LONG TERM BEHAVIOUR FOR SOLUTIONS OF SINGULAR PARABOLIC EQUATIONS

BY

FRANCESCO RAGNEDDA, STELLA PIRO VERNIER: CAGLIARI

V.V.: FLORENCE

PROTOTYPE EQUATIONS

$$\begin{cases} u_t = \Delta u^m & \text{in } \mathbb{R}^N \times [0, \infty[\\ u(\cdot, 0) = \delta & \left(\frac{N-2}{N}\right)_+ < m < 1 \end{cases}$$

$$\begin{cases} u_t = \operatorname{Div}(|Du|^{p-2} Du) & \text{in } \mathbb{R}^N \times [0, \infty[\\ u(\cdot, 0) = \delta & \frac{2N}{N+1} < p < 2 \end{cases}$$

SOME REMARKS

- $u_0 \in L^1$
- UNDER $\frac{2N}{N+1}$ OR $\left(\frac{N-2}{N}\right)_+$ NO REGULARIZING EFFECTS

EXPLICIT SOLUTIONS (BARENBLATT)

1) $$t^{-\frac{N}{\lambda}}\left\{1+\gamma_p\left(\frac{|x|}{t^{\frac{1}{\lambda}}}\right)^{\frac{p}{p-1}}\right\}^{\frac{p-1}{p-2}} = \mathcal{B}_p \qquad t>0$$

$$\lambda = N(p-2)+p$$

$$\gamma_p = \left(\frac{1}{\lambda}\right)^{\frac{1}{p-1}}\frac{2-p}{p}$$

2) $$t^{-\frac{N}{k}}\left\{1+\gamma_m\left(\frac{|x|}{t^{\frac{1}{k}}}\right)^{2}\right\}^{\frac{1}{m-1}} = \mathcal{B}_m \qquad t>0$$

$$k = N(m-1)+2$$

$$\gamma_m = \left(\frac{1}{k}\right)^{\frac{1}{m}}\frac{1-m}{2m}$$

Why singular?

$u_t = m\, \mathrm{Div}\,(u^{m-1} \nabla u)$

If u solution $\not\Rightarrow$ Πu solution

The long term behaviour different if

Ω bounded domain.

$$\begin{cases} u_t = \Delta_p u & \\ u = 0 \quad \text{on } \partial\Omega & \Omega \times [0,\infty[\\ u(0,x) = u_0(x) \geq 0 & \end{cases}$$

Multiply by u and $\int_\Omega dx$

$$\frac{d}{dt} \|u\|_2^2 + 2 \|\nabla u\|_p^p = 0$$

$$\|u\|_2 \leq |\Omega|^{\frac{N(p-2)+2p}{2Np}} \|u\|_{\frac{Np}{N-p}} \leq \gamma |\Omega|^{\frac{N(p-2)+2p}{2Np}} \|\nabla u\|_p$$

$\Downarrow$

$$\frac{d}{dt} \|u\|_2 + \gamma \|u\|_2^{p-1} \leq 0 \Rightarrow$$

$$\|u\|_2(t) \leq c\,(T-t)^{\frac{1}{2-p}}$$

EXTINCTION TIME

$$\mathcal{E}(u) = \left(\frac{\|\nabla u\|_p}{\|u\|_2}\right)^p$$ is not increasing ↘

$$B_p(\Omega) \leq \mathcal{E}(u)(t) \leq \mathcal{E}u(0)$$

best Sobolev constant

$$\frac{d}{dt}\|u\|^2 + 2\|\nabla u\|^p = 0$$

$$\frac{d}{dt}\|u\|^2 + 2\,\mathcal{E}(u)(t)\,\|u\|_2^p = 0$$

$$\Rightarrow \|u\|_2(t) \sim c\,(T-t)^{\frac{1}{2-p}}$$

THE CASE $\Omega = \mathbb{R}^N$ is DIFFERENT

- NO EXTINCTION TIME

- DEPENDS ON N

LONG TERM BEHAVIOUR

v SOLUTION OF $v_t = \Delta v^m$

Let

$$u(x,t) = \alpha(t)^N v(\alpha(t)x, \beta(t))$$

where

$$\alpha(t) = e^t \qquad \beta(t) = \frac{1}{k}(e^{tk} - 1)$$

$$\Rightarrow \quad u_t = \operatorname{div}(xu + \nabla u^m)$$ FOKKER-PLANCK EQUATION

ONLY ONE STATIONARY SOLUTION

FLUX $= 0$ $\qquad xu + \nabla u^m = 0$

$$xu + \nabla u^m = u\left(\frac{m}{m-1}\nabla u^{m-1} + x\right) =$$

$$= \frac{m}{m-1} u \nabla\left(u^{m-1} - \left(c_2 + \frac{1-m}{2m}|x|^2\right)\right)$$

$$\Rightarrow u = \left(c_2 + \frac{\cancel{1}-m}{m}|x|^2\right)^{\frac{1}{m-1}} = u_\infty$$

$$u(x,t) \xrightarrow{?} u_\infty$$

A LOT OF RESULTS:

VAZQUEZ, KAMIN, TOSCANI, CARILLO, MARKOWICH ...

ENTROPY METHOD

$$H(u) = \int_{\mathbb{R}^n} \left(|x|^2 u + \frac{2}{m-1} u^m \right)$$

$$H(u \mid u_\infty) = H(u) - H(u_\infty) \geq 0$$

$$I = \int_{\mathbb{R}^n} u \left| x + \frac{m}{m-1} \nabla u^{m-1} \right|^2 dx$$

$$\frac{d}{dt} H(u(t) \mid u_\infty) = -2I$$

$$\frac{d}{dt} I(u(t)) = -2 I(t) - R(t)$$

$$R(t) \geq 0$$

$$\Rightarrow \quad 0 \leq H(u(t) \mid u_\infty) \leq I(u(t))$$

$$\Rightarrow \quad \frac{d}{dt} H \leq -2H$$

$\Rightarrow$ CONVERGENCE (RATE OF CONVERGENCE)

SHARP AND CLEVER METHOD.

$\Rightarrow$ SPECIAL EQUATIONS

$\Rightarrow$ THE RESULT SHOULD HOLD FOR MORE GENERAL OPERATORS

$$u_t = \operatorname{div} A(x,t,u,Du)$$

$$A(x,t,\varsigma,z)\cdot z \geq C_0 |z|^p$$

$$|A(x,t,\varsigma,z)| \leq C_1 |z|^{p-1}$$

$$\big(A(x,t,\varsigma,z_1) - A(x,t,\varsigma,z_2)\big)(z_1 - z_2) \geq 0$$

$$|A(x,t,\varsigma_1,z) - A(x,t,\varsigma_2,z)| \leq L|\varsigma_1 - \varsigma_2|(1+|z|^{p-1})$$

- MONOTONICITY TO HAVE EXISTENCE
- STRUCTURE CONDITIONS TO HAVE POSITIVITY, COMPARISON

POROUS MEDIUM

$$A(x,t,u,Du)\cdot Du \geq c_0 |u|^{1-m} |D|u^m|^2|$$

$$|A(x,t,u,Du)| \leq c_1 |D|u^m||^2$$

Theorem

Let u be solution of

$$\begin{cases} u_t = \mathrm{Div}\,(A(x,t,u,Du)) & \mathbb{R}^n \times [0,\infty[\\ u(0,x) = \delta \end{cases}$$

$\Rightarrow \exists \; K_1, K_2 > 0$:

$$K_2 B(x,t) \leq u(x,t) \leq K_1 \mathcal{B}(x,t)$$

POINTWISE ESTIMATE

(K. TUUSI - G. MINGIONE ; DIFFERENT CONTEXT)

REMARKS

- IT WORKS FOR GENERAL DATA $u_0 \in L^1$ $u_0 \geq 0$
- FROM THE ESTIMATES ON u $\Rightarrow$ ESTIMATES ON THE ∇u (P-LAPLACE)
- FROM THE ESTIMATE ON u $\Rightarrow$ ESTIMATES ON $D^k u$ (POROUS MEDIUM) $\Rightarrow$ ANALITICITY

  (KWONG- DIBENEDETTO - V) $\left(t_0\, u^{m-1}(x_0,t_0)\right)^{\frac{1}{m}}$ $\frac{t_0}{2}$ $\Rightarrow$ ESTIMATES
- IT CAN QUANTITATIVELY ESTIMATED THE TIME WHEN THE SOLUTION REALIZES THAT A DOMAIN IS NOT UNBOUNDED
- WE USE ESTIMATES ONLY ON P-LAPLACEAN AND POROUS MEDIUM $\rightarrow$ THE ESTIMATES ON FOKKER-PLANCK ARE A BY-PRODUCT

TOOLS:

$L^1 - L^1$ ESTIMATES

$L^1 - L^\infty$ ESTIMATES

DE GIORGI'S LEMMATA

RECENT HARNACK INEQUALITIES.

ESTIMATES FROM ABOVE ("EASY" PART)

$$\sup_{B_\rho(y) \times [s,t]} u \leq \frac{\gamma}{(t-s)^{\frac{N}{\lambda}}} \left(\int_{B_{2\rho}(y)} u(x, 2s-t) \right)^{\frac{p}{\lambda}} + \gamma \left(\frac{t-s}{\rho^p} \right)^{\frac{1}{2-p}}$$

$$\Rightarrow \quad u(0,t) \leq c\, t^{-\frac{N}{\lambda}}$$

$$\Rightarrow \quad u(x,1) \leq c\, |x|^{\frac{p}{p-2}}$$

ESTIMATES FROM BELOW

I° STEP : CONSERVATION OF THE LOCAL MASS

$$\sup_{s<\tau<t} \fint_{B_\rho(y)} u(x,\tau)\,dx \le \bar\gamma \fint_{B_{2\rho}} u(x,t)\,dx + \bar\gamma \left(\frac{t-s}{\rho^r}\right)^{\frac{1}{2-r}}$$

II STEP :

FROM L^1-L^1 ESTIMATES + L^∞ ESTIMATES $\Rightarrow$

CAN APPLY DE GIORGI'S LEMMA

LEMMA

ASSUME

$$\left|\left[u(\cdot,t) \geq M\right] \cap B_\rho(y)\right] \geq \alpha |B_\rho|$$

$\forall \quad s - \varepsilon M^{2-p} \rho^r < t < s$

$\Rightarrow \exists\, \xi \in (0,1)$

$$u(x,t) \geq \xi M \qquad \forall x \in B_{2\rho}(y)$$

$\forall \quad s - \frac{1}{2}\varepsilon M^{2-r} \rho^r < t \leq s$

De Giorgi's Lemma $\Rightarrow$ $u \geq c\, t^{-N/\alpha}$ close to the origin

$\underline{\text{III}}$ STEP:

PROVE THAT $u(t,x) \sim |x|^{\frac{p}{p-2}}$ $|x| \to +\infty$

CONSEQUENCE OF A RECENT HARNACK INEQUALITY (DIBENEDETTO - GIANNAZZA - V)

Assume $u \geq 0$ be a solution of a singular parabolic equation $\Rightarrow$

$$\gamma\, u(x_0, t_0) \leq \inf_{B_\rho(x_0)} u(\cdot, t)$$

$$\forall\, t : t_0 - (c\, u(x_0, t_0))^{2-p} \rho^p \leq t$$

$$\leq t_0 + (c\, u(x_0, t_0))^{2-p} \rho^p$$

<u>ELLIPTIC HARNACK FOR A PARABOLIC EQUATION.</u>