \), SFL-Dirichlet)
    De Pablo, Quirós, Rodriguez [NLTMA 2016]. (RFL-rough kernels \( \mathbb{R}^N \))
  - **Classical Solutions:**
    Vázquez, De Pablo, Quirós, Rodriguez [JEMS 2016] (RFL on \( \mathbb{R}^N \))
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  - **Higher regularity:** \( C^{\infty}_x \) and \( C^{\alpha} \) up to the boundary:
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Assume (R), hypothesis 2 of the interior regularity and in addition that $2s > \gamma$. Then $u$ is Hölder continuous up to the boundary.

More precisely, for all $0 < T_0 < T_2 < T_1$ there exists a constant $C > 0$ such that

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\|u\|_{C^{\frac{\gamma}{m} \vartheta, \frac{\gamma}{m}}((T_2, T_1) \times \Omega)} \leq C \quad \text{with} \quad \vartheta := 2s - \gamma \left(1 - \frac{1}{m}\right).
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- Since $u(t, x) \asymp \Phi_1(x)^{1/m} \asymp \text{dist}(x, \partial \Omega)^{\gamma/m}$, the spacial Hölder exponent is sharp, while the Hölder exponent in time is the natural one by scaling. ($2s > \gamma$ implies $\sigma = 1$)
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$$\|u\|_{C^{\varsigma/(m\theta), \theta} \left( (T_2, T_1) \times \Omega \right)} \leq C$$

with $\theta := 2s - \gamma \left(1 - \frac{1}{m}\right)$.

- Since $u(t, x) \asymp \Phi_1(x)^{1/m} \asymp \text{dist}(x, \partial\Omega)^{\gamma/m}$, the spatial Hölder exponent is sharp, while the Hölder exponent in time is the natural one by scaling. ($2s > \gamma$ implies $\sigma = 1$)

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Higher Interior Regularity for RFL.

**Theorem. (Higher interior regularity in space) (M.B., A. Figalli, X. Ros-Oton)**

Under the running assumptions \( (R) \), then \( u \in C_x^\infty((0, \infty) \times \Omega) \).

More precisely, let \( k \geq 1 \) be any positive integer, and \( d(x) = \text{dist}(x, \partial \Omega) \), then, for any \( t \geq t_0 > 0 \) we have

\[
|D_x^k u(t, x)| \leq C [d(x)]^{s-m-k},
\]

where \( C \) depends only on \( N, s, m, k, \Omega, t_0 \), and \( \|u_0\|_{L^1_{\Phi_1}(\Omega)} \).

- Higher regularity in time is a difficult open problem. It is connected to higher order boundary regularity in \( t \). To our knowledge also open for the local case \( s = 1 \).
- When \( m = 1 \) (FHE) \( u_t + (-\Delta)_{|\Omega})^s u = 0 \) on \( (0, 1) \times B_1 \) we have \( u \in C_x^\infty \)
  \[
  \|u\|_{C_x^{k,\alpha}((1/2,1)\times B_{1/2})} \leq C \|u\|_{L^\infty((0,1)\times \mathbb{R}^N)}, \quad \text{for all } k \geq 0.
  \]

  *Analogous estimates in time do not hold* for \( k \geq 1 \) and \( \alpha \in (0, 1) \).

  Indeed, one can construct a solution to the (FHE) which is bounded in all of \( \mathbb{R}^N \), but which is not \( C^1 \) in \( t \) in \( (1/2, 1) \times B_{1/2} \). [Chang-Lara, Davila, JDE (2014)]

- Our techniques allow to prove regularity also in unbounded domains, and also for operator with more general kernels.
- Also the “classical/local” case \( s = 1 \) works after the waiting time \( t_* \):
  \[
  u \in C_{x,t}^{1/m, 1/2m}\left(\overline{\Omega} \times [t_*, T]\right), C_x^\infty((0, \infty) \times \Omega) \text{ and } C_{t}^{1,\alpha}([t_0, T] \times K) .
  \]
Higher Interior Regularity for RFL.

**Theorem. (Higher interior regularity in space)** (M.B., A. Figalli, X. Ros-Oton)

Under the running assumptions \( (R) \), then \( u \in C^\infty_x((0, \infty) \times \Omega) \).

More precisely, let \( k \geq 1 \) be any positive integer, and \( d(x) = \text{dist}(x, \partial \Omega) \), then, for any \( t \geq t_0 > 0 \) we have

\[
|D^k u(t, x)| \leq C [d(x)]^{\frac{s}{m} - k},
\]

where \( C \) depends only on \( N, s, m, k, \Omega, t_0 \), and \( \|u_0\|_{L^1_1(\Omega)} \).

- Higher regularity in time is a difficult open problem. It is connected to higher order boundary regularity in \( t \). To our knowledge also open for the local case \( s = 1 \).
- When \( m = 1 \) (FHE) \( u_t + (-\Delta|_\Omega)^s u = 0 \) on \( (0, 1) \times B_1 \) we have \( u \in C^\infty_x \)

\[
\|u\|_{C^k, \alpha_x((\frac{1}{2},1) \times B_{1/2})} \leq C\|u\|_{L^\infty((0,1) \times \mathbb{R}^N)}, \quad \text{for all } k \geq 0.
\]

*Analogous estimates in time do not hold* for \( k \geq 1 \) and \( \alpha \in (0, 1) \).

Indeed, one can construct a solution to the (FHE) which is bounded in all of \( \mathbb{R}^N \), but which is not \( C^1 \) in \( t \) in \( (\frac{1}{2}, 1) \times B_{1/2} \). [Chang-Lara, Davila, JDE (2014)]

- Our techniques allow to prove regularity also in unbounded domains, and also for operator with more general kernels.
- Also the “classical/local” case \( s = 1 \) works after the waiting time \( t_* \):

\[
u \in C^m_{x,t} \left( \frac{1}{2m} [\Omega \times [t_*, T]] \right), C^\infty_x((0, \infty) \times \Omega) \text{ and } C^1_t,\alpha([t_0, T] \times K).
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The End

Thank You!!!
Grazie Mille!!!
Muchas Gracias!!!