The role of Wolff potentials in the analysis of degenerate parabolic equations

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Part 1: Ellipticity
The classical potential estimates

Consider the model case

\[-\Delta u = \mu \text{ in } \mathbb{R}^n\]
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We have

$$u(x) = \int G(x, y) \mu(y) \, dy$$
The classical potential estimates

Consider the model case

$$-\Delta u = \mu \quad \text{in} \quad \mathbb{R}^n$$

We have

$$u(x) = \int G(x, y) \mu(y) \, dy$$

where

$$G(x, y) \approx \begin{cases} |x - y|^{2-n} & \text{se } n > 2 \\ -\log |x - y| & \text{se } n = 2 \end{cases}$$
Previous formula gives

\[ |u(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x - y|^{n-2}} = l_2(|\mu|)(x) \]
Previous formula gives

\[ |u(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x - y|^{n-2}} = I_2(|\mu|)(x) \]

while, after differentiation, we obtain

\[ |Du(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x - y|^{n-1}} = I_1(|\mu|)(x) \]
What happens in the nonlinear case?

- For instance for nonlinear equations with linear growth

\[-\text{div } a(Du) = \mu\]

that is equations well posed in $W^{1,2}$ \((p\text{-growth and } p = 2)\)

- And degenerate ones like

\[-\text{div } (|Du|^{p-2}Du) = \mu\]
In bounded domains one uses

\[ I^\mu_\beta(x, R) := \int_0^R \frac{\mu(B(x, \varrho))}{\varrho^{n-\beta}} \frac{d\varrho}{\varrho} \quad \beta \in (0, n] \]

since

\[ I^\mu_\beta(x, R) \lesssim \int_{B_R(x)} \frac{d\mu(y)}{|x - y|^{n-\beta}} = I_\beta(\mu B(x, R))(x) \leq I_\beta(\mu)(x) \]

for non-negative measures.
We consider equations
\[-\text{div } a(Du) = \mu\]
under the assumptions
\[
\begin{align*}
|a(z)| + |\partial a(z)||z| &\leq L|z|^{p-1} \\
\nu^{-1}|z|^{p-2}|\lambda|^2 &\leq \langle \partial a(x, z)\lambda, \lambda \rangle
\end{align*}
\]
with
\[p \geq 2\]
this last bound is assumed in order to keep the exposition brief.
The nonlinear Wolff potential is defined by

$$W_{\beta,p}^\mu(x, R) := \int_0^R \left( \frac{|\mu|(B(x, \varrho))}{\varrho^{n-\beta p}} \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \quad \beta \in (0, n/p]$$

which for $p = 2$ reduces to the usual Riesz potential

$$I_{\beta}^\mu(x, R) := \int_0^R \frac{\mu(B(x, \varrho))}{\varrho^{n-\beta}} \frac{d\varrho}{\varrho} \quad \beta \in (0, n]$$

The nonlinear Wolff potential plays in nonlinear potential theory the same role the Riesz potential plays in the linear one.
A fundamental estimate

For solutions to $\operatorname{div} (|Du|^{p-2}Du) = \mu$ with $p \leq n$ we have

\[ |u(x)| \lesssim W_{\mu, p}(x,R) + \left( -\int_{B(x,\varrho)} |u|^p \right)^{1/(p-1)} \]

where $W_{\mu, p}(x,R) := \int_R^0 \left( |\mu|_{B(x,\varrho)} \right)^{1/(p-1)} d\varrho$.

For $p = 2$ we have $W_{\mu, 2} = I_{\mu}$. Another approach to this result has been given by Trudinger & Wang (Amer. J. Math. 02).
For solutions to \( \text{div} \left( |Du|^{p-2} Du \right) = \mu \) with \( p \leq n \) we have

**Theorem (Kilpeläinen-Malý, Acta Math. 94)**

\[
|u(x)| \lesssim W_{1,p}^\mu(x, R) + \left( \int_{B(x, R)} |u|^{p-1} \, dy \right)^{\frac{1}{p-1}}
\]
A fundamental estimate

For solutions to $\text{div} \left( |Du|^{p-2} Du \right) = \mu$ with $p \leq n$ we have

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$$|u(x)| \lesssim W_{1,p}^\mu(x, R) + \left( \int_{B(x,R)} |u|^{p-1} \, dy \right)^{\frac{1}{p-1}}$$

where

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For $p = 2$ we have $W_{1,p}^\mu = I_{2}^\mu$
A fundamental estimate

For solutions to $\text{div } (|Du|^{p-2} Du) = \mu$ with $p \leq n$ we have

$$|u(x)| \lesssim W_{1,p}^\mu(x, R) + \left( \int_{B(x, R)} |u|^{p-1} \, dy \right)^{\frac{1}{p-1}}$$

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For $p = 2$ we have $W_{1,p}^\mu = I_2^\mu$

Another approach to this result has been given by Trudinger & Wang (Amer. J. Math. 02)
Integral estimates follow via Wolff inequalities

We have

\[ \mu \in L^q \iff W_{\beta,p}^{\mu} \in L^{\frac{pq(p-1)}{n-qp\beta}} \quad q \in (1, n) \]

with related explicit estimates, also in Marcinkiewicz spaces.

Such a property allows to reduce the study of integrability of solutions to that of nonlinear potentials.

The key is the following inequality:

\[
\int_0^\infty \left( \frac{|\mu|(B(x, \varrho))}{\varrho^{n-\beta p}} \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \lesssim I_{\beta}\left\{ I_{\beta}(|\mu|)^{\frac{1}{p-1}} \right\}(x)
\]

the last quantity is called Havin-Maz’ja potential.
The potential gradient estimate for $p = 2$

\[
|D_\xi u(x)| \lesssim \mathcal{I}_1^\mu(x, R) + \int_{B(x, R)} |D_\xi u| \, dy
\]

holds for almost every point $x$ and $\xi \in \{1, \ldots, n\}$

**Theorem (Min., JEMS 2011)**

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The role of Wolff potentials in the degenerate problems
A general potential gradient estimate

Theorem (Duzaar & Min., Amer. J. Math. 2011)

$|Du(x)| \lesssim W_{1/p, p}^{\mu}(x, R) + \int_{B(x, R)} |Du| \, dy$

holds for almost every point $x$
Theorem (Duzaar & Min., Amer. J. Math. 2011)

\[ |Du(x)| \lesssim W_{1/p,p}^\mu (x, R) + \int_{B(x,R)} |Du| \, dy \]

holds for almost every point \( x \)

This means

\[ |Du(x)| \leq c \int_0^R \left( \frac{\|\mu\|(B(x, \varrho))}{\varrho^{n-1}} \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} + c \int_{B(x,R)} |Du| \, dx \]
General potential gradient estimate

Theorem (Duzaar & Min., Amer. J. Math. 2011)

\[ |Du(x)| \lesssim W_{1/p,p}^\mu(x, R) + \int_{B(x,R)} |Du| \, dy \]

holds for almost every point \( x \)
First remarks

Since

$$\mu \in L^q \implies W_{1/p,p}^\mu \in L^{\frac{nq(p-1)}{n-q}} \quad q \in (1, n)$$

therefore, for instance

$$\mu \in L^q \implies Du \in L^{\frac{nq(p-1)}{n-q}}_{\text{loc}}(\Omega) \quad q \in (1, n)$$
- Since

\[ \mu \in L^q \implies W_{1/p,p}^{\mu} \in L^{\frac{nq(p-1)}{n-q}} \quad q \in (1, n) \]

therefore, for instance

\[ \mu \in L^q \implies Du \in L_{\text{loc}}^{\frac{nq(p-1)}{n-q}} (\Omega) \quad q \in (1, n) \]

- More in general, estimates in all rearrangement invariant spaces follow, recovering all those already known for the model case \( \text{div} \left( |Du|^{p-2}Du \right) = \mu \), and fixing \textbf{open borderline cases}
The two potential estimates are

\[ |u(x)| \lesssim W_{1,p}^{\mu}(x, R) + c \int_{B(x,R)} |u| \, dy \]

and

\[ |Du(x)| \leq c W_{1/p,p}^{\mu}(x, R) + \int_{B(x,R)} |Du| \, dy \]

They basically provide size estimates on \( u \) and \( Du \).

The aim is now to provide estimates on the oscillations of solutions and/or alternatively, on intermediate derivatives.
The following definition is due to DeVore & Sharpley (Mem. AMS, 1982)

Let $\alpha \in (0, 1]$, $q \geq 1$, and let $\Omega \subset \mathbb{R}^n$ be a bounded open subset. A measurable function $v$, finite a.e. in $\Omega$, belongs to the Calderón space $C_q^\alpha(\Omega)$ if and only if there exists a nonnegative function $m \in L^q(\Omega)$ such that

$$|v(x) - v(y)| \leq [m(x) + m(y)]|x - y|^\alpha$$

holds for almost every couple $(x, y) \in \Omega \times \Omega$. 
Calderón spaces of DeVore & Sharpley

- In other words
  \[ m(x) \approx \partial^\alpha v(x) \]
- Indeed DeVore & Sharpley take
  \[ M_\#^\alpha v(x) = \sup_{B(x,\varrho)} \varrho^{-\alpha} \int_{B(x,\varrho)} |v(y) - (v)_{B(x,\varrho)}| \, dy \]
- For \( \alpha \in (0, 1) \) and \( q > 1 \) we have
  \[ W^{\alpha,q} \subset C^{\alpha,q} \subset W^{\alpha-\varepsilon,q} \]
  therefore such spaces, although not being of interpolation type, are just another way to say “fractional differentiability”
A “universal potential estimate”

**Theorem (Kuusi & Min.)**

The estimate

\[
|u(x) - u(y)| 
\leq c \left[ W^{1-\frac{\alpha(p-1)}{p},p}(x, R) + W^{1-\frac{\alpha(p-1)}{p},p}(y, R) \right] |x - y|^\alpha 
+ c \int_{B_R} |u| d\xi \cdot \left( \frac{|x - y|}{R} \right)^\alpha
\]

holds uniformly in \( \alpha \in [0, 1] \), whenever \( x, y \in B_{R/4} \)

- The cases \( \alpha = 0 \) and \( \alpha = 1 \) give back the two known potential estimates as endpoint cases
The homogeneous case

- The estimate tells that

$$\partial^\alpha u(x) \lesssim W_{1-\frac{\alpha(p-1)}{p},p}(x,R)$$
The homogeneous case

- The estimate tells that
  \[ \partial^\alpha u(x) \lesssim W_1^{\mu} W_{1-\frac{\alpha(p-1)}{p}}(x, R) \]

- The case \( \mu = 0 \) reduces to the classical estimate
  \[ |u(x) - u(y)| \leq \int_{B_R} |u| \, d\xi \cdot \left( \frac{|x - y|}{R} \right)^\alpha \]
The homogeneous case

- The estimate tells that
  \[
  \partial^{\alpha} u(x) \lesssim W_{1 - \frac{\alpha(p-1)}{p}, p}^{\mu} (x, R)
  \]

- The case \( \mu = 0 \) reduces to the classical estimate
  \[
  |u(x) - u(y)| \leq \int_{B_R} |u| \, d\xi \cdot \left( \frac{|x - y|}{R} \right)^{\alpha}
  \]

- In the case \( p = 2 \) we have
  \[
  |u(x) - u(y)| \leq c \left[ I_{2-\alpha}^{\mu} (x, R) + I_{2-\alpha}^{\mu} (y, R) \right] |x - y|^{\alpha}
  + c \int_{B_R} |u| \, d\xi \cdot \left( \frac{|x - y|}{R} \right)^{\alpha}
  \]
  which in the classical case \(-\Delta u = \mu\) can be derived directly from the standard representation formula via potentials.
The second universal potential estimate

Theorem (Kuusi & Min.)

The estimate

\[
|Du(x) - Du(y)| 
\leq c \left[ W_{\mu}^{1 - \frac{(1+\alpha)(p-1)}{p}}(x, R) + W_{\mu}^{1 - \frac{(1+\alpha)(p-1)}{p}}(y, R) \right] |x - y|^\alpha 
\]

\[
+ c \int_{B_R} |Du| d\xi \cdot \left( \frac{|x - y|}{R} \right)^\alpha 
\]

holds whenever \( \alpha < \alpha_M \), whenever \( x, y \in B_{R/4} \)

- The case \( \alpha = 0 \) gives back the gradient potential estimate
Part 2: Parabolicity
The model case is here given by

$$u_t - \text{div} (|Du|^{p-2} Du) = \mu,$$

more in general we consider

$$u_t - \text{div} a(Du) = \mu.$$

A basic reference for existence and a priori estimates is the work of Boccado, Dall’Aglio, Galloüet and Orsina, J. Funct. Anal., 1997.
Problematic aspects

- For basic scaling reasons the previous potential estimates do not hold in the case $p \neq 2$
- For the case $p = 2$ it holds

Theorem (Duzaar & Min., Amer. J. Math. 2011)

$$|Du(x, t)| \lesssim |\mu|^1(x, t; r) + \int_{Q_r(x, t)} |Du| \, dz$$

holds for almost every point $(x, t)$ and $\xi \in \{1, \ldots, n\}$
Here $I^{|\mu|}_{1}(x, t; r)$ denotes the parabolic Riesz potential

$$I^{|\mu|}_{\beta}(x, t; r) := \int_{0}^{r} \frac{|\mu|(Q_\varrho(x, t))}{\varrho^{N-\beta}} \frac{d\varrho}{\varrho}, \quad \beta < N := n + 2$$

and

$$Q_\varrho(x, t) = B(x, \varrho) \times (t - \varrho^2, t)$$

is a standard parabolic cylinder

**The case $p \neq 2$ is a very different story**
Recall the elliptic estimate

Theorem (Duzaar & Min., Amer. J. Math. 2011)

\[ |Du(x)| \lesssim W^{\mu}_{1/p,p}(x, R) + \int_{B(x,R)} |Du| \, dy \]

holds for almost every point \( x \)
Recall the elliptic estimate

\[ |Du(x)| \lesssim W_{1/p,p}^\mu (x, R) + \int_{B(x,R)} |Du| \, dy \]

holds for almost every point \( x \)

This means

\[ |Du(x)| \lesssim \int_0^R \left( \frac{|\mu|(B(x, \varrho))}{\varrho^{n-1}} \right)^{1/p-1} \frac{d\varrho}{\varrho} + \int_{B(x,R)} |Du| \, dx \]
Another way to say the same thing

if

$$c \int_{B(x,R)} |Du| \, dy + c W_{1/p,p}^{\mu}(x, R) \leq \lambda$$

then

$$|Du(x)| \leq \lambda$$
The basic analysis is the following: consider intrinsic cylinders

\[ Q^\lambda_\varrho(x, t) = B(x, \varrho) \times (t - \lambda^{2-p} \varrho^2, t) \]

where it happens that

\[ |Du| \approx \lambda \quad \text{in} \quad Q^\lambda_\varrho(x, t) \]

then the equation behaves as

\[ u_t - \lambda^{p-2} \Delta u = 0 \]

that is, scaling back in the same cylinder, as the heat equation.

On intrinsic cylinders estimates “ellipticize”; in particular, they become homogeneous.
DiBenedetto’s intrinsic estimate

- The effect of intrinsic geometry

**Theorem (DiBenedetto & Friedman, Crelle J. 85)**

*There exists a universal constant $c \geq 1$ such that*

$$c \left( \int_{Q_r^\lambda(x,t)} |Du|^{p-1} \, dz \right)^{1/(p-1)} \leq \lambda$$

*then*

$$|Du(x, t)| \leq \lambda$$
Define the intrinsic Wolff potential such that

$$W_{\chi}^{\mu}(x, t; r) := \int_0^r \left[ \frac{|\mu|(Q^\chi(x, t))}{\lambda^{2-p}Q^{N-1}} \right]^{1/(p-1)} \frac{dQ}{Q}$$

Note that

$$W_{\chi}^{\mu}(x, t; r) = I_{1}^{\mu}(x, t; r) \quad \text{when } p = 2$$

and

$$W_{\chi}^{\mu}(x, t; r) = W_{1/p, p}^{\mu}(x, r) \quad \text{when } \mu \text{ is time independent}$$

this is the elliptic case
The parabolic Wolff gradient bound

**Theorem (Kuusi & Min.)**

There exists a universal constant \( c \geq 1 \) such that

\[
cW^\mu_\lambda (x, t; r) + c \left( \int_{Q^\lambda_r (x, t)} |Du|^{p-1} \, dz \right)^{1/p-1} \leq \lambda
\]

then

\[|Du(x, t)| \leq \lambda\]

- When \( \mu \equiv 0 \) this reduces to the sup estimate of DiBenedetto & Friedman (Crelles J. 84)
Consider the equation
\[ u_t - \text{div} \left( |Du|^{p-2} Du \right) = \delta, \]
where \( \delta \) denotes the Dirac unit mass charging the origin.

The so-called Barenblatt (fundamental solution) is
\[
B_p(x, t) = \begin{cases} 
  t^{-n/\theta} \left( c_b - \theta^{1-p} \left( \frac{p-2}{p} \right) \left( \frac{|x|}{t^{1/\theta}} \right)^{p-1} \right)^{p-1} & t > 0 \\
  0 & t \leq 0 
\end{cases}
\]
for \( \theta = n(p - 2) + p \) and a suitable constant \( c_b \) such that
\[
\int_{\mathbb{R}^n} B_p(x, t) \, dx = 1 \quad \forall \ t > 0
\]
A direct computation shows the following upper optimal upper bound

$$|DB_p(x, t)| \leq ct^{-(n+1)/\theta}$$

The intrinsic estimate above exactly reproduces this upper bound

This decay estimate is indeed reproduced for all those solutions that are initially compactly supported
A priori estimates

The previous bound always implies a priori estimates on standard parabolic cylinders

Theorem (Kuusi & Min.)

\[
|Du(x, t)| \lesssim \left[ \int_0^r \left( \frac{|\mu| (Q_\varrho(x, t))}{\varrho^{N-1}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho} \right]^{p-1}
+ \int_{Q_r(x,t)} (|Du| + s + 1)^{p-1} \, dz
\]

holds for every standard parabolic cylinder \(Q_r\)
Theorem (Kuusi & Min.)

\[ |Du(x, t)| \lesssim [W_1^{\mu}(x, t; r)]^{p-1} + \int_{Q_r(x, t)} (|Du| + s + 1)^{p-1} \, dx \, dt \]

holds for every standard parabolic cylinder \( Q_r \)

- The scaling deficit exponent \( p - 1 \) appears
Occurrence of deficit scaling exponents

For solutions to

\[ u_t - \Delta_p u = \text{div} (|F|^{p-2} F) \]


\[
\left( \int_{Q_r} |Du|^q \, dz \right)^{\frac{1}{q}} \lesssim \left[ \left( \int_{Q_{2r}} |Du|^p \, dz \right)^{\frac{1}{p}} + \left( \int_{Q_{2r}} |F|^q \, dz \right)^{\frac{1}{q}} + 1 \right]^{\frac{p}{2}}
\]

for every \( q \geq p \)

- The scaling deficit exponent \( p/2 \) appears
A priori estimates

Theorem (Kuusi & Min.)

It \( \mu \) is time independent than

\[
|Du(x, t)| \lesssim W_{1/p,p}^{\mu}(x, r) + \int_{Q_r} (|Du| + s + 1)^{p-1} \, dx \, dt
\]

holds for every standard parabolic cylinder \( Q_r \).
A basic tool

Theorem (Kuusi & Min., General regularity estimate)

\[ A, B, q \geq 1 \quad \text{and} \quad \varepsilon \in (0, 1). \]

Then there exists a constant \( \delta_\varepsilon \in (0, 1/2) \)

\[ \frac{\lambda}{B} \leq \sup_{Q_{\delta_\varepsilon}^\lambda} \| Dw \| \leq \sup_{Q_r^\lambda} \| Dw \| \leq A\lambda \]

holds, then

\[ E_q(Dw, \delta_\varepsilon Q_r^\lambda) \leq \varepsilon E_q(Dw, Q_r^\lambda) \]

holds, where

\[ E_q(Dw, Q_\varrho^\lambda) := \left( \int_{Q_\varrho^\lambda} |Dw - (Dw)_{Q_\varrho^\lambda}|^q \, dx \, dt \right)^{1/q} \]
Theorem (Kuusi & Min.)

Assume that

$$\lim_{r \to 0} \sup_{(x,t) \in \Omega_T} W_1^\mu(x, t; r) = 0$$

holds, then \( Du \) is continuous in \( Q_T \).
Gradient continuity via potentials

**Theorem (Kuusi & Min.)**

Assume that

\[
\lim_{r \to 0} \sup_{(x,t) \in \Omega_T} W_1^\mu(x, t; r) = 0
\]

holds, then \( Du \) is continuous in \( Q_T \)

Previous assumption reads as

\[
\lim_{r \to 0} \sup_{(x,t) \in \Omega_T} \int_0^r \left( \frac{|\mu|((Q_\varrho(x, t)))}{\varrho^{N-1}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho} = 0
\]
Gradient continuity via potentials

Theorem (Kuusi & Min.)

Assume that

$$\lim_{r \to 0} \sup_{(x,t) \in \Omega_T} W_1^\mu(x, t; r) = 0$$

holds, then $Du$ is continuous in $Q_T$

Previous assumption reads as

$$\lim_{r \to 0} \sup_{(x,t) \in \Omega_T} \int_0^r \left( \frac{|\mu|(Q_\varphi(x, t))}{\varphi^{N-1}} \right)^{1/(p-1)} \frac{d\varphi}{\varphi} = 0$$

i.e. the convergence is uniform
Theorem (Kuusi & Min., General regularity estimate)

\[ A, B, q \geq 1 \quad \text{and} \quad \varepsilon \in (0, 1). \]

Then there exists a constant \( \delta_{\varepsilon} \in (0, 1/2) \)

\[ \frac{\lambda}{B} \leq \sup_{Q_{\delta_{\varepsilon}r}^\lambda} \| Dw \| \leq \sup_{Q_r^\lambda} \| Dw \| \leq A\lambda \]

holds, then

\[ E_q(Dw, \delta_{\varepsilon} Q_r^\lambda) \leq \varepsilon E_q(Dw, Q_r^\lambda) \]

with

\[ \delta_{\gamma} = \frac{1}{c(A)} \left( \frac{\varepsilon}{B} \right)^{1/\alpha}, \]
Gradient continuity via potentials

Theorem (Kuusi & Min.)

Assume that

\[ |\mu|_1(Q_\varrho) \lesssim \varrho^{N-1+\delta} \]

holds, then there exists \( \alpha \), depending on \( \delta \), such that

\[ Du \in C^{0,\alpha} \quad \text{locally in } Q_T \]
Thanks for the attention (self-portrait of Serena Nono)

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