Asymptotic behavior of the degenerate p–Laplacian equation on bounded domains

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p-Laplacian Equation

$\longrightarrow$ Describing the behavior of nonnegative solutions of the p-Laplacian Equation (PLE) for large times.

$$\begin{cases} u_\tau(\tau,x) = \Delta_p u(\tau,x) & \text{for } \tau > 0 \text{ and } x \in \Omega, \\ u(0,x) = u_0(x) & \text{for } x \in \Omega, \\ u(\tau,x) = 0 & \text{for } \tau > 0 \text{ and } x \in \partial\Omega. \end{cases}$$

where:

- $p > 2$.
- $\Omega \in \mathbb{R}^N$ is a bounded connected domain with regular boundary.
- initial data: $u_0 \geq 0, u_0 \in L^r(\Omega),\ r \geq 1$.

Typical nonlinear diffusion models:

- the Porous Medium Equation(PME) :

$$u_t = \Delta u^m, \quad m > 1$$

- the p-Laplacian Equation(PLE):

$$u_t = \Delta_p u, \quad p > 2$$

Different behavior depending on p:

- $1 < p < 2$: extinction in finite time.
- $p > 2$: positivity for all times when $u_0 \geq 0$.

Definition. Weak solution of the Dirichlet problem (PLE):

$$u \in C(0,T;L^2(\Omega)) \cap L^p(0,T;W^{1,p}(\Omega)),$$

s.t. for all $t \in (0,T]$

$$\int \int_{\Omega_t} \left(-u\varphi_t + |\nabla u|^{p-2}\nabla u \nabla \varphi\right) dx d\tau = \int_\Omega u_0(x)\varphi(0,x)dx,$$

for every $T > 0$ and for all bounded test function

$$\varphi \in W^{1,2}(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega)),\ \varphi \geq 0.$$

- Is is known by standard semigroup theory that there exists a unique non-negative weak solution u of the PLE with good regularity properties and satisfies Maximum Principle.

Asymptotic behavior for the PME (J.L.Vázquez, Mon.Math.,2004)

There exists a unique self-similar solution of the PME of the form

$$U(\tau,x)=\tau^{-1/(m-1)}f(x),\ \tau\in(0,+\infty),x\in\Omega,$$

such that if $u\geq 0$ is a any weak solution of the PME we have

$$\lim_{\tau\to+\infty}\tau^{1/(m-1)}|u(\tau,x)-U(\tau,x)|=\lim_{\tau\to+\infty}|\tau^{1/(m-1)}u(\tau,x)-f(x)|=0,$$

unless u is trivial, $u\equiv 0$. The convergence is uniform in space and monotone non-decreasing in time. Moreover, the asymptotic profile f is the unique non-negative solution of the stationary problem:

$$\Delta f^m(x)+\frac{1}{m-1}f(x)=0,\ x\in\Omega,\ \ f(x)=0,\ x\in\partial\Omega$$

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There exists a unique self-similar solution of problem (PLE) of the form

$$U(\tau, x) = \tau^{-1/(p-2)} f(x), \ \tau \in (0, +\infty), x \in \Omega,$$

Then

$$\lim_{\tau\to+\infty} \tau^{1/(p-2)} |u(\tau, x) - U(\tau, x)| = \lim_{\tau\to+\infty} |\tau^{1/(p-2)} u(\tau, x) - f(x)| = 0,$$

unless u is trivial, $u \equiv 0$. The convergence is uniform in space and monotone non-decreasing in time. Moreover, the asymptotic profile f is the unique non-negative solution of the stationary problem:

$$\Delta_p f(x) + \frac{1}{p-2} f(x) = 0, \ x \in \Omega, \ \ f(x) = 0, \ x \in \partial\Omega$$

Estimates

- Bénilan-Crandall type estimates:

$$\|u_\tau(\tau,\cdot)\|_{L^q(\Omega)} \le \frac{1}{(p-2)\tau}\|u_0(\cdot)\|_{L^q(\Omega)}, \;\; q \ge 1.$$

$$\|u_\tau(\tau+s,\cdot)\|_{L^q(\Omega)} \le \frac{1}{(p-2)(\tau+s)}\|u(s,\cdot)\|_{L^q(\Omega)}, \;\; q \ge 1.$$

- Smoothing effects: for $\forall r \ge 1$ there exists $C > 0$ maybe depending on Ω s.t.

$$\|u(t,\cdot)\|_{L^\infty(\Omega)} \le C_0 \frac{\|u(s,\cdot)\|_{L^r(\Omega)}^{rp\vartheta_r}}{(t-s)^{N\vartheta_r}}, \quad \vartheta_r = \frac{1}{rp+(p-2)N}.$$

- Absolute bound: there exists $C = C(\Omega) \ge 0$ s.t.

$$\|u(t,\cdot)\|_{L^\infty(\Omega)} \le Ct^{-1/(p-2)}, t \in (0,+\infty).$$

Sketch of the proof

Idea: the separate variables solution of the PLE :

$$U(\tau,x)=\tau^{-1/(p-2)}f(x), \quad t\geq 0,\ x\in\Omega,$$

where

$$\Delta_p f(x)+\frac{1}{p-2}f(x)=0,\ x\in\Omega,\ f(x)=0,\ x\in\partial\Omega.$$

Method of rescaling and time transformation:

$$v(\tau,x)=\tau^{-\frac{1}{p-2}}v(t,x),\ \tau=e^t.$$

Rescaled problem:

$$\begin{cases} v_t(t,x)=\Delta_p v(t,x)+\frac{1}{p-2}v(t,x), & \text{for } t\in\mathbb{R} \text{ and } x\in\Omega,\\ v(0,x)=v_0(x)=u(x,1), & \text{for } x\in\Omega,\\ v(t,x)=0, & \text{for } t\in\mathbb{R} \text{ and } x\in\partial\Omega.\end{cases}$$

- Bounded and regular initial data $v_0(x)=u(x,1)$.

Convergence

The main tools are the a-priori estimates rewritten as:

$$u(\tau, x) \le C\tau^{-1/(p-2)} \quad \text{and} \quad u_\tau(\tau, x) \ge -C\frac{u}{(p-2)\tau}$$

In the new variable:

$$0 \le v \le C \quad \text{and} \quad v_t \ge 0.$$

$\Longrightarrow$ $\forall x \in \Omega$ there exists the limit

$$\lim_{t\to\infty} v(t,x) = f(x)$$

and this convergence is monotone non-decreasing

$\Longrightarrow$ $f(x)$ is nontrivial and bounded.

$\Longrightarrow$ $v(t,\cdot) \to f$ strong in $L^q(\Omega), 1 \le q < \infty$.

The limit is a stationary solution.

Test function: $\phi(x) \in C_c^\infty(\Omega)$. Fixe $T_0 > 0$ and let $t_2 = t_1 + T_0$.

$$\int_\Omega v(t_2)\phi dx - \int_\Omega v(t_1)\phi dx =$$
$$= -\int_{t_1}^{t_2}\int_\Omega |\nabla v|^{p-2}\nabla v \nabla\phi dxdt + \frac{1}{p-2}\int_{t_1}^{t_2}\int_\Omega v\phi dxdt.$$

Let $t_1 \to \infty$. Then

$$0 = -T_0\int_\Omega |\nabla f|^{p-2}\nabla f \nabla\phi dx + T_0\int_\Omega f\phi dx,$$

$\Longrightarrow$ f is a weak solution of the stationary problem

$$\boxed{-\Delta_p f(x) = \frac{1}{p-2}f(x), \ x \in \Omega.}$$

Also we prove the uniqueness of the stationary solution.

Difference from the PME Case

Difficult convergence:

$$\int_0^{T_0} \int_\Omega |\nabla v(t+n,x)|^{p-2} \nabla v(t+n,x) \nabla \phi \, dx dt \longrightarrow T_0 \int_\Omega |\nabla f|^{p-2} \nabla f \nabla \phi \, dx.$$

Idea:

1. Convergence in measure of gradients:

$$\nabla v(t,\cdot) \to \nabla f(\cdot) \text{ when } t \to \infty \text{ in measure.}$$

2. Energy estimate: $\int_\Omega |\nabla v(t,x)|^p dx \leq M, \quad \forall t \in \mathbb{R}.$

Then $(1) + (2) \Longrightarrow \nabla v(t,\cdot) \to \nabla f(\cdot) \quad$ a.e. in Ω.

Brezis,Cont.Nonl.Funct.An.,1971

Let A be a maximal monotone operator on a Hilbert space H. Let Z_n and W_n be measurable functions from Ω (a finite measure space) into H. Assume $Z_n \to Z$ a.e. on Ω and $W_n \rightharpoonup W$ weakly in $L^1(\Omega; H)$. If $W_n(x) \in A(Z_n(x))$ a.e. on Ω, then $W(x) \in A(Z(x))$ a.e. on Ω.

Our case:

- $\Omega_1 = [0, T_0) \times \Omega$ (finite measure space), $H = \mathbb{R}^N$ (Hilbert space).
- $A : H \to H, \quad A(Z) = |Z|^{p-2} Z$ maximal monotone operator.
- $Z_n(t,x) = \nabla v(t+n,x) : \Omega_1 \to H$,
- $W_n(t,x) = A(Z_n(t,x)) = |\nabla v(t+n,x)|^{p-2} \nabla v(t+n,x) : \Omega_1 \to H$.

$$\boxed{\text{Lemma} \implies W_n(t,x) \rightharpoonup W(t,x) \text{ weakly in } L^1(\Omega_1; H).}$$

Better convergence

Uniform Convergence:

$$\boxed{v(t,x) = \tau^{1/(p-2)} u(\tau,x) \to f(x),\ \ \tau = e^t.}$$

- Idea $\longrightarrow$ Second type of rescaling - fixed rate rescaling:

$$u_\lambda(\tau,x) = \lambda^{\frac{1}{p-2}} u(\lambda\tau), \quad \lambda > 0.$$

- u_λ is still a solution of (PLE).
- On $\Omega \times (\tau_1, \tau_2)$ the family $\{u_\lambda\}_{\lambda>0}$ is equicontinuous (because of the Hölder continuity and the a-priori estimates).
- Ascoli Arzelà Theorem $\Longrightarrow$ uniform convergence on subsequences $(u_{\lambda_j})_j$.
- Remark: $u_\lambda(1,x) = v(\log\lambda, x)$
- $v(\log\lambda_j, x)$ converges uniformly
- The limit $v(t,x) \to f$ is unique $\Longrightarrow$ $v(t,x) \to f$ uniformly.

Rate of convergence for the PME

Hypothesis (H):

1. Ω is a bounded arcwise connected open set with compact closure and regular boundary.
2. u_0 is a nonnegative Lipschitz function defined on $\overline{\Omega}$ such that $u_0 = 0$ on $\partial\Omega$.

Rate of convergence for the PME (Aronson & Peletier, J.Diff.Eq.1981)

Assume that Ω and u_0 satisfy (H). Then $\exists\ \mathcal{C} \in [0, +\infty)$ which depends only on the data such that

$$|(1+t)^{1/(m-1)} u(t,x) - f(x)| \leq \mathcal{C} f(x) (1+t)^{-1} \quad \text{in } \overline{\Omega} \times [0, +\infty).$$

Rate of convergence for the PLE

Assume that Ω and u_0 satisfy (H). Then $\exists\, \mathcal{C} \in [0, +\infty)$ which depends only on the data such that

$$\left|(1+t)^{1/(p-2)} u(t,x) - f(x)\right| \leq \mathcal{C} f(x)(1+t)^{-1} \quad \text{in } \overline{\Omega} \times [0, +\infty).$$

$$\Longrightarrow \quad u(t,x) = U(t,x)\left(1 + \mathcal{O}\left(\frac{1}{t}\right)\right).$$

Consider

- the separate variables solution of the PLE : $U(\tau, x) = \tau^{-1/(p-2)} f(x)$.
- the *rescaled solution* of the PLE: $v(t,x) = \tau^{1/(p-2)} u(\tau, x), \quad \tau = e^t$.

Convergence in relative error

Assume that Ω and u_0 satisfy (H). Then

$$\lim_{\tau \to \infty} \left\| \frac{u(\tau, \cdot)}{U(\tau, \cdot)} - 1 \right\|_{L^\infty(\Omega)} = \lim_{t \to \infty} \left\| \frac{v(t, \cdot)}{f(\cdot)} - 1 \right\|_{L^\infty(\Omega)} = 0.$$

Main steps:

1. **Upper bound.** Prove there exists a constant $\tau_1 > 0$ depending only on p, d, u_0 and Ω s.t.

$$0 \le u(t,x) \le (\tau_1 + t)^{-1/(p-2)} f(x), \quad x \in \Omega, t \ge 0.$$

2. **Positivity.** Prove that even if u_0 has compact support there exists $T_0 > 0$ depending only on p, d, u_0 and Ω s.t.

$$u(t,x) > 0, \quad x \in \Omega, t > T_0.$$

3. **Lower bound.** Prove there exist $T^* \ge 0$ and $\tau_0 > 0$ depending only on p, d, u_0 and Ω s.t.

$$u(t,x) \ge (\tau_0 + t)^{-1/(p-2)} f(x), x \in \overline{\Omega}, t \ge T^*.$$

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Self-Similar solutions for the PLE

Barenblatt solutions:

$$\mathcal{U}(x,t;a,\tau) = c(t+\tau)^{-\alpha}\left\{\left[a^{\frac{p}{p-1}} - \left(|x|(t+\tau)^{-\beta}\right)^{\frac{p}{p-1}}\right]_+\right\}^{\frac{p-1}{p-2}},$$

$$\alpha = \beta N,\ \alpha = \frac{N}{(p-2)N+p},\ \beta = \frac{1}{(p-2)N+p},\ c = \left(\frac{1}{(p-2)N+p}\left(\frac{p-2}{p}\right)^{p-1}\right)^{\frac{1}{p-2}}.$$

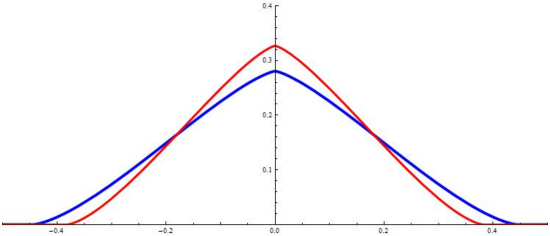

Figure: Barenblatt solutions at $t_1 > t_2$ in $N = 1$

Separate variable solutions:

$$\boxed{U(t,x) = (t+\tau)^{-1/(p-2)} f(x), x \in \Omega,}$$

where τ is a fixed positive parameter and f is the solution of the elliptic equation

$$\Delta_p f + \frac{1}{p-2} f = 0 \text{ in } \Omega, \quad f = 0 \text{ on } \partial\Omega.$$

Intermediate self-similar solutions:

$$\mathcal{V}(x,t;c,\tau) = (t+\tau)^{-\alpha}[g(\eta,c)]_+, \quad \eta = |x|(t+\tau)^{-\beta},$$

$$\begin{cases} \alpha g(\eta) + \beta\eta g'(\eta) + \frac{N-1}{\eta}|g'(\eta)|^{p-2}g'(\eta) + (p-1)|g'(\eta)|^{p-2}g''(\eta) = 0, \ \eta > 0, \\ g(0) = c, \quad g'(0) = 0. \end{cases}$$

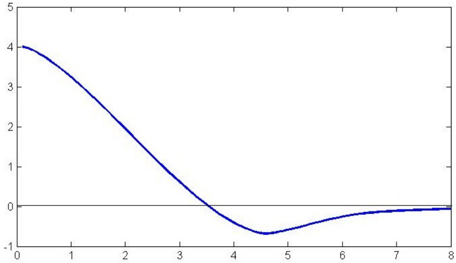

Figure: Solving the Cauchy problem for $N = 2, p = 3$.

Barenblatt solutions $\mathcal{U}(x,t;a,\tau) \longrightarrow$ to describe the behavior inside Ω:

1. Good point: compactly supported and they propagate in time.
2. Good point: solutions in the whole space.
3. Landing contact is flat.

Intermediate family $\mathcal{V}(x,t;c,\tau) \longrightarrow$ to describe the behavior up to $\partial\Omega$:

1. Good point: compactly supported and they propagate in time.
2. They are subsolutions of the PLE in Ω .
3. Landing contact not so flat.

Separate variables solutions $U(t,x)$:

1. Good point: correct boundary behavior.
2. Bad Point: they do not propagate.

Upper bound

Prove there exists a constant $\tau_1 > 0$ depending only on p, d, u_0 and Ω s.t.

$$0 \leq u(t,x) \leq (\tau_1 + t)^{-1/(p-2)} f(x), \quad x \in \Omega, t \geq 0.$$

$\longrightarrow$ Use Comparison Principle between u and the separate variable solution

$$U(t,x) = (\tau_1 + t)^{-1/(p-2)} f(x),$$

for an appropriate constant $\tau_1 > 0$ chosen s.t.

$$\tau_1 f(x) \geq u_0(x) \text{ in } \overline{\Omega}.$$

Positivity

Prove that even if u_0 has compact support there exists $T_0 > 0$ depending only on p, d, u_0 and Ω s.t.

$$u(t,x) > 0, \quad x \in \Omega, t > T_0.$$

- Sufficient to prove the existence of $T \geq 0$ s.t $u(T,x) > 0, \quad x \in \Omega.$
- Prove positivity inside Ω at a time T' using a Lemma about transmitting positivity between neighboring balls.
- Prove positivity up to the boundary using the uniform continuity of $u(\cdot, T')$ in a compact subset of Ω.

Positivity $\longleftrightarrow$ Comparison from below with a Barenblatt solution

Lower bound

Prove there exist $T^* \geq 0$ and $\tau_0 > 0$ depending only on p, d, u_0 and Ω s.t.

$$u(t,x) \geq (\tau_0 + t)^{-1/(p-2)} f(x), x \in \overline{\Omega}, t \geq T^*.$$

Idea:

Prove that

$$\underbrace{u(T^*, \cdot) \geq k_1 \phi(\cdot)}_{\text{Comparison with } \mathcal{V}} \quad \text{and} \quad \underbrace{\phi(\cdot) \geq k_2 f(\cdot)}_{\text{Comparison Principle}} \quad \text{in } \overline{\Omega},$$

where

$$\begin{cases} \Delta_p f + \frac{1}{p-2} f = 0 \text{ in } \Omega \\ f = 0 \text{ on } \partial\Omega. \end{cases} \quad \text{and} \quad \begin{cases} \Delta_p \phi = 1 \text{ in } \Omega \\ \phi = 0 \text{ on } \partial\Omega. \end{cases}$$

Future work:

1. Extend the result for Fast Diffusion case of the p–Laplacian equation:

$$\boxed{u_t = \Delta_p u, \quad 1 < p < 2.}$$

$\longrightarrow$ extinction in finite time, no conservation of mass.
$\longrightarrow$ PME case: BGV-2001.

2. Extend the result for the doubly nonlinear equation

$$\boxed{u_t = \Delta_p u^m.}$$

$\longrightarrow$ The separate variable solutions:

$$U(t,x) = t^{\frac{1}{m(p-1)-1}} f(x),$$

$$\Delta_p f^m = cf \text{ in } \Omega, \quad f = 0 \text{ on } \partial\Omega.$$

Thank you for your attention!