Radial solutions to the Emden-Fowler equation on the hyperbolic space

Gabriele Grillo - Politecnico di Milano (Italy)

joint work with Matteo Bonforte - Filippo Gazzola - Juan Luis Vázquez

UAM - Madrid, September 20, 2011
The hyperbolic space $\mathbb{H}^n$ - a short reminder

Several coordinates models are possible. E.g. we can identify $\mathbb{H}^n$ with $\mathbb{R}^n - 1 \times \mathbb{R}^+ \mathbb{R}^+$ endowed with the metric $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$, $x \in \mathbb{R}^n - 1$, $y > 0$.

The Ricci curvature is shown to be constant and negative. Other important facts:

Volume element: $dV = y^{-n} dx dy$;

Laplacian (on functions): $\Delta = y^2 (\Delta x + \frac{\partial^2}{\partial y^2}) - (n-2)y \frac{\partial}{\partial y}$;

Riemannian distance $d$ between two points: $\cosh(d^2) = \left[ |x_1 - x_2|^2 + (y_1 - y_2)^2 \right]^{1/2}$.
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- Volume element: $d\text{Vol} = y^{-n} dx \, dy$;
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  $$\Delta = y^2 \left( \Delta_x + \frac{\partial^2}{\partial y^2} \right) - (n - 2)y \frac{\partial}{\partial y};$$
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  $$\Delta = y^2 (\Delta_x + \partial^2 / \partial y^2) - (n - 2)y (\partial / \partial y);$$
- Riemannian distance $d$ between two points:
  $$\cosh \left( \frac{d}{2} \right) = \left[ \frac{|x_1 - x_2|^2 + (y_1 - y_2)^2}{4y_1y_2} \right]^{1/2}.$$
Let, given a fixed \( a \in \mathbb{H}^n \), \( \varrho(x) = d(a, x) \). The above formulas imply that \( \Delta \varrho = (n - 1) \cosh \varrho \). Hence, for a function depending on \( \varrho \) only:

Further crucial functional analytic properties:

\[
\|u\|_2^2 \leq C \|\nabla u\|_2^2 \quad \text{(Sobolev inequality: related to the curvature bound and the behaviour of the Green's function).}
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- \( \sigma_{L^2}(-\Delta) = \left[ \frac{(n-1)^2}{4}, +\infty \right) \) (Poincaré-type inequality).
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Further crucial functional analytic properties:

- \( \sigma_{L^2(-\Delta)} = \left[ \frac{(n-1)^2}{4}, +\infty \right) \) (Poincaré-type inequality).
- \( \|u\|_{2n/(n-2)} \leq C \|\nabla u\|_2 \) (Sobolev inequality: related to the curvature bound and the behaviour of the Green’s function).
The radial setting

The quantity $\varrho$ can be used to give the hyperbolic space the structure of a **model manifold**: given a pole $o$, the metric has the form

$$ds^2 = d\varrho^2 + f(\varrho)^2 d\omega^2,$$

for an appropriate function $f$, where $\varrho$ is the Riemannian distance from the pole $o$ and $d\omega^2$ is the canonical metric on $S^{n-1}$.
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The volume element is $d\mu = (\sinh \varrho)^{n-1} \, d\varrho \, d\sigma$, where $d\sigma$ is the volume element on $S^{n-1}$. 

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The Emden-Fowler equation on the hyperbolic space

Consider the following nonlinear elliptic equation

\[ \Delta u + |u|^{p-1}u = 0 \quad \text{in } \mathbb{H}^n, \]

on the hyperbolic space \( \mathbb{H}^n \).
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$\Delta := \text{div}\nabla$ is the Laplace-Beltrami operator on $\mathbb{H}^n$ and we take $p > 0$. 

A function is radial if it depends on the Riemannian distance $r$ from a pole $o$.

Our purpose is classifying smooth radial solutions which satisfy the ODE

$$u''(\varrho) + (n-1)(\coth \varrho)u'(\varrho) + |u(\varrho)|^{p-1}u(\varrho) = 0 \quad \text{for } \varrho > 0,$$

together with the initial conditions $u(0) = \alpha$, $u'(0) = 0$. 

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The study of this problem was initiated by

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for the slightly more general equation \( \Delta u + \lambda u + |u|^{p-1}u = 0 \) in the range \( p \in (1, \frac{n+2}{n-2}) \). They consider energy solutions in \( \text{H}^1(\mathbb{H}^n) \). Here variational methods can be successfully employed.

Later, Punzo (JDE 2011) studied the Dirichlet problems on balls and related evolution equations, also considered by Bandle, Pozio, Tesei (JDE 2011, to appear).
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EUCLIDEAN CASE

We discuss our results in comparison with those regarding radial solutions to the intensively studied Euclidean problem $\Delta u + u^p = 0$, namely with solutions to

$$u''(\varrho) + \frac{n-1}{\varrho} u'(\varrho) + |u(\varrho)|^{p-1} u(\varrho) = 0 \quad \text{for } \varrho > 0,$$

together with the initial conditions $u(0) = \alpha, u'(0) = 0$. 
THE SUPERCRITICAL CASE

THEOREM

For any $p \geq n + 2n - 2$, the equation

$$\Delta u + |u|^{p-1}u = 0 \text{ in } H^n,$$

admits infinitely many positive radial solutions $u(\varrho)$ and infinitely many negative solutions. All radial solutions $u$ with $u(0) > 0$, $u'(0) = 0$, are everywhere positive and decay polynomially at infinity with the following rates

$$\lim_{\varrho \to +\infty} \varrho^{1/(p-1)}u(\varrho) = c(n, p) := (n - 1)\varrho^{-1/(p-1)}$$

$$\lim_{\varrho \to +\infty} u'(\varrho)u(\varrho) = \lim_{r \to +\infty} u''(\varrho)u'(\varrho) = 0.$$

For $u(0) < 0$, $u'(0) = 0$, the solutions are everywhere negative and decay polynomially with the opposite limit $-c(n, p)$. In particular, any radial solution $u$ belongs to $L^q(\mu)$ only for $q = +\infty$.

EUCLIDEAN CASE: result qualitatively similar but solutions decay differently, like $\varrho^{-2}/(p-1)$.
THE SUPERCritical CASE

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For \( u(0) < 0, u'(0) = 0 \), the solutions are everywhere negative and decay polynomially with the opposite limit \(-c(n, p)\). In particular, any radial solution \( u \) belongs to \( L^q(\mu) \) only for \( q = \infty \).

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**REMARK** As a byproduct of our proof we obtain the following non-existence result for solutions to the Dirichlet problem in a ball:

**COROLLARY** If \( p \geq \frac{n+2}{n-2} \), then for any radius \( R > 0 \), the equation

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In the case \( p = \frac{n+2}{n-2} \) this result is already known (Stapelkamp, Proceedings (Rolduc/Gaeta) 2002).
Plot of some solutions when $d = 3$, $p = 6$ (supercritical case).
Phase plot of some solutions when $d = 3, p = 6$ (supercritical case).
METHOD OF PROOF: A POHOŽAEV-TYPE FUNCTIONAL

Let \( \phi_n(\varrho) = \int_{\varrho_0}^{\varrho} \sinh(s)^{n-1} \, ds \) and for all solution to 
\[ u''(\varrho) + (n-1)\coth(\varrho)u'(\varrho) + |u(\varrho)|^{p-1}u(\varrho) = 0 \]
define \( \Psi(\varrho) := \phi_n(\varrho)(u'(\varrho)^2 + |u(\varrho)|^p + 1)^{\frac{1}{p}} + \sinh(\varrho)^{n-1}u(\varrho)u'(\varrho)^{p+1}. \)

• If \( p \geq n + \frac{2}{n-2} \), then \( \Psi'(\varrho) < 0 \) and \( \Psi(\varrho) < 0 \) for all \( \varrho > 0 \).

• If \( 1 < p < n + \frac{2}{n-2} \), then \( \exists R_n,p > 0 \) s.t. \( \Psi'(\varrho) > 0 \) for all \( \varrho < R_n,p \), \( \Psi'(\varrho) < 0 \) for all \( \varrho > R_n,p \) so that \( \varrho \mapsto \Psi(\varrho) \) is eventually decreasing and admits a limit as \( \varrho \to +\infty \).

• If \( 0 < p \leq 1 \), then \( \Psi'(\varrho) > 0 \) and \( \Psi(\varrho) > 0 \) for all \( \varrho > 0 \).
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\[ \cdot \text{If } 1 < p < n + \frac{2}{n-2}, \text{ then } \exists R_n, p > 0 \text{ s.t. } \Psi'(\varrho) > 0 \text{ for all } \varrho < R_n, p \text{ and } \Psi'(\varrho) < 0 \text{ for all } \varrho > R_n, p \text{ so that } \varrho \mapsto \Psi(\varrho) \text{ is eventually decreasing and admits a limit as } \varrho \to +\infty. \]

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$$\Psi(\varrho) := \phi_n(\varrho) \left( \frac{u'(\varrho)^2}{2} + \frac{|u(\varrho)|^{p+1}}{p+1} \right) + (\sinh \varrho)^{n-1} \frac{u(\varrho)u'(\varrho)}{p+1}.$$

• If $p \geq \frac{n+2}{n-2}$, then $\Psi'(\varrho) < 0$ and $\Psi(\varrho) < 0$ for all $\varrho > 0$.

• If $1 < p < \frac{n+2}{n-2}$, then there exists $R_{n,p} > 0$ such that $\Psi'(\varrho) > 0$ for all $\varrho < R_{n,p}$, $\Psi'(\varrho) < 0$ for all $\varrho > R_{n,p}$, so that $\varrho \mapsto \Psi(\varrho)$ is eventually decreasing and admits a limit as $\varrho \to +\infty$.

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- If \( 0 < p \leq 1 \), then \( \Psi'(\varrho) > 0 \) and \( \Psi(\varrho) > 0 \) for all \( \varrho > 0 \).
THE SUBCRITICAL CASE

A first main difference with the supercritical case is the existence of a positive global solution having fast decay at infinity.

THEOREM (Mancini-Sandeep)

Let \( 1 < p < \frac{n+2}{n-2} \). There exists a unique function \( U \in H^1(\mathbb{R}^n) \) which is a radial positive and bounded solution to the equation

\[
\Delta U + |U|^{p-1}U = 0 \quad \text{in} \quad \mathbb{R}^n.
\]

The function \( U \) is (radially) decreasing and \( \exists c > 0 \) such that

\[
\lim_{\rho \to +\infty} e^{(n-1)\rho} U(\rho) = c.
\]

Of course, \( \exists! \) negative ground state which is given by \(-U\).

EUCLIDEAN CASE:

\( \not\exists \) positive solutions.

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Of course, there exists a negative ground state which is given by $-U$.

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We use the ground state $U$ in the classification of all radial solutions. We restrict ourselves to the case $u(0) = \alpha > 0$.

**THEOREM**

Let $1 < p < n + \frac{n - 2}{2}$ and let $U$ be the unique positive ground state. Each local solution $u$ satisfying $0 < u(0) < U(0)$ can be extended as a positive solution for $0 < \rho < \infty$, hence generating a positive radial solution to $\Delta u + |u|^{p-1}u = 0$ in $H^1_r(\mathbb{H}^n)$.

Moreover, there exists a unique $\rho_0 > 0$ such that $u(\rho_0) = U(\rho_0)$ and the asymptotic behavior is given by $\lim_{\rho \to +\infty} \rho^{1/(p-1)}u(\rho) = c(n, p)$, the same constant of the supercritical case.

None of these slow-decaying solutions belongs to the energy space $H^1_r(\mathbb{H}^n)$. 

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We use the ground state $U$ in the classification of all radial solutions. We restrict ourselves to the case $u(0) = \alpha > 0$.

**THEOREM** Let $1 < p < \frac{n+2}{n-2}$ and let $U$ be the unique positive ground state. Each local solution $u$ satisfying

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**Theorem** Let \( 1 < p < \frac{n+2}{n-2} \) and let \( U \) be the unique positive ground state. Each local solution \( u \) satisfying

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the same constant of the supercritical case. None of these slow-decaying solutions belongs to the energy space \( H^1_r(\mathbb{H}^n) \).
A second main difference is the presence of sign-changing solutions.

**Theorem**

Let $1 < p < n + 2n - 2$ and let $U$ be the unique positive ground state. If $u(0) > U(0)$, then $u$ is sign-changing.

Moreover:

(i) if $u(0) = \alpha > U(0)$ and if $\varrho_\alpha$ denotes the first zero of $u$, then $\alpha \rightarrow \varrho_\alpha$ is strictly decreasing from $(U(0), \infty)$ into $(0, \infty)$;

(ii) any radial sign-changing solution has finitely many zeros;

(iii) $\exists$ infinitely many radial sign-changing solutions $u \not\in H^1(H^n)$, having exactly one zero, and satisfying (same constant!)

$$\lim_{\varrho \to +\infty} \varrho e^{(n-1)\varrho U(\varrho)} = c(n, p).$$

(iv) $\forall k \geq 1 \exists$ infinitely many solutions having exactly $k$ zeros;

(v) $\forall$ radial sign-changing solution $u \in H^1(H^n) \exists c \in \mathbb{R}$ s.t.

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**Euclidean Case:** all sign-changing radial solutions have infinitely many zeros (see Pucci-Serrin, Asympt. Anal. 1991).
A second main difference is the presence of sign-changing solutions.

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SOME COMMENTS

– We can identify the solution $U$ with the separatrix between the sign-changing class from the globally positive radial solutions in hyperbolic space. In particular all radial solutions $u$ satisfying $u(0) > U(0)$ change sign.

– The $L^\infty$-norm $U(0)$ of the variational solution $U$ is the optimal a priori bound for all positive radial and global solutions in the subcritical case. Sign-changing solutions have no a priori bound.

– Item (iv) can be complemented with the statement that for any integer $k \geq 1$ there exists $\alpha_k > 0$ such that if $u(0) > \alpha_k$, then the solution has at least $k$ zeros.

– Mancini-Sandeep proved that the corresponding Dirichlet problem admits a unique radial positive solution in any ball of finite radius.

– Bhakta-Sandeep proved that there exist infinitely many sign-changing solutions which can be chosen to be radial and belonging to $H^1_0(H^n)$.
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Emden-Fowler equation on the hyperbolic space
CONJECTURES

$u(0) \rightarrow 0$

$u > 0$ $U(0)$ $u$ has 1 zero

$U(1)(0)$ $u$ has 2 zeros

$U(2)(0)$ $u$ has $k$ zeros

The solutions $\{U_k\}$ have finite (increasing and divergent) energy.

This conjecture is motivated by our proof: we show that $u$ may enter from infinity once at a time as $u(0)$ increases.

Numerics shows that in the supercritical case and for large dimensions and $p$ large the solutions are ordered and do not intersect. The corresponding result is true in the Euclidean setting.

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\[ u > 0 \]

0 \hspace{1cm} U(0) \hspace{1cm} u(0) \rightarrow
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SOME EXPLICIT GROUND STATES

\[ U(\varrho) = \left[ n^2 \left( n^2 - 1 \right) \right]^{n-1} \left( 1 + \cosh \varrho \right)^{n-1} \] for \( p = \frac{n}{n-1} \)

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\[ U(\varrho) = \left( n \left( n^2 - 1 \right) \right)^\frac{n}{n+1} \left( n^2 - 1 \right)^{\frac{n-1}{2}} \left( \sinh 2\varrho + 1 \right)^{n/2} \left( n^2 - 1 \right)^{\frac{n-1}{2}} \] for \( p = \frac{n+3}{n-1} \).

These are the extremals for the best constant in the inequalities

\[ \| u \|_q \leq C \| \nabla u \|_2, \quad q = \frac{2}{n-1}, \quad q = \frac{2}{n}, \quad q = \frac{2}{n+2} \]

Such inequalities are true by interpolation between the Sobolev and Poincaré inequalities on \( H^1 \).

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Plot of some positive solutions when $d = 3$, $p = 2$ (subcritical case). The special exponentially decaying solution $U$ corresponds to the blue line ($U(0) = 6$)
Phase plot of some positive solutions when $d = 3$, $p = 2$ (subcritical case). The special exponentially decaying solution $U$ corresponds to the blue line ($U(0) = 6$)
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Let $0 < p < 1$. Then there exists no positive radial solution to
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Phase plot of one sign-changing solution when $d = 3$, $p = \frac{1}{2}$ (sublinear case).
SOME OPEN PROBLEMS (among many others)

Which are the conditions on curvature which determine the properties of solutions of the Emden-Fowler equation proved here?

Consider the parabolic equation

$$\dot{u} = \Delta u^m$$

with $$m < 1$$ in $$H^m_n$$.

The special solution $$U$$ found before gives rise to an integrable, separable variable solution, vanishing in finite time $$T$$:

$$U(\varrho, t) = c U(\varrho)^{1/m} (T - t)^{1/(1 - m)}$$.

This is known to be an attractor for more general solutions in the Euclidean case: $$u - 1 \to 0$$ in $$L^\infty$$ as $$t \to T$$ (see M. Bonforte's talk). We believe that similar results can be proved here in the range $$m \in (n - 2, 1)$$.
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Gabriele Grillo - Politecnico di Milano (Italy)  Emden-Fowler equation on the hyperbolic space
THANK YOU FOR YOUR ATTENTION!