Fractional nonlinear degenerate diffusion equations on bounded domains

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[BSV] M. B., Y. SIRE, J. L. VÁZQUEZ, Existence, Uniqueness and Asymptotic behaviour for fractional porous medium equations on bounded domains.
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Outline of the talk

- The setup of the problem and first pointwise estimates
- Upper Estimates
- Lower bounds and Harnack inequalities
- Existence, uniqueness and asymptotic behaviour of solutions

The setup of the problem and first pointwise estimates

- **Introduction**
- **About the operator $\mathcal{L}$**
- **About the inverse operator $\mathcal{L}^{-1}$**
- **The “dual” formulation of the problem**
- **Monotonicity for the nonlinear flow**
- **The fundamental pointwise estimates**

Introduction

Homogeneous Dirichlet Problem for Fractional Nonlinear Degenerate Diffusion Equations

$$\text{(HDP)} \qquad \begin{cases} u_t + \mathcal{L}\, F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

where:

- $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N \geq 1$.
- The linear operator $\mathcal{L}$ will be a fractional Laplacian on domains,

$$\boxed{\mathcal{L} = (-\Delta_\Omega)^s}\,, \qquad \text{with } 0 < s \leq 1.$$

  Indeed, a wider class of linear (fractional) operators can be treated.
- The nonlinearity is typically $F(u) = |u|^{m-1}u$, with $m > 1$.
  We deal with Degenerate diffusion of Porous Medium type.
  More general classes of "degenerate" nonlinearities F are allowed.
- The homogeneous boundary condition is posed on the lateral boundary, which may take different forms, depending on the particular choice of the operator $\mathcal{L}$.

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Reminder about the fractional Laplacian operator on $\mathbb{R}^N$

We have several equivalent definitions for $(-\Delta_{\mathbb{R}^N})^s$:

1. By means of **Fourier Transform**,

$$((-\Delta_{\mathbb{R}^N})^s f)\widehat{\ }(\xi) = |\xi|^{2s} \hat{f}(\xi) \,.$$

This formula can be used for positive and negative values of s.

2. By means of an **Hypersingular Kernel**:
if $0 < s < 1$, we can use the representation

$$(-\Delta_{\mathbb{R}^N})^s g(x) = c_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{g(x) - g(z)}{|x-z|^{N+2s}} \, dz,$$

where $c_{N,s} > 0$ is a normalization constant.

3. **Spectral definition**, in terms of the heat semigroup associated to the standard Laplacian operator:

$$(-\Delta_{\mathbb{R}^N})^s g(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_{\mathbb{R}^N}} g(x) - g(x) \right) \frac{dt}{t^{1+s}} .$$

About the operator $\mathcal{L}$

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Fractional Laplacian operators on bounded domains

There are different definitions for the fractional Laplacian on bounded domains, which turn out to be not equivalent.

The Spectral Fractional Laplacian operator (SFL)

$$(-\Delta_\Omega)^s g(x) = \sum_{j=1}^{\infty} \lambda_j^s \, \hat{g}_j \, \phi_j(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_\Omega} g(x) - g(x) \right) \frac{dt}{t^{1+s}}.$$

- Δ_Ω is the classical Dirichlet Laplacian on the domain Ω
- EIGENVALUES: $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \lambda_{j+1} \le \ldots$ and $\lambda_j \asymp j^{2/N}$.
- EIGENFUNCTIONS: ϕ_j are as smooth as the boundary of Ω allows, namely when $\partial\Omega$ is C^k, then $\phi_j \in C^\infty(\Omega) \cap C^k(\overline{\Omega})$ for all $k \in \mathbb{N}$.

$$\hat{g}_j = \int_\Omega g(x)\phi_j(x)\,\mathrm{d}x\,, \qquad \text{with} \qquad \|\phi_j\|_{\mathrm{L}^2(\Omega)} = 1\,.$$

Lateral boundary conditions for the SFL

$$u(t,x) = 0\,, \qquad \text{in } (0,\infty) \times \partial\Omega\,.$$

Fractional Laplacian operators on bounded domains

Definition via the hypersingular kernel in $\mathbb{R}^N$, "restricted" to functions that are zero outside Ω.

The Restricted Fractional Laplacian operator (RFL)

$$(-\Delta_{|\Omega})^s g(x) = c_{d,s}\,\text{P.V.}\int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{d+2s}}\,\mathrm{d}z\,, \qquad \text{with } \operatorname{supp}(g) \subseteq \overline{\Omega}\,.$$

where $s \in (0,1)$ and $c_{N,s} > 0$ is a normalization constant.

- $(-\Delta_{|\Omega})^s$ is a self-adjoint operator on $\mathrm{L}^2(\Omega)$ with a discrete spectrum:
- EIGENVALUES: $0 < \overline{\lambda}_1 \le \overline{\lambda}_2 \le \ldots \le \overline{\lambda}_j \le \overline{\lambda}_{j+1} \le \ldots$ and $\overline{\lambda}_j \asymp j^{2s/N}$.
  Eigenvalues of the RFL are bigger than the ones of SFL: $\lambda_j^s \le \overline{\lambda}_j$ for all $j \in \mathbb{N}$.
- EIGENFUNCTIONS: $\overline{\phi}_j$ are the normalized eigenfunctions, are only Hölder continuous up to the boundary, namely $\overline{\phi}_j \in C^s(\overline{\Omega})$.

Lateral boundary conditions for the RFL

$$u(t,x) = 0\,, \qquad \text{in } (0,\infty)\times\left(\mathbb{R}^N \setminus \Omega\right).$$

Remark. Both for the SFL and the RFL there is another possible definition using the so-called Caffarelli-Silvestre extension.

About the inverse operator $\mathcal{L}^{-1}$

Reminder about Green functions

Notation. Let (λ_k, Φ_k) be the eigenvalues and eigenfunctions of $\mathcal{L}$. Recall that:

$$\Phi_1(x) \asymp \mathrm{dist}(x, \partial\Omega)^\gamma \qquad \text{with } \gamma = 1 \text{ for the SFL and } \gamma = s \text{ for the RFL.}$$

The inverse $\mathcal{L}^{-1}$ has a symmetric kernel $G_\Omega(x,y)$, which is the Green function:

$$\mathcal{L}^{-1}f(x_0) := \sum_{k=1}^{+\infty} \lambda_k^{-1} \hat{f}_k \Phi_k(x_0) = \int_\Omega G_\Omega(x, x_0) f(x) \,\mathrm{d}x\,.$$

When dealing with the SFL or RFL, it is well-known that the Green function satisfy the following estimates for all $x, x_0 \in \Omega$:

$$\text{(Type I)} \qquad 0 \le G_\Omega(x,x_0) \le \frac{c_{1,\Omega}}{|x-x_0|^{N-2s}} \sim G_{\mathbb{R}^N}(x,x_0)\,,$$

(Type II)

$$c_{0,\Omega}\Phi_1(x)\Phi_1(x_0) \le G_\Omega(x,x_0) \le \frac{c_{1,\Omega}}{|x-x_0|^{N-2s}} \left(\frac{\Phi_1(x)}{|x-x_0|^\gamma} \wedge 1\right)\left(\frac{\Phi_1(x_0)}{|x-x_0|^\gamma} \wedge 1\right).$$

with $\gamma = 1$ for the SFL and $\gamma = s$ for the RFL.
It is hopeless to resume the huge literature about estimates on Green functions.

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It is hopeless to resume the huge literature about estimates on Green functions.

The "dual" formulation of the problem

Recall the homogeneous Dirichlet problem:

$$
\text{(HDP)} \qquad \begin{cases} \partial_t u = -\mathcal{L}\, F(u)\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}
$$

We can formulate a "dual problem", using the inverse $\mathcal{L}^{-1}$ as follows

$$
\partial_t U = -F(u)\,,
$$

where

$$
U(t,x) := \mathcal{L}^{-1}[u(t,\cdot)](x) = \int_\Omega \mathbb{K}(x,y)u(t,y)\,\mathrm{d}y\,.
$$

This formulation encodes the lateral boundary conditions in the inverse operator $\mathcal{L}^{-1}$.

Remark. This formulation has been used before by Pierre, Vázquez [...] to prove (in the $\mathbb{R}^N$ case) uniqueness of the "fundamental solution", i.e. the solution corresponding to $u_0 = \delta_{x_0}$, known as the Barenblatt solution.

The "dual" formulation of the problem

Definition of Weak Dual solutions

Recall that

$$\|f\|_{\mathrm{L}^1_{\Phi_1}(\Omega)} = \int_\Omega f(x)\Phi_1(x)\,\mathrm{d}x\,, \quad \text{and} \quad \mathrm{L}^1_{\Phi_1}(\Omega) := \{f:\Omega\to\mathbb{R}\,\big|\,\|f\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}<\infty\}\,.$$

Weak Dual Solutions

A function u is a *weak dual* solution to the (HDP) if:
$u\in C([0,\infty):\mathrm{L}^1_{\Phi_1}(\Omega))\,,\ F(u)\in \mathrm{L}^1\left((0,T):\mathrm{L}^1_{\Phi_1}(\Omega)\right)$, and moreover
$u(0,x)=u_0\in \mathrm{L}^1_{\Phi_1}(\Omega))$.
The following identity holds for every ψ with $\psi/\Phi_1\in C^1_c((0,T):\mathrm{L}^\infty(\Omega))$:

$$\int_0^T\int_\Omega \mathcal{L}^{-1}(u)\,\frac{\partial\psi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t-\int_0^\infty\int_\Omega F(u)\,\psi\,\mathrm{d}x\,\mathrm{d}t=0.$$

We will need a special class of weak dual solutions:

The class $\mathcal{S}_p$ of weak dual solutions

We consider a class $\mathcal{S}_p$ of nonnegative weak dual solutions u to the (HDP) with initial data in $u_0\in\mathrm{L}^1_{\Phi_1}(\Omega)$, such that (i) the map $u_0\mapsto u(t)$ is order preserving in $\mathrm{L}^1_{\Phi_1}(\Omega)$; (ii) for all $t>0$ we have $u(t)\in L^p(\Omega)$ for some $p\geq 1$.

Monotonicity estimates for powers

The nonlinear flow has a very important monotonicity property, which is related to the m-homogeneity of the equation. Benilan and Crandall proved the following estimates for the case $F(u) = u^m$, with $m > 1$.

Monotonicity estimates

Every mild solution $u \geq 0$ corresponding to an initial datum $u_0 \in \mathrm{L}^1(\Omega)$, satisfies the following differential estimate

$$u_t \geq -\frac{u}{(m-1)t} \qquad \text{in the sense of distributions in } (0, \infty) \times \Omega\,.$$

Alternatively, we have the following monotonicity in time, namely the function

$$t \mapsto t^{\frac{1}{m-1}} u(t,x) \qquad \text{is nondecreasing in } t > 0 \text{ for a.e. } x \in \Omega\,.$$

The fundamental pointwise estimates I: the pure power case

Theorem (M.B. and J. L. Vázquez, 2013)

Let $0 \le u \in \mathcal{S}_p$, with $p > N/2s$. Then,

$$\int_\Omega u(t,x) G_\Omega(x,x_0)\,\mathrm{d}x \le \int_\Omega u_0(x) G_\Omega(x,x_0)\,\mathrm{d}x \qquad \text{for all } t > 0\,.$$

Moreover, for almost every $0 \le t_0 \le t_1$ and almost every $x_0 \in \Omega$, we have

$$\frac{t_0^{\frac{m}{m-1}}}{t_1^{\frac{m}{m-1}}}(t_1 - t_0)\, u^m(t_0, x_0) \le \int_\Omega \big[u(t_0, x) - u(t_1, x) \big]\, G_\Omega(x, x_0)\,\mathrm{d}x \le c_m \frac{t_1^{\frac{m}{m-1}}}{t_0^{\frac{1}{m-1}}}\, u^m(t_1, x_0)$$

with $c_m = m - 1$

Remark. As a consequence of the above inequality and Hölder inequality, we have that $\mathcal{S}_p = \mathcal{S}_\infty$, when $p > N/2s$.

A more general setup

Upper Estimates

- **Absolute upper bounds**
  - **Absolute bounds**
  - **Sharp upper boundary behaviour**
- **Smoothing Effects**
  - L^1-L^∞ **Smoothing Effects**
  - $L^1_{\Phi_1}$-L^∞ **Smoothing Effects**
  - **Backward in time Smoothing effects**

Theorem. (Absolute upper estimate and boundary behaviour)
(M.B. & J. L. Vázquez, 2013)

Let u be a weak dual solution. Then, there exists universal constants $K_1, K_2 > 0$ such that the following estimates hold true: Type I estimates imply

$$\|u(t)\|_{L^\infty(\Omega)} \le \frac{K_1}{t^{\frac{1}{m-1}}}, \qquad \text{for all } t > 0 .$$

Moreover, Type II estimates imply

$$u(t,x) \le K_2 \frac{\Phi_1(x)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}} \qquad \text{for all } t > 0 \text{ and } x \in \Omega .$$

Remark.

- This is a very strong regularization *independent* of the initial datum u_0.
- The boundary estimates are sharp, since we will obtain lower bounds with matching powers.
- This bounds give a sharp time decay for the solution, but only for large times, say $t \ge 1$. For small times we will obtain a better time decay when $0 < t < 1$, in the form of smoothing effects

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Sketch of the proof of Absolute Bounds

• STEP 1. *First upper estimates.* Recall the pointwise estimate:

$$\left(\frac{t_0}{t_1}\right)^{\frac{m}{m-1}} (t_1 - t_0)\, u^m(t_0, x_0) \le \int_\Omega u(t_0, x) G_\Omega(x, x_0)\,\mathrm{d}x - \int_\Omega u(t_1, x) G_\Omega(x, x_0)\,\mathrm{d}x\,.$$

for any $u \in \mathcal{S}_p$, all $0 \le t_0 \le t_1$ and all $x_0 \in \Omega$. Choose $t_1 = 2t_0$ to get

$$(*) \qquad \boxed{u^m(t_0, x_0) \le \frac{2^{\frac{m}{m-1}}}{t_0} \int_\Omega u(t_0, x) G_\Omega(x, x_0)\,\mathrm{d}x\,.}$$

Recall that $u \in \mathcal{S}_p$ with $p > N/(2s)$, means $u(t) \in \mathrm{L}^p(\Omega)$ for all $t > 0$, so that:

$$u^m(t_0, x_0) \le \frac{c_0}{t_0} \int_\Omega u(t_0, x) G_\Omega(x, x_0)\,\mathrm{d}x \le \frac{c_0}{t_0} \|u(t_0)\|_{\mathrm{L}^p(\Omega)} \|G_\Omega(\cdot, x_0)\|_{\mathrm{L}^q(\Omega)} < +\infty$$

since $G_\Omega(\cdot, x_0) \in \mathrm{L}^q(\Omega)$ for all $0 < q < N/(N-2s)$, so that $u(t_0) \in \mathrm{L}^\infty(\Omega)$ for all $t_0 > 0$.

• STEP 2. Let us estimate the r.h.s. of $(*)$ as follows:

$$u^m(t_0, x_0) \le \frac{c_0}{t_0} \int_\Omega u(t_0, x) G_\Omega(x, x_0)\,\mathrm{d}x \le \|u(t_0)\|_{\mathrm{L}^\infty(\Omega)} \frac{c_0}{t_0} \int_\Omega G_\Omega(x, x_0)\,\mathrm{d}x\,.$$

Taking the supremum over $x_0 \in \Omega$ of both sides, we get:

$$\boxed{\|u(t_0)\|^{m-1}_{\mathrm{L}^\infty(\Omega)} \le \frac{c_0}{t_0} \sup_{x_0 \in \Omega} \int_\Omega G_\Omega(x, x_0)\,\mathrm{d}x \le \frac{K_1^{m-1}}{t_0}} \qquad \square$$

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$$\|u(t_0)\|^{m-1}_{\mathrm{L}^\infty(\Omega)} \le \frac{c_0}{t_0} \sup_{x_0\in\Omega} \int_\Omega G_\Omega(x,x_0)\,dx \le \frac{K_1^{m-1}}{t_0} \qquad \square$$

Define the exponents:

$$\vartheta_{1,\gamma} = \frac{1}{2s + (N+\gamma)(m-1)} \qquad \text{and} \qquad \vartheta_1 = \vartheta_{1,0} = \frac{1}{2s + N(m-1)}$$

Theorem. (Smoothing effects) (M.B. & J. L. Vázquez, 2013)

There exist universal constants $K_3, K_4 > 0$ such that the following estimates hold.

L^1-L^∞ SMOOTHING EFFECT: are consequence of Type I bounds

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{K_3}{t^{N\vartheta_1}} \|u(t)\|_{\mathrm{L}^1(\Omega)}^{2s\vartheta_1} \le \frac{K_3}{t^{N\vartheta_1}} \|u_0\|_{\mathrm{L}^1(\Omega)}^{2s\vartheta_1} \qquad \text{for all } t > 0\,.$$

$\mathrm{L}^1_{\Phi_1}$-L^∞ SMOOTHING EFFECT: are consequence of Type II bounds; for all $t > 0$:

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{K_4}{t^{(N+\gamma)\vartheta_{1,\gamma}}} \|u(t)\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{2s\vartheta_{1,\gamma}} \le \frac{K_4}{t^{(N+\gamma)\vartheta_{1,\gamma}}} \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{2s\vartheta_{1,\gamma}}\,.$$

- A novelty is that we get instantaneous smoothing effects.
- Also the weighted smoothing effect is new (as far as we know).
- The time decay is better for small times $0 < t < 1$ than the one given by absolute bounds, namely

$$(N+\gamma)\vartheta_{1,\gamma} = \frac{N+\gamma}{2 + (N+\gamma)(m-1)} < \frac{1}{m-1}\,.$$

Define the exponents:

$$\vartheta_{1,\gamma} = \frac{1}{2s + (N+\gamma)(m-1)} \qquad \text{and} \qquad \vartheta_1 = \vartheta_{1,0} = \frac{1}{2s + N(m-1)}$$

Theorem. (Smoothing effects) (M.B. & J. L. Vázquez, 2013)

There exist universal constants $K_3, K_4 > 0$ such that the following estimates hold.
L^1-L^∞ SMOOTHING EFFECT: are consequence of Type I bounds

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{K_3}{t^{N\vartheta_1}} \|u(t)\|_{\mathrm{L}^1(\Omega)}^{2s\vartheta_1} \le \frac{K_3}{t^{N\vartheta_1}} \|u_0\|_{\mathrm{L}^1(\Omega)}^{2s\vartheta_1} \qquad \text{for all } t > 0\,.$$

$\mathrm{L}^1_{\Phi_1}$-L^∞ SMOOTHING EFFECT: are consequence of Type II bounds; for all $t > 0$:

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{K_4}{t^{(N+\gamma)\vartheta_{1,\gamma}}} \|u(t)\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{2s\vartheta_{1,\gamma}} \le \frac{K_4}{t^{(N+\gamma)\vartheta_{1,\gamma}}} \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{2s\vartheta_{1,\gamma}}\,.$$

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Theorem. (Backward Smoothing effects) (M.B. & J. L. Vázquez, 2013)

There exists a universal constant $K_4 > 0$ such that for all $t, h > 0$

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{K_4}{t^{(d+\gamma)\vartheta_{1,\gamma}}} \left(1 \vee \frac{h}{t}\right)^{\frac{2s\vartheta_{1,\gamma}}{m-1}} \|u(t+h)\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{2s\vartheta_{1,\gamma}} .$$

Proof. By the monotonicity estimates , the function $u(x,t)t^{1/(m-1)}$ is non-decreasing in time for fixed x, therefore using the smoothing effect, we get for all $t_1 \ge t$:

$$\begin{aligned}\|u(t)\|_{\mathrm{L}^\infty(\Omega)} &\le \frac{K_4}{t^{(N+1)\vartheta_{1,\gamma}}} \left(\int_\Omega u(t,x)\Phi_1(x)\,\mathrm{d}x\right)^{2s\vartheta_{1,\gamma}} \\ &\le \frac{K_4}{t^{(N+1)\vartheta_{1,\gamma}}} \left(\frac{t_1^{\frac{1}{m-1}}}{t^{\frac{1}{m-1}}} \int_\Omega u(t_1,x)\Phi_1(x)\,\mathrm{d}x\right)^{2s\vartheta_{1,\gamma}}\end{aligned}$$

where K_4 is as in the smoothing effects. Finally, let $t_1 = t + h$. □

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Lower bounds and Harnack inequalities

- Quantitative positivity estimates
- Weighted $L^1_{\Phi_1}$ estimates
- Harnack inequalities
- Estimates for Elliptic equations

Theorem. (Lower absolute and boundary estimates)
(M.B. & J. L. Vázquez, 2013)

Let let $m > 1$ and let $u \ge 0$ be a weak dual solution to the Dirichlet problem (1), corresponding to the initial datum $0 \le u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Then, there exist constants $L_0(\Omega), L_1(\Omega) > 0$, so that, setting

$$t_* = \frac{L_0(\Omega)}{\left(\int_\Omega u_0 \Phi_1 \,\mathrm{d}x\right)^{m-1}} ,$$

we have that for all $t \ge t_*$ and all $x_0 \in \Omega$, the following inequality holds:

$$\boxed{u(t,x_0) \ge L_1(\Omega)\, \frac{\Phi_1(x_0)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}} .}$$

The constants $L_0(\Omega), L_1(\Omega) > 0$, depend on N, m, s and on Ω , but not on u (or any norm of u); they have an explicit form.

Remarks.

- Recall that Φ_1 is the first eigenfunction of $\mathcal{L}$ and satisfies:

$$\Phi_1(x) \asymp \mathrm{dist}(x, \partial\Omega)^\gamma \wedge 1 \qquad \text{for all } x \in \Omega.$$

  Therefore, the lower boundary behaviour of $u(t, \cdot)$ is:

$$u(t,x) \geq \frac{L_1}{t_0^{\frac{1}{m-1}}} \left(\mathrm{dist}(x_0, \partial\Omega)^{\frac{\gamma}{m}} \wedge 1\right), \qquad \text{for all } t_0 \geq t_* \geq 0 \text{ and } x_0 \in \Omega\,.$$

- This boundary behaviour is sharp because we have upper bounds with matching powers of Φ_1.
- t_* is an estimate the time that it takes to fill the hole: if u_0 is concentrated close to the border (leaves an hole in the middle of Ω), then $\int_\Omega u_0\Phi_1\,\mathrm{d}x$ is small, therefore t_* becomes very large, therefore it takes a lot of time to fill the hole.
- These estimates can also be rewritten as Aronson-Caffarelli type estimates:

$$\text{either} \quad t \leq t_* = \frac{L_0}{\left(\int_\Omega u_0\Phi_1\,\mathrm{d}x\right)^{m-1}}\,, \quad \text{or} \quad u(t,x_0) \geq L_1\,\frac{\Phi_1(x_0)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}} \quad \forall t \geq t_*\,,$$

  which gives, for all $t \geq 0$ and all $x_0 \in \Omega$:

$$u(t,x_0) \geq \frac{L_1\Phi_1(x_0)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}}\left[1 - \left(\frac{t_*}{t}\right)^{\frac{1}{m-1}}\right].$$

- *Open problem:* find precise lower bounds for small times, $0 < t < t_*$.

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Proposition. (Weighted L^1-estimates)

Under the current assumptions on m and u, the integral $\int_\Omega u(t,x)\Phi_1(x)\,dx$ is monotonically non-increasing in time and for all $0 \le \tau_0 \le \tau, t < +\infty$ we have

$$\int_\Omega u(\tau,x)\Phi_1(x)\,dx \le \int_\Omega u(t,x)\Phi_1(x)\,dx + K_5\,|t-\tau|^{2s\vartheta_{1,\gamma}} \left(\int_\Omega u(\tau_0)\Phi_1\,dx\right)^{2s(m-1)\vartheta_{1,\gamma}+1}$$

where $K_5 := \lambda_1 K_4/(2s\vartheta_{1,\gamma})$ and $K_4 > 0$ is the constant in the smoothing effects .

Remark. Notice that, contrary to the usual monotonicity, we can allow $\tau \le t$.

Proposition. (Almost $L^1_{\Phi_1}$-contractivity)

For ordered solutions $u \ge v$, we have that for all $0 \le \tau_0 \le \tau, t < +\infty$

$$\int_\Omega [u(\tau,x)-v(\tau,x)]\Phi_1(x)\,dx \le \int_\Omega [u(t,x)-v(t,x)]\Phi_1(x)\,dx + K_5[u(\tau_0)]\,|t-\tau|^{2s\vartheta_{1,\gamma}} \int_\Omega [u(\tau_0,x)-v(\tau_0,x)]\Phi_1\,dx$$

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Corollary. (Backward in time $L^1_{\Phi_1}$ lower bounds)

For all

$$0 \le \tau_0 \le t \le \tau_0 + \frac{1}{K_6 \left(\int_\Omega u(\tau_0)\Phi_1 \,dx\right)^{m-1}}$$

we have

$$\frac{1}{2}\int_\Omega u(\tau_0,x)\Phi_1(x)\,dx \le \int_\Omega u(t,x)\Phi_1(x)\,dx\,.$$

where $K_6 = (2K_5)^{1/(2s\vartheta_{1,1})} > 0$ and K_5 is as in the above Proposition .

Corollary. (Absolute lower bounds for the $L^1_{\Phi_1}$ norm)

The choice $\tau_0 = 0$ and $t = K_6^{-1}\left(\int_\Omega u_0\Phi_1\,dx\right)^{-(m-1)}$ gives

$$t^{\frac{1}{m-1}}\int_\Omega u(t,x)\Phi_1(x)\,dx \ge \frac{t^{\frac{1}{m-1}}}{2}\int_\Omega u_0(x)\Phi_1(x)\,dx = \frac{1}{2K_6^{m-1}}$$

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Theorem. (Global Harnack Principle) (M.B. & J. L. Vázquez, 2013)

There exist universal constants $H_0, H_1, L_0 > 0$ such that setting

$$t_* = \frac{L_0}{\left(\int_\Omega u_0 \Phi_1 \, \mathrm{d}x\right)^{m-1}},$$

we have that for all $t \geq t_*$ and all $x \in \Omega$, the following inequality holds:

$$H_0 \frac{\Phi_1(x)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq H_1 \frac{\Phi_1(x)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}}$$

Recall that Φ_1 is the first eigenfunction of $\mathcal{L}$.

Remarks.

- This inequality implies local Harnack inequalities of elliptic type
- As a corollary we get the sharp asymptotic behaviour (Part 4)

Solutions u to the parabolic problem inherit the Harnack inequality for Φ_1:

$$\sup_{x\in B_R(x_0)} \Phi_1(x) \le \mathcal{H} \inf_{x\in B_R(x_0)} \Phi_1(x) \qquad \forall B_R(x_0) \in \Omega.$$

The constant $\mathcal{H} > 0$ is universal (and explicit at least when $s = 1$, cf. [BGV-2012]).

Theorem. (Local Harnack Inequalities of Elliptic Type)
(M.B. & J. L. Vázquez, 2013)

There exist universal constants $H_0, H_1, L_0 > 0$ such that setting $t_* = L_0 \|u_0\|_{L^1_{\Phi_1}(\Omega)}^{-(m-1)}$, we have that for all $t \ge t_*$ and all $B_R(x_0) \in \Omega$, the following inequality holds:

$$\sup_{x\in B_R(x_0)} u(t,x) \le \frac{H_1\, \mathcal{H}^{\frac{1}{m}}}{H_0} \inf_{x\in B_R(x_0)} u(t,x)$$

Corollary. (Local Harnack Inequalities of Backward Type)

Under the runninig assumptions, for all $t \ge t_*$ and all $B_R(x_0) \in \Omega$, we have:

$$\sup_{x\in B_R(x_0)} u(t,x) \le 2\frac{H_1\, \mathcal{H}^{\frac{1}{m}}}{H_0} \inf_{x\in B_R(x_0)} u(t+h,x) \qquad \text{for all } 0 \le h \le t_* \,.$$

We consider the homogeneous Dirichlet problem

$$(1) \qquad \begin{cases} \mathcal{L}(V^m) = \lambda V\,, & \text{in } \Omega\,, \\ V = 0\,, & \text{on } \partial\Omega\,, \end{cases}$$

under the running assumptions on $\Omega, m, s, N, \mathcal{L}$.

Theorem. (Bounds and boundary behaviour for the elliptic problem)

Let $V \geq 0$ be a very weak solution to the Dirichlet Problem (1), then there exist universal positive constants h_0 and h_1 such that the following estimates hold true for all $x_0 \in \Omega$:

$$h_0 \|V\|_{\mathrm{L}^1_{\Phi_1}} \Phi_1(x_0) \leq V^m(x_0) \leq h_1 \Phi_1(x_0)\,,$$

where $h_1 = c_{5,\Omega} \lambda^{1/(m-1)}$ and $h_0 = c_{0,\Omega}\lambda$, with $c_{5,\Omega}$ given in Lemma "Integral Green function estimates II" and $c_{0,\Omega}$ is the constant in the Type II lower estimates.

Existence, uniqueness and asymptotic behaviour of solutions

- **Existence and uniqueness theory**
  - **Existence and uniqueness theory in $\mathrm{L}^1_{\Phi_1}$**
  - **Reminder about fractional Sobolev spaces on bounded domains**
  - **Existence and uniqueness theory in $H^*(\Omega)$**
  - **Existence for the elliptic problem via parabolic methods**
- **Asymptotic behaviour of nonnegative solutions**
  - **The rescaled flow and stationary solutions**
  - **Convergence to the stationary profile**
  - **The Friendly Giant and convergence with optimal rate**

Theorem. (Existence and Uniqueness in $\mathrm{L}^1_{\Phi_1}$) (M.B. & J. L. Vázquez, 2013-14)

For every nonnegative $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$ there exists *a unique minimal weak dual solution* to the Dirichlet problem:

$$\text{(HDP)} \qquad \begin{cases} u_t + \mathcal{L}\,F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

Such a solution is obtained as the monotone limit of the semigroup solutions that exist and are unique when the initial data are in $\mathrm{L}^1(\Omega)$. The minimal weak dual solution is continuous in the weighted space $u \in C([0,\infty) : \mathrm{L}^1_{\Phi_1}(\Omega))$. Moreover, it belongs to the class $\mathcal{S}$.

This is a consequence of the upper estimates and of the almost-contractivity in $\mathrm{L}^1_{\Phi}(\Omega)$.

We now pass to a more general framework, and we prove existence and uniqueness.

Consider the Problem

$$\begin{cases} u_t + \mathcal{L}(\varphi(u)) = 0 & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x) & \text{in } \Omega \\ u(t,x) = 0 & \text{on } (0,+\infty)\times\Gamma \end{cases}$$

$\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous, smooth and increasing function.
Assume moreover that $\varphi' > 0$, $\varphi(\pm\infty) = \pm\infty$ and $\varphi(0) = 0$.
The leading example is $\varphi(u) = |u|^{m-1}u$ with $m > 0$.
$\mathcal{L}$ is a linear operator with eigenelements $(\lambda_{k,s}, \phi_{k,s})$.
We study the above problem in the framework of fractional Sobolev spaces:

$$H(\Omega) = \{u = \sum_{k=1}^{\infty} u_k\phi_{s,k} \in L^2(\Omega) \;:\; \|u\|_H^2 = \sum_{k=1}^{\infty} \lambda_{s,k}|u_k|^2 < +\infty\} \subset L^2(\Omega)$$

and let $H^*(\Omega)$ be the topological dual of $H(\Omega)$.
Under some assumption on $\mathcal{L}$, essentially that $\lambda_k \le C^k$ for some $C > 0$,
we can identify $H(\Omega)$ in terms of more familiar spaces:

$$H(\Omega) = \begin{cases} H_0^s(\Omega)\,, & \text{if } \frac{1}{2} < s \le 1\,, \\ H_{00}^{1/2}(\Omega)\,, & \text{if } s = \frac{1}{2}\,, \\ H^s(\Omega)\,, & \text{if } 0 < s < \frac{1}{2}\,. \end{cases}$$

There is another possible characterization of the space $H(\Omega)$,

$$H(\Omega) = \dot{\mathrm{H}}^s(\overline{\Omega}) = \{\, u \in H^s(\mathbb{R}^d) \mid \mathrm{supp}(u) \subset \overline{\Omega}\,\}$$

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Assume moreover that $\varphi' > 0$, $\varphi(\pm\infty) = \pm\infty$ and $\varphi(0) = 0$.
The leading example is $\varphi(u) = |u|^{m-1}u$ with $m > 0$.
$\mathcal{L}$ is a linear operator with eigenelements $(\lambda_{k,s}, \phi_{k,s})$.
We study the above problem in the framework of fractional Sobolev spaces:

$$H(\Omega) = \{u = \sum_{k=1}^{\infty} u_k \phi_{s,k} \in L^2(\Omega) \;:\; \|u\|_H^2 = \sum_{k=1}^{\infty} \lambda_{s,k}|u_k|^2 < +\infty\} \subset L^2(\Omega)$$

and let $H^*(\Omega)$ be the topological dual of $H(\Omega)$.
Under some assumption on $\mathcal{L}$, essentially that $\lambda_k \le C^k$ for some $C > 0$,
we can identify $H(\Omega)$ in terms of more familiar spaces:

$$H(\Omega) = \begin{cases} H_0^s(\Omega)\,, & \text{if } \frac{1}{2} < s \le 1\,, \\ H_{00}^{1/2}(\Omega)\,, & \text{if } s = \frac{1}{2}\,, \\ H^s(\Omega)\,, & \text{if } 0 < s < \frac{1}{2}\,. \end{cases}$$

There is another possible characterization of the space $H(\Omega)$,

$$H(\Omega) = \dot{\mathrm{H}}^s(\overline{\Omega}) = \{\, u \in H^s(\mathbb{R}^d) \mid \mathrm{supp}(u) \subset \overline{\Omega}\,\}$$

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Theorem. (Existence and Uniqueness in H^*) (M.B. , Y. Sire, J. L. Vázquez, 2014)

For every $u_0 \in H^*(\Omega)$ there exists a unique solution $u \in C([0,T] : H^*(\Omega))$ of Problem 2 for every $T > 0$, i.e. the solution is global in time. We also have

$$t\,\varphi(u) \in \mathrm{L}^\infty(0,T \,:\, H^*(\Omega)), \quad t\,\partial_t u \in \mathrm{L}^\infty(0,T \,:\, H^*(\Omega)).$$

We also have $u\varphi(u) \in L^1((0,T)\times\Omega)$. The solution map $S_t : u_0 \mapsto u(t)$ defines a semigroup of (non-strict) contractions in $H^*(\Omega)$, i. e.,

$$\|u(t) - v(t)\|_{H^*(\Omega)} \le \|u(0) - v(0)\|_{H^*(\Omega)},$$

which turns out to be also compact in $H^*(\Omega)$.

Remarks.

- The nonlinearity φ is more general than F, treated in the previous parts of the talk.
- We consider any $m > 0$ and *solutions with any sign.*
- The proof is based on an abstract result of H. Brezis about generation of semigroups on Hilbert spaces.

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In the rest of the talk we consider the nonlinearity $\varphi(u) = |u|^{m-1}u$ with $m > 1$.

Theorem. (Asymptotic behaviour) (M.B. , Y. Sire, J. L. Vázquez, 2014)

There exists a unique nonnegative selfsimilar solution of the Dirichlet Problem (2)

$$U(\tau, x) = \frac{S(x)}{\tau^{\frac{1}{m-1}}},$$

for some bounded function $S : \Omega \to \mathbb{R}$. Let u be any nonnegative H^*-solution to the Dirichlet Problem (2) , then we have (unless $u \equiv 0$)

$$\lim_{\tau\to\infty} \tau^{\frac{1}{m-1}} \|u(\tau, \cdot) - U(\tau, \cdot)\|_{L^\infty(\Omega)} = 0 .$$

The previous theorem admits the following corollary.

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Let $m > 1$. There exists a unique H^* solution to the elliptic problem

$$\begin{cases} \mathcal{L}(S^m) = \dfrac{S}{m-1} & \text{in } \Omega, \\ S(x) = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

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Let u be any nonnegative H^*-solution to the Dirichlet Problem , then we have (unless $u \equiv 0$) that there exist $t_0 > 0$ of the form

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such that for all $t \geq t_0$ we have

$$\left\| \frac{u(t,\cdot)}{U(t,\cdot)} - 1 \right\|_{\mathrm{L}^\infty(\Omega)} \leq \frac{2}{m-1} \, \frac{t_0}{t_0 + t} \, .$$

We remark that the constant $\overline{k} > 0$ only depends on m, d, s, and $|\Omega|$ and has explicit expressions given in the proof.

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- We provide two different proofs of the above result.
- One proof is based on the construction of the so-called Friendly-Giant solution, namely the solution with initial data $u_0 = +\infty$, and is based on the Global Harnack Principle of Part 3
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The End

Daedanhi Gamsahabnida!!!

Grazie Mille!!!

Muchas Gracias!!!

Thank You!!!

Merci Beaucoup!!!

A general class of linear operators

We may consider any linear operator $\mathcal{L} : \mathrm{dom}(\mathcal{L}) \subseteq \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ which is densely defined and such that

(A1) $\mathcal{L}$ is $m-$accretive on $\mathrm{L}^1(\Omega)$,

(A2) If $0 \le f \le 1$ then $0 \le \mathrm{e}^{-t\mathcal{L}} f \le 1$, or equivalently,

(A3) If β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$, $u \in \mathrm{dom}(\mathcal{L})$, $\mathcal{L}u \in \mathrm{L}^p(\Omega)$, $1 \le p \le \infty$, $v \in \mathrm{L}^{p/(p-1)}(\Omega)$, $v(x) \in \beta(u(x))$ a.e , then

$$\int_\Omega v(x)\mathcal{L}u(x)\,\mathrm{d}x \ge 0$$

Remarks.

- These assumptions are needed to obtain the existence (and uniqueness) of semi-group (mild) solutions for the nonlinear equation $u_t = \mathcal{L}F(u)$, through a variant of the celebrated Crandall-Liggett theorem, as done by Benilan, Crandall and Pierre.
- Actually, (A1) and (A2) imply that $\mathrm{dom}(\mathcal{L})$ is dense.

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Assumptions on the inverse in the general case

We will assume that the operator $\mathcal{L}$ has an inverse $\mathcal{L}^{-1}$ with a kernel $\mathbb{K}$ such that

$$\mathcal{L}^{-1}f(x) = \int_\Omega \mathbb{K}(x,y) f(y)\, \mathrm{d}y\,,$$

and that satisfies (one of) the following estimates for some $\gamma, s \in (0,1]$ and $c_{i,\Omega} > 0$

(K1) $$0 \le \mathbb{K}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}}$$

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where $\mathrm{d}(x) := \mathrm{dist}(x, \partial\Omega)$.

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- It is easy to see that (K2) implies $(K3)$, more precisely, $(K2)$ implies that Φ_1 behaves like $\Phi_1 \asymp \mathrm{dist}(\cdot,\partial\Omega)^\gamma$.
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Reminder about Mild solutions and their properties

Mild solutions, or semigroup solutions have been obtained by Benilan, Crandall and Pierre via Crandall-Liggett type theorems; the underlying idea is the use of an Implicit Time Discretization (ITD) method: consider the following partition of $[0, T]$

$$t_k = \frac{k}{n}T\,, \quad \text{for any } 0 \le k \le n\,, \quad \text{with } t_0 = 0\,, t_n = T\,, \text{and} \quad h = t_{k+1} - t_k = \frac{T}{n}\,.$$

For any $t \in (0, T)$, the (unique) semigroup solution $u(t, \cdot)$ is obtained as the limit in $\mathrm{L}^1(\Omega)$ of the solutions $u_{k+1}(\cdot) = u(t_{k+1}, \cdot)$ which solve the following elliptic equation (u_k is the datum, is given by the previous iterative step)

$$h\mathcal{L}F(u_{k+1}) + u_{k+1} = u_k \qquad \text{or equivalently} \qquad \frac{u_{k+1} - u_k}{h} = -\mathcal{L}F(u_{k+1})\,.$$

Usually such solutions are difficult to treat since a priori they are merely very weak solutions. We can prove the following result:

Semigroup solutions with $u_0 \in \mathrm{L}^p$ are weak dual solutions

Let u be the unique mild solution corresponding to the initial datum $u_0 \in \mathrm{L}^p(\Omega)$ with $p \ge 1$. Then u is a weak dual solution and is contained in the class $\mathcal{S}_p$.

Monotonicity estimates in the general case

When the nonlinearity F is not a pure power, the homogeneity fails, therefore one expects a lack of monotonicity. Crandall and Pierre have proven monotonicity estimate under some assumptions on F.

(N1) Assume $F \in C^1(\mathbb{R} \setminus \{0\})$ and $F' \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R} \setminus \{0\})$ and there exists $\mu_0, \mu_1 \in (0,1]$ such that

$$\mu_0 \le \frac{F(r)F''(r)}{[F'(r)]^2} \le \mu_1 \qquad \text{a.e. } r > 0\,.$$

Theorem (M. Crandall and M. Pierre, JFA 1982)

Let $\mathcal{L}$ satisfy (A1) and (A2) and let F as satisfy (N1). Then for all nonnegative $u_0 \in \mathrm{L}^1(\Omega)$, there exists a unique mild solution u to equation $u_t + \mathcal{L}F(u) = 0$, and the function

$$(3) \qquad t \mapsto t^{\frac{1}{\mu_0}} F(u(t,x)) \qquad \text{is nondecreasing in } t > 0 \text{ for a.e. } x \in \Omega\,.$$

Moreover, the semigroup is contractive on $\mathrm{L}^1(\Omega)$ and $u \in C([0,\infty) : \mathrm{L}^1(\Omega))$.

We notice that (3) is a weak formulation of the monotonicity inequality:

$$\partial_t u \ge -\frac{1}{\mu_0\, t}\frac{F(u)}{F'(u)}$$

The fundamental pointwise estimates II: the general case

Theorem (M.B. and J. L. Vázquez, 2014)

Let $0 \le u \in \mathcal{S}_p$, with $p > N/2s$. Then,

$$\int_\Omega u(t,x)\mathbb{K}(x,x_0)\,\mathrm{d}x \le \int_\Omega u_0(x)\mathbb{K}(x,x_0)\,\mathrm{d}x \qquad \text{for all } t > 0\,.$$

Moreover, for almost every $0 < t_0 \le t_1 \le t$ and almost every $x_0 \in \Omega$, we have

$$\left(\frac{t_0}{t_1}\right)^{\frac{1}{\mu_0}} (t_1 - t_0)\, F(u(t_0,x_0)) \le \int_\Omega u(t_0,x)\mathbb{K}(x,x_0)\,\mathrm{d}x - \int_\Omega u(t_1,x)\mathbb{K}(x,x_0)\,\mathrm{d}x$$

$$\le (m_0 - 1)\,\frac{t^{\frac{1}{\mu_0}}}{t_0^{\frac{1-\mu_0}{\mu_0}}}\, F(u(t,x_0))\,.$$

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