Applications of two new functional inequalities to fractional diffusion I

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Fractional porous medium equation

(FPME) $u_t + (-\Delta)^{\sigma/2}(|u|^{m-1}u) = 0, \quad x \in \Omega,\, t > 0$

- DOMAIN: $\Omega = \mathbb{R}^N$ *or* Ω bounded
- PARAMETERS: $0 < \sigma < 2,\, m > 0$
- INITIAL DATA: $u(\cdot, 0) = f \in L^1(\Omega)$
- BOUNDARY DATA (Ω BOUNDED): $u = 0, \quad x \in \partial\Omega,\, t > 0$

A non-negative, self-adjoint, linear operator

$$A^{\alpha}u(x) = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} \left(e^{-tA}u(x) - u(x)\right) \frac{dt}{t^{1+\alpha}}$$

- $\lambda^{\alpha} = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} \left(e^{-t\lambda} - 1\right) \frac{dt}{t^{1+\alpha}}, \qquad \lambda > 0$

$$(-\Delta)^{\sigma/2}u(x) = \frac{1}{\Gamma(-\frac{\sigma}{2})} \int_0^{\infty} \left(e^{t\Delta}u(x) - u(x)\right) \frac{dt}{t^{1+\frac{\sigma}{2}}}$$

Non-local operator

$$\Omega = \mathbb{R}^N$$

- $\mathcal{F}\big((-\Delta)^{\sigma/2}u\big)(\xi) = |\xi|^\sigma\,\mathcal{F}(u)(\xi)$
- $(-\Delta)^{\sigma/2}u(x) = C_{N,\sigma}\text{P.V.}\int_{\mathbb{R}^N}\frac{u(x+y)-u(x)}{|y|^{N+\sigma}}\,dy$

Ω bounded, homogeneous Dirichlet B. C.

- $u = \sum_{k=1}^{\infty} u_k\varphi_k \qquad \Rightarrow \qquad (-\Delta)^{\sigma/2}u = \sum_{k=1}^{\infty}\lambda_k^{\sigma/2}u_k\varphi_k$

$$\begin{cases} -\Delta\varphi_k = \lambda_k\varphi_k, & x\in\Omega,\\ \varphi_k = 0, & x\in\partial\Omega \end{cases}$$

Motivation

- Non-linear generalization of the *fractional heat equation* (FHE)

$$u_t + (-\Delta)^{\sigma/2} u = 0$$

- Non-local generalization of the *porous medium equation* (PME)

$$u_t - \Delta u^m = 0$$

(Other possible generalizations [Caffarelli-Vázquez, 2009])

- Hydrodynamic limit of zero range processes [Jara, 2009]

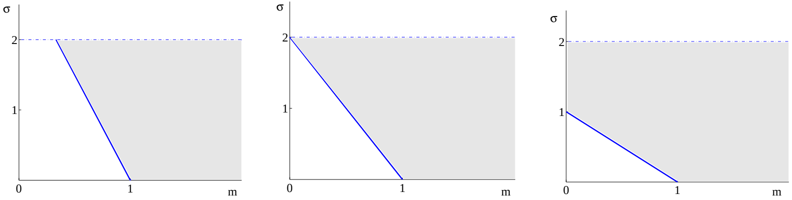
Main results, $m > m_* \equiv (N-\sigma)_+/N$

THEOREM:

$f \in L^1(\mathbb{R}^N) \Rightarrow$ Exists a unique *weak* solution

- SOME PROPERTIES:
  - $\partial_t u \in L^\infty((\tau,\infty) : L^1(\mathbb{R}^N))$ for every $\tau > 0$
  - Conservation of mass
  - L^1-L^∞ smoothing effect
  - Positivity for $f \geq 0$
  - Hölder continuity if either $m \geq 1$ or $f \geq 0$
  - Continuous dependence on the parameters and initial data

Supercritical region: $m > m_* \equiv (N-\sigma)_+/N$



$N \geq 3$ $\quad N = 2$ $\quad N = 1$

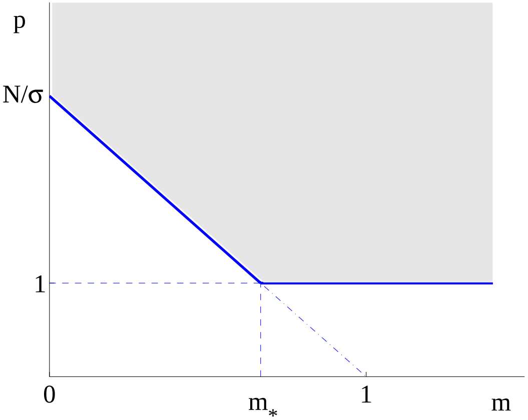
THEOREM:

$$f \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N),\ p > p_*(m) = (1-m)N/\sigma$$
$$\Downarrow$$

Exists a unique *strong* solution

- SOME PROPERTIES:
  - Conservation of mass if $m = m_*$, extinction in finite time if $m < m_*$
  - L^p-L^∞ smoothing effect
  - Positivity up to the extinction time for $f \ge 0$
  - Hölder continuity if $f \ge 0$ up to the extinction time

Existence region of strong solutions



Mild solutions [Crandall-Pierre, 1982]

THEOREM:

$$m > 0,\ \sigma \in (0,2),\ f \in L^1(\mathbb{R}^N)$$
$$\Downarrow$$
Exists a unique *mild* solution (ITD)

- Abstract construction:
  - Not enough information to prove that mild $\Rightarrow$ weak
  - No estimates $\Rightarrow$ no further properties
- However [dPQRV, Preprint]: THEOREM: mild $\Rightarrow$ *very* weak

Integration by parts / Functional space

- $$\begin{aligned}\int_{\mathbb{R}^N} (-\Delta)^{\sigma/2}\psi\,\varphi &= \int_{\mathbb{R}^N} |\xi|^{\sigma}\hat{\psi}\,\hat{\varphi} = \int_{\mathbb{R}^N} |\xi|^{\sigma/2}\hat{\psi}|\xi|^{\sigma/2}\,\hat{\varphi} \\ &= \int_{\mathbb{R}^N} (-\Delta)^{\sigma/4}\psi\,(-\Delta)^{\sigma/4}\varphi\end{aligned}$$

- $$\|\psi\|_{\dot{H}^{\sigma/2}} = \left(\int_{\mathbb{R}^N} |\xi|^{\sigma}|\hat{\psi}|^2\,d\xi\right)^{1/2} = \|(-\Delta)^{\sigma/4}\psi\|_2$$

Weak/strong solutions

Weak (L^1-energy) solution

- $u \in C([0,\infty) : L^1(\mathbb{R}^N))$, $|u|^{m-1}u \in L^2_{\text{loc}}((0,\infty) : \dot{H}^{\sigma/2}(\mathbb{R}^N))$

- $\displaystyle\int_0^\infty \int_{\mathbb{R}^N} u \frac{\partial \varphi}{\partial t}\, dxds - \int_0^\infty \int_{\mathbb{R}^N} (-\Delta)^{\sigma/4}(|u|^{m-1}u)(-\Delta)^{\sigma/4}\varphi\, dxds = 0$,
  $\forall \varphi \in C^1_{\mathsf{c}}(\mathbb{R}^N \times (0,\infty))$

- $u(\cdot, 0) = f$ a.e.

Strong solution

- Weak (L^1-energy) solution

- $\partial_t u \in L^\infty((\tau,\infty) : L^1(\mathbb{R}^N))$ for every $\tau > 0$

Difficulties

- $(-\Delta)^{\sigma/4}(\varphi\psi) = ?$
- $(-\Delta)^{\sigma/4}(\varphi \circ \psi) = ?$

- $(-\Delta)^{\sigma/4}\varphi$ not compactly supported even when φ is
  - $(-\Delta)^{\sigma/4}$ *non-local* operator

σ-harmonic extension [Caffarelli-Silvestre, 2007]

$$v = \mathrm{E}(g) : \quad \begin{cases} L_\sigma v \equiv \mathrm{div}(y^{1-\sigma}\nabla v) = 0, & \mathbb{R}^{N+1}_+ = \{x \in \mathbb{R}^N,\ y > 0\}, \\ v(x,0) = g(x), & x \in \mathbb{R}^N. \end{cases}$$

- $v(x,y) = \displaystyle\int_{\mathbb{R}^N} P(x-\xi,y)g(\xi)\,d\xi, \qquad P(x,y) = d_{N,\sigma}\frac{y^\sigma}{(|x|^2+|y|^2)^{(N+\sigma)/2}}$
- $$\lim_{y\to 0^+} y^{1-\sigma}\frac{\partial v}{\partial y} = \sigma \lim_{y\to 0^+}\frac{v(x,y)-v(x,0)}{y^\sigma}$$

$$= \sigma \lim_{y\to 0^+}\int_{\mathbb{R}^N}\frac{P(x-\xi,y)}{y^\sigma}(g(\xi)-g(x))\,d\xi = -\frac{\sigma d_{N,\sigma}}{C_{N,\sigma}}(-\Delta)^{\sigma/2}g$$

$$\frac{\partial v}{\partial y^\sigma} \equiv \mu_\sigma \lim_{y\to 0^+} y^{1-\sigma}\frac{\partial v}{\partial y} = -(-\Delta)^{\sigma/2}g$$

An equivalent local problem

- $w = \mathrm{E}(|u|^{m-1}u), \quad u = |\operatorname{Tr}(w)|^{\frac{1}{m}-1}\operatorname{Tr}(w)$

$$\begin{cases} L_\sigma w = 0, & (x,y) \in \mathbb{R}^{N+1}_+,\ t > 0, \\ \dfrac{\partial w}{\partial y^\sigma} - \dfrac{\partial |w|^{\frac{1}{m}-1}w}{\partial t} = 0, & x \in \mathbb{R}^N, y = 0,\ t > 0, \\ w = |f|^{m-1}f, & x \in \mathbb{R}^N, y = 0,\ t = 0. \end{cases}$$

- Some proofs (e.g. existence) are easier in this formulation!
- Dynamical boundary conditions (Amann, Escher, Fila, Vitillaro, ...)
- [Athanasopoulos-Caffarelli, 2009]:

u bounded (*weak*) solution + $\left\{ \begin{array}{c} m > 1 \\ \text{or} \\ m < 1 + \text{positivity} \end{array} \right\} \Rightarrow$ continuity

Weak (L^1-energy) solution

- $u \in C([0,\infty) : L^1(\mathbb{R}^N))$, $w \in L^2_{\text{loc}}((0,\infty) : X^\sigma(\mathbb{R}^{N+1}_+))$;
- $$\int_0^\infty \int_{\mathbb{R}^N} u \frac{\partial \varphi}{\partial t}\, dxds - \mu_\sigma \int_0^\infty \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma} \langle \nabla w, \nabla \varphi \rangle \, dxdyds = 0,$$
  $\forall \varphi \in C^1_0\left(\overline{\mathbb{R}^{N+1}_+} \times (0,\infty)\right)$;
- $u(\cdot, 0) = f$ a.e.

► $$\|v\|_{X^\sigma} = \left(\mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma} |\nabla v|^2 \, dxdy \right)^{1/2}$$

Extension / trace operators

- $\mathrm{E} : \dot{H}^{\sigma/2}(\mathbb{R}^N) \to X^\sigma(\mathbb{R}_+^{N+1})$ isometry [Caffarelli-Silvestre, 2007]

- $$\mu_\sigma \int_{\mathbb{R}_+^{N+1}} y^{1-\sigma} \langle \nabla \mathrm{E}(\psi), \nabla \mathrm{E}(\varphi) \rangle = \int_{\mathbb{R}^N} (-\Delta)^{\sigma/4} \psi \, (-\Delta)^{\sigma/4} \varphi$$

- $$\mathrm{Tr}(\Phi_1) = \mathrm{Tr}(\Phi_2)$$
$$\Downarrow$$
$$\int_{\mathbb{R}_+^{N+1}} y^{1-\sigma} \langle \nabla \mathrm{E}(\psi), \nabla \Phi_1 \rangle = \int_{\mathbb{R}_+^{N+1}} y^{1-\sigma} \langle \nabla \mathrm{E}(\psi), \nabla \Phi_2 \rangle$$

- $\mathrm{Tr} : X^\sigma(\mathbb{R}_+^{N+1}) \to \dot{H}^{\sigma/2}(\mathbb{R}^N)$ surjective and continuous

- TRACE EMBEDDING: $\|\Phi\|_{X^\sigma} \geq \|\mathrm{E}(\mathrm{Tr}(\Phi))\|_{X^\sigma} = \|\mathrm{Tr}(\Phi)\|_{\dot{H}^{\sigma/2}}$

THEOREM.

$$f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \quad u \text{ strong solution}$$

- $m > 0$, $p > \max\{1, p_*(m)\}$, $\qquad p_*(m) = (1-m)N/\sigma$

$$\sup_{x\in\mathbb{R}^N} |u(x,t)| \le C\, t^{-N/(N(m-1)+\sigma p)} \|f\|_p^{\sigma p/(N(m-1)+\sigma p)}$$

- $m > m_*$

$$\sup_{x\in\mathbb{R}^N} |u(x,t)| \le C\, t^{-N/(N(m-1)+\sigma))\gamma} \|f\|_1^{\sigma/(N(m-1)+\sigma)}$$

- Hardy-Littlewood-Sobolev's inequality: $1 < r < N/\gamma,\ \ 0 < \gamma < 2$

$$\|v\|_{r_1} \leq C\|(-\Delta)^{\gamma/2}v\|_r, \qquad r_1 = \frac{Nr}{N-\gamma r}$$

- $r = 2$, $\gamma = \sigma/2$, $\sigma < N \qquad \Rightarrow \qquad \dot{H}^{\sigma/2}(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-\sigma}}(\mathbb{R}^N)$

- $N = 1 \leq \sigma < 2$?

Stroock-Varopoulos inequality (local case)

- $q > 1$:

$$\begin{aligned}\int_{\mathbb{R}^N} |v|^{q-2} v(-\Delta) v &= \int_{\mathbb{R}^N} \langle \nabla(|v|^{q-2} v), \nabla v \rangle \\ &= \frac{4(q-1)}{q^2} \int_{\mathbb{R}^N} |\nabla |v|^{q/2}|^2 \\ &= \frac{4(q-1)}{q^2} \int_{\mathbb{R}^N} \left| (-\Delta)^{1/2} |v|^{q/2} \right|^2\end{aligned}$$

Stroock-Varopoulos inequality

THEOREM:

$$\int_{\mathbb{R}^N} (|v|^{q-2}v)(-\Delta)^{\sigma/2}v \geq \frac{4(q-1)}{q^2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\sigma/4}|v|^{q/2}\right|^2, \quad q>1$$

$$\begin{aligned}
\int_{\mathbb{R}^N} (|v|^{q-2}v)(-\Delta)^{\sigma/2}v &= \int_{\mathbb{R}^N} (-\Delta)^{\sigma/4}(|v|^{q-2}v)(-\Delta)^{\sigma/4}v \\
&= \mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma}\langle \nabla\, \mathrm{E}(|v|^{q-2}v), \nabla\, \mathrm{E}(v)\rangle \\
&= \mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma}\langle \nabla(|\,\mathrm{E}(v)|^{q-2}\,\mathrm{E}(v)), \nabla\, \mathrm{E}(v)\rangle \\
&= \mu_\sigma \frac{4(q-1)}{q^2}\int_{\mathbb{R}^{N+1}_+} y^{1-\sigma}|\nabla(|\,\mathrm{E}(v)|^{q/2})|^2 \\
&\geq \frac{4(q-1)}{q^2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\sigma/4}|v|^{q/2}\right|^2
\end{aligned}$$

Smoothing effect, $\sigma < N$: proof (Moser's iteration)

- Multiply by $|u|^{p_k-2}u$, integrate on $\mathbb{R}^N \times (t_k, t_{k+1})$, $t_k = (1-2^{-k})t$:

$$\int_{\mathbb{R}^N} |u|^{p_k}(\cdot, t_k) = \int_{t_k}^{t_{k+1}} \int_{\mathbb{R}^N} (-\Delta)^{\sigma/2}(|u|^{m-1}u)|u|^{p_k-2}u + \int_{\mathbb{R}^N} |u|^{p_k}(\cdot, t_{k+1})$$

$$\geq C \int_{t_k}^{t_{k+1}} \|(-\Delta)^{\sigma/4}|u|^{\frac{p_k+m-1}{2}}(\cdot,\tau)\|_2^2 \, d\tau \qquad \text{(Stroock-Varopoulos)}$$

$$\geq C \int_{t_k}^{t_{k+1}} \|u(\cdot,\tau)\|^{p_k+m-1}_{\frac{N(p_k+m-1)}{N-\sigma}} \, d\tau \qquad \text{(Hardy-Littlewood-Sobolev)}$$

$$\geq C 2^{-(k+1)} t \|u(\cdot, t_{k+1})\|^{p_k+m-1}_{\frac{N(p_k+m-1)}{N-\sigma}} \qquad (L^p\text{-decay})$$

Smoothing effect, $\sigma < N$: proof (Moser's iteration)

- $\|u(\cdot,t_{k+1})\|_{\frac{N(p_k+m-1)}{N-\sigma}} \leq \left(\frac{c}{t}\right)^{\frac{1}{p_k+m-1}} 2^{\frac{k+1}{p_k+m-1}} \|u(\cdot,t_k)\|_{p_k}^{\frac{p_k}{p_k+m-1}}$

- $p_{k+1} \equiv \dfrac{N(p_k+m-1)}{N-\sigma} > p_k$ if $p_0 = p > \dfrac{(1-m)N}{\sigma} = p_*(m)$

- Iteration

A Nash-Gagliardo-Nirenberg type inequality

THEOREM:

$$p \geq 1,\ r > 1,\ 0 < \gamma < \min\{N, 2\}$$

$$\|v\|_{r_2}^{\alpha+1} \leq C\|(-\Delta)^{\gamma/2} v\|_r \, \|v\|_p^{\alpha}, \qquad r_2 = \frac{N(rp + r - p)}{r(N - \gamma)},\ \alpha = \frac{p(r-1)}{r}$$

Proof. Stroock-Varopoulos + Hardy-Littlewood-Sobolev + Hölder

- $r = 2,\ \gamma = \sigma/2,\ \sigma < 2N \quad \Rightarrow \quad \dot{H}^{\sigma/2}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \hookrightarrow L^{\frac{N(p+2)}{2N-\sigma}}(\mathbb{R}^N)$

Smoothing effect: proof (Moser's iteration)

- Multiply by $|u|^{p_k-2}u$, integrate on $\mathbb{R}^N \times (t_k, t_{k+1})$, $t_k = (1-2^{-k})t$:

$$\int_{\mathbb{R}^N} |u|^{p_k}(\cdot, t_k) = \int_{t_k}^{t_{k+1}} \int_{\mathbb{R}^N} (-\Delta)^{\sigma/2}(|u|^{m-1}u)|u|^{p_k-2}u + \int_{\mathbb{R}^N} |u|^{p_k}(\cdot, t_{k+1})$$

$$\geq C \int_{t_k}^{t_{k+1}} \|(-\Delta)^{\sigma/4}|u|^{\frac{p_k+m-1}{2}}(\cdot,\tau)\|_2^2\, d\tau \qquad \text{(Stroock-Varopoulos)}$$

$$\geq \frac{C}{\|u(\cdot,t_k)\|_{p_k}^{p_k}} \int_{t_k}^{t_{k+1}} \|u(\cdot,\tau)\|_{p_k}^{p_k} \|(-\Delta)^{\sigma/4}|u|^{\frac{p_k+m-1}{2}}(\cdot,\tau)\|_2^2\, d\tau \; (L^p\text{-decay})$$

$$\geq \frac{C}{\|u(\cdot,t_k)\|_{p_k}^{p_k}} \int_{t_k}^{t_{k+1}} \|u(\cdot,\tau)\|_{\frac{N(2p_k+m-1)}{2N-\sigma}}^{2p_k+m-1}\, d\tau \; \text{(Nash-Gagliardo-Nirenberg)}$$

$$\geq \frac{C2^{-(k+1)}t}{\|u(\cdot,t_k)\|_{p_k}^{p_k}} \|u(\cdot,t_{k+1})\|_{\frac{N(2p_k+m-1)}{2N-\sigma}}^{2p_k+m-1} \qquad (L^p\text{-decay})$$

Smoothing effect: proof (Moser's iteration)

- $\|u(\cdot,t_{k+1})\|_{\frac{N(2p_k+m-1)}{2N-\sigma}} \le \left(\frac{c}{t}\right)^{\frac{1}{2p_k+m-1}} 2^{\frac{k+1}{2p_k+m-1}} \|u(\cdot,t_k)\|_{p_k}^{\frac{2p_k}{2p_k+m-1}}$

- $p_{k+1} \equiv \dfrac{N(2p_k+m-1)}{2N-\sigma} > p_k$ if $p_0 = p > \dfrac{(1-m)N}{\sigma} = p_*(m)$

- Iteration

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