Radial solutions to the Emden-Fowler equation on the hyperbolic space

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joint work with Matteo Bonforte - Filippo Gazzola - Juan Luis Vázquez

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- Volume element: $d\mathrm{Vol} = y^{-n} dx\, dy$;
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  $\Delta = y^2 \left(\Delta_x + \partial^2/\partial y^2\right) - (n-2)y(\partial/\partial y)$;
- Riemannian distance d between two points:
  $\cosh\left(\frac{d}{2}\right) = \left[\frac{|x_1 - x_2|^2 + (y_1 - y_2)^2}{4 y_1 y_2}\right]^{1/2}$.

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$$\begin{aligned} \Delta f(\varrho) &= f''(\varrho) + (n-1) \coth \varrho \, f'(\varrho) \\ &= \frac{1}{(\sinh \varrho)^{n-1}} \frac{d}{d\varrho} \left[(\sinh \varrho)^{n-1} f'(\varrho) \right] . \end{aligned}$$

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- $\sigma_{L^2}(-\Delta) = \left[\frac{(n-1)^2}{4}, +\infty\right)$ (Poincaré-type inequality).
- $\|u\|_{2n/(n-2)} \le C\|\nabla u\|_2$ (Sobolev inequality: related to the curvature bound and the behaviour of the Green's function).

The radial setting

The quantity ϱ can be used to give the hyperbolic space the structure of a **model manifold**: given a pole o, the metric has the form

$$ds^2 = d\varrho^2 + f(\varrho)^2 d\omega^2,$$

for an appropriate function f, where ϱ is the Riemannian distance from the pole o and $\mathrm{d}\omega^2$ is the canonical metric on $\mathbb{S}^{n-1}$.

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The volume element is $d\mu = (\sinh \varrho)^{n-1} \, d\varrho \, d\sigma$, where $d\sigma$ is the volume element on $\mathbb{S}^{n-1}$.

The Emden-Fowler equation on the hyperbolic space

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A function is **radial** if it depends on the Riemannian distance r from a pole o.
Our purpose is classifying smooth radial solutions which satisfy the ODE

$$u''(\varrho) + (n-1)(\coth \varrho) u'(\varrho) + |u(\varrho)|^{p-1} u(\varrho) = 0 \qquad \text{for } \varrho > 0\,,$$

together with the initial conditions $u(0) = \alpha$, $u'(0) = 0$.

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for the slightly more general equation $\Delta u + \lambda u + |u|^{p-1}u = 0$ in the range $p \in (1, \frac{n+2}{n-2})$. They consider **energy solutions** in $H^1(\mathbb{H}^n)$. Here variational methods can be successfully employed.

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Later, Punzo (JDE 2011) studied the Dirichlet problems on balls and related evolution equations, also considered by Bandle, Pozio, Tesei (JDE 2011, to appear).

EUCLIDEAN CASE

We discuss our results in comparison with those regarding radial solutions to the intensively studied Euclidean problem $\Delta u + u^p = 0$, namely with solutions to

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$$\lim_{\varrho \to +\infty} \varrho^{1/(p-1)} u(\varrho) = c(n,p) := \left(\frac{n-1}{p-1} \right)^{1/(p-1)}$$

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For $u(0) < 0$, $u'(0) = 0$, the solutions are everywhere negative and decay polynomially with the opposite limit $-c(n,p)$. In particular, any radial solution u belongs to $\mathrm{L}^q(\mu)$ only for $q = \infty$.

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EUCLIDEAN CASE: result qualitatively similar but solutions decay differently, like $\varrho^{-2/(p-1)}$.

REMARK As a byproduct of our proof we obtain the following non-existence result for solutions to the Dirichlet problem in a ball:

COROLLARY If $p \geq \frac{n+2}{n-2}$, then for any radius $R > 0$, the equation

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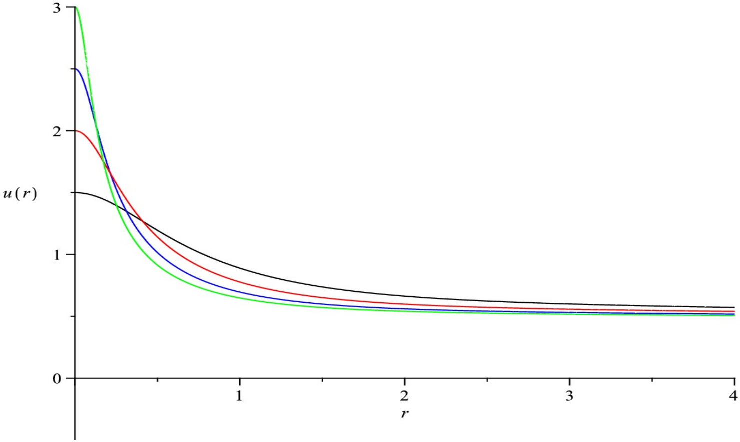
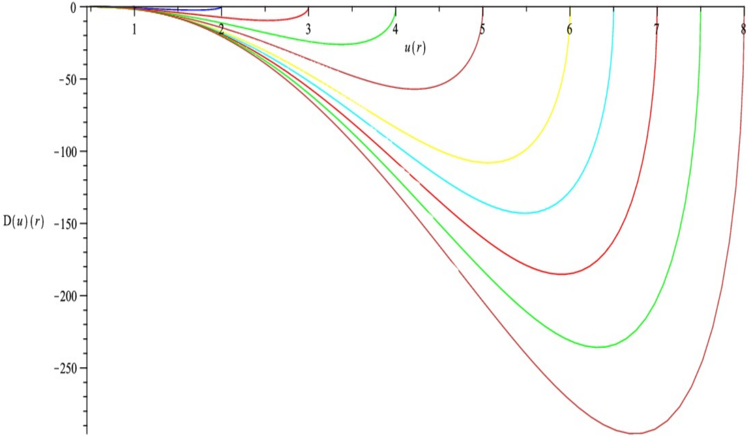
In the case $p = \frac{n+2}{n-2}$ this result is already known (Stapelkamp, Proceedings (Rolduc/Gaeta) 2002)



Plot of some solutions when $d = 3$, $p = 6$ (supercritical case).



Phase plot of some solutions when $d = 3$, $p = 6$ (supercritical case).

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$$\Psi(\varrho) := \phi_n(\varrho)\left(\frac{u'(\varrho)^2}{2} + \frac{|u(\varrho)|^{p+1}}{p+1}\right) + (\sinh\varrho)^{n-1}\frac{u(\varrho)u'(\varrho)}{p+1}.$$

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- If $p \geq \frac{n+2}{n-2}$, then $\Psi'(\varrho) < 0$ and $\Psi(\varrho) < 0$ for all $\varrho > 0$.
- If $1 < p < \frac{n+2}{n-2}$, then $\exists R_{n,p} > 0$ s.t. $\Psi'(\varrho) > 0$ for all $\varrho < R_{n,p}$, $\Psi'(\varrho) < 0$ for all $\varrho > R_{n,p}$ so that $\varrho \mapsto \Psi(\varrho)$ is eventually decreasing and admits a limit as $\varrho \to +\infty$.

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- If $0 < p \leq 1$, then $\Psi'(\varrho) > 0$ and $\Psi(\varrho) > 0$ for all $\varrho > 0$.

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Moreover, there exists a unique $\varrho_0 > 0$ such that $u(\varrho_0) = U(\varrho_0)$ and the asymptotic behavior is given by

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(iii) $\exists$ infinitely many radial sign-changing solutions $u \notin H^1(\mathbb{H}^n)$, having exactly one zero, and satisfying (same constant!)

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EUCLIDEAN CASE: all sign-changing radial solutions have infinitely many zeros (see Pucci-Serrin, Asympt. Anal. 1991).

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– Bhakta-Sandeep proved that there exist infinitely many sign-changing solutions which can be chosen to be radial and belonging to $H^1(\mathbb{H}^n)$.

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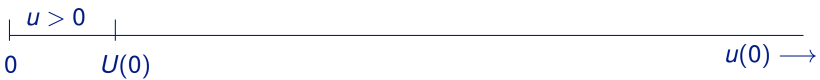

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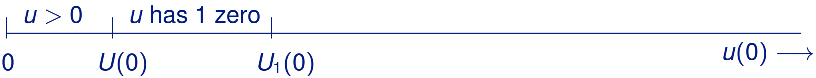

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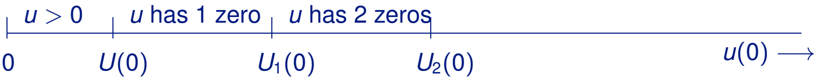

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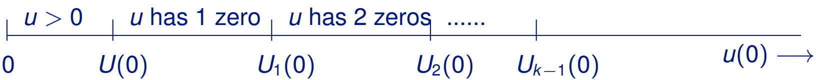

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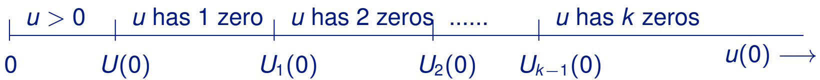

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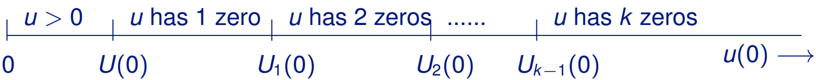

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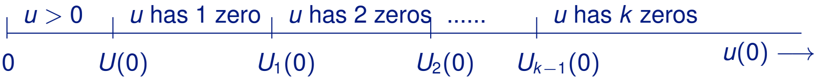

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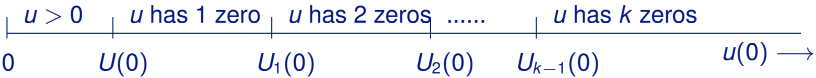

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This conjecture is motivated by our proof: we show that **zeros of u may enter from infinity once at a time as $u(0)$ increases**.

Numerics shows that in the supercritical case and for large dimensions and p large the solutions are ordered and do not intersect. The corresponding result is true in the Euclidean setting.

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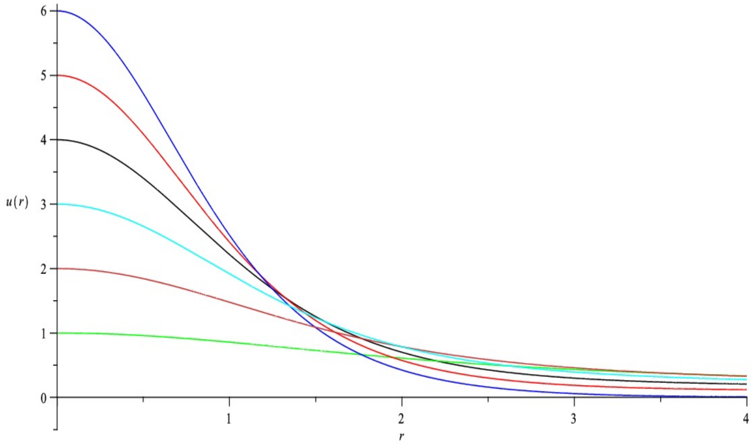
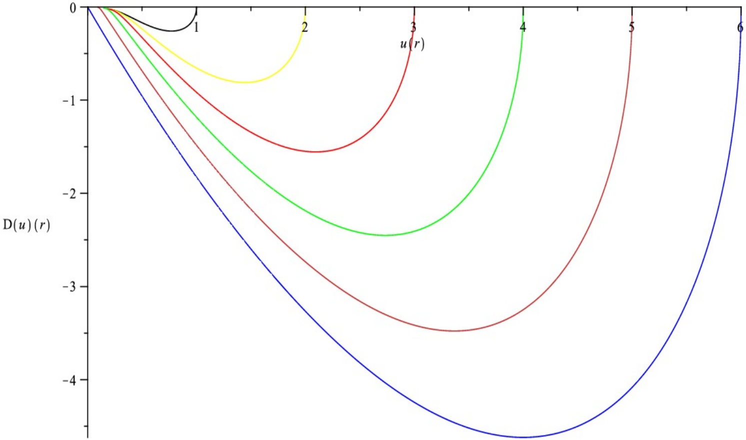
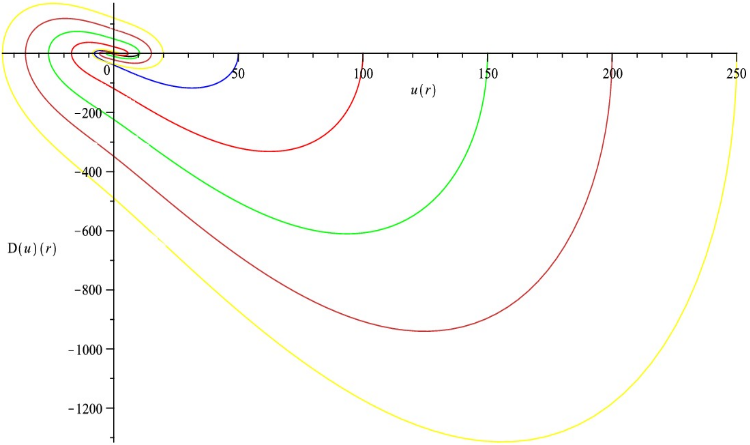
Such inequalities are true by interpolation between the Sobolev and Poincaré inequalities on $\mathbb{H}^n$.



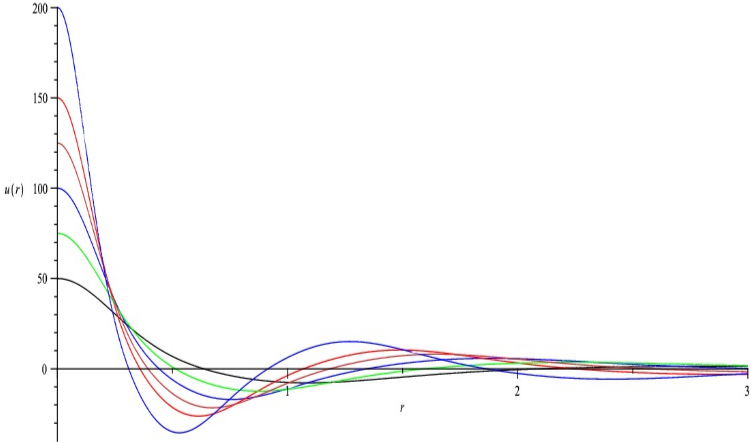
Plot of some positive solutions when $d = 3$, $p = 2$ (subcritical case). The special exponentially decaying solution U corresponds to the blue line ($U(0) = 6$)



Phase plot of some positive solutions when $d = 3$, $p = 2$ (subcritical case). The special exponentially decaying solution U corresponds to the blue line ($U(0) = 6$)



Phase plot of some sign-changing solutions when $d = 3$, $p = 2$ (subcritical case).



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EUCLIDEAN CASE: nonexistence of positive solutions, all sign-changing solutions have infinitely many zeros,

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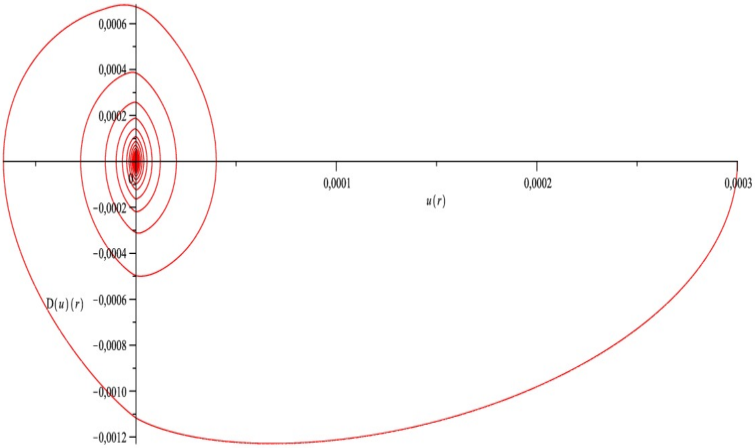

Phase plot of one sign-changing solution when $d = 3$, $p = \frac{1}{2}$ (sublinear case).

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This is known to be an attractor for more general solutions in the Euclidean case: $\frac{u}{\mathcal{U}} - 1 \to 0$ in L^∞ as $t \to T$ (see M. Bonforte's talk). We believe that similar results can be proved here in the range $m \in \left(\frac{n-2}{n+2}, 1\right)$.

THANK YOU FOR YOUR ATTENTION!