A Discrete Bernoulli Problem

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- Joint work with: M. Gualdani (UT Austin), H. Shahgholian (KTH)
- Aim: New free boundary problems.

Motivation - (exterior) Bernoulli problem

Given $K \subset \mathbb{R}^N$ convex bounded set and $\omega > 0$ constant, find a solution (u, Ω) of

$$(P_B) \qquad \begin{cases} \Delta_p\, u &= 0 \qquad \text{in } \Omega \setminus \overline{K}, \\ u &= 1 \qquad \text{in } \overline{K}, \\ u &= 0 \qquad \text{in } \partial\Omega, \\ |\nabla u| &= \omega \qquad \text{for all } x \in \partial\Omega. \end{cases}$$

Theorem (Henrot-Shahgholian)

$\exists!$ smooth solution, $\partial\Omega \in \mathcal{C}^{2,\alpha}$.

Applications: (Flucher-Rupmf, ...)

- Free surfaces in ideal fluid dynamics.
- Galvanization processes.
- Optimal insulation.

Interesting question: numerical approximation.

The exterior discrete Bernoulli problem

Problem: Fix $K \subset \mathbb{R}^N$ convex open bounded. Given constants $l \in (0,1)$, $\lambda > 0$, find a function u and a convex open bounded domain $\Omega \in \mathbb{R}^N$, $\Omega \supset \overline{K}$, solution of

$$(P_E) \qquad \begin{cases} \Delta_p u = 0 & \text{in } \Omega \setminus \overline{K}, \\ u = 1 & \text{in } \overline{K}, \\ u = 0 & \text{on } \partial\Omega, \\ \operatorname{dist}(x, \{u = l\}) = \lambda & \text{for all } x \in \partial\Omega. \end{cases}$$

Theorem A (existence and uniqueness)

$\exists!$ solution, $u \in \mathcal{C}^{1,\alpha}$, $\partial\Omega \in \mathcal{C}^{1,1}$ convex, Problem: Let D be a smooth bounded domain, u solution of

$$\begin{cases} -\Delta u = 1 \text{ in } D, \\ \quad u = 0 \text{ on } \partial D, \\ |\nabla u| = \omega \text{ on } \partial D. \quad (*) \end{cases}$$

Theorem (Serrin, Weinberger)

Then D is a ball.

New problem: Substitute condition $(*)$ by

$$\mathrm{dist}(x, \{u = l\}) = \lambda \text{ for all } x \in \partial D$$

Theorem (Shahgholian)

Then D is a ball.

Inspiration: price formation model (G.-Gualdani)

Some preliminaries

p-Laplacian:

$$\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u), \quad 1 < p < \infty.$$

- $u \in \mathcal{C}^{1,\alpha}_{loc}(U)$.
- U exterior cone condition $\Rightarrow \quad u \in \mathcal{C}^{\alpha}(\overline{U})$.
- U domain $\mathcal{C}^{1,\alpha} \quad \Rightarrow u \in \mathcal{C}^{\beta}(\overline{U})$.

Let $K \subset \Omega \subset \mathbb{R}^N$, K, Ω convex open.

- u_Ω is the *p*-capacitary potential in $\Omega \backslash K$ if

$$\begin{cases} \Delta_p\, u_\Omega = 0 \text{ in } \Omega \setminus \overline{K}, \\ u_\Omega = 1 \quad \text{on } \overline{K}, \\ u_\Omega = 0 \quad \text{on } \partial\Omega. \end{cases}$$

- (Lewis) Let $l \in [0, 1)$. The level sets $\{u_\Omega > l\}$ are convex.

Beurling's method

- Subsolutions
  $\mathcal{A} = \{\Omega \text{ convex}, \Omega \supset \overline{K} \mid \sup_{x\in\partial\Omega} \mathrm{dist}(x, \{u_\Omega = l\}) \leq \lambda\}$
- Strict subsolutions
  $\mathcal{A}_0 = \{\Omega \text{ convex}, \Omega \supset \overline{K} \mid \inf_{x\in\partial\Omega} \mathrm{dist}(x, \{u_\Omega = l\}) < \lambda\}$
- Supersolutions
  $\mathcal{B} = \{\Omega \text{ convex}, \Omega \supset \overline{K} \mid \inf_{x\in\partial\Omega} \mathrm{dist}(x, \{u_\Omega = l\}) \geq \lambda\}$

Ideas of the proof

- $\mathcal{A}_0$, $\mathcal{B}$ are nonempty.
- $\Omega_1, \Omega_2 \in \mathcal{B} \quad \Rightarrow \quad \Omega_1 \cap \Omega_2 \in \mathcal{B}$.
- Stability: Let $\Omega_1 \supset \Omega_2 \supset \ldots$ sequence in $\mathcal{B}$ and $\Omega = \overline{\cap \Omega_k}$ $\Rightarrow \quad \Omega \in \mathcal{B}$.
- Note that $u_k \to u \in \mathcal{C}^{1,\alpha}_{\text{loc}}$ but only $\mathcal{C}^{\alpha}$ up to the boundary.
- Let Ω minimal set in $\mathcal{B}$. Then (Ω, u) satisfies the distance property.
- Regularity.

Uniqueness

Main idea: Lavrent'ev rescaling method.

- Suppose $\exists$ two solutions, (u_1, Ω_1) and (u_2, Ω_2).
- Let $\epsilon < 1$ and rescale $u_2^\epsilon(x) = u_2(\frac{x}{\epsilon})$ so that $\Omega_2^\epsilon \subset \Omega_1$, until they touch.
- Comparison principle $\quad \Rightarrow \quad u_2^\epsilon \leq u_1$
- If $x^0 \in \partial\Omega_2^\epsilon \cap \partial\Omega_1$,
  $\epsilon\lambda = \mathrm{dist}(x^0, \{u_2^\epsilon = l\}) \geq \mathrm{dist}(x^0, \ \{u_1 = l\}) = \lambda$.
- Contradiction.

Convergence to Bernoulli

Theorem B

Let λ_n, l_n such that $l_n = \omega\lambda_n$. Then the solution to the discrete Bernoulli problem (u_n, Ω_n) converges to (u, Ω) solution of Bernoulli.

Proof:

- Easy to see that $u_n \to u$ in $\mathcal{C}^{1,\alpha}_{\text{loc}}$, $\mathcal{C}^{\alpha}$ up to the boundary.
- Let $S_n := \{0 \leq u_n \leq l_n\}$. Since level sets are convex, the distance function is superharmonic in the set S_n.
- Comparison principle yields that $u_n \leq d_n$ everywhere in S_n.
- $|\nabla u_n| \leq |\nabla d| = \omega$ on $\partial\Omega_n$.
- Now get equality.

The interior problem

Problem: given a convex open bounded set $\Omega \subset \mathbb{R}^N$, $l \in (0,1)$, $\lambda > 0$, find a function u_K and a convex open bounded domain $K \subset \Omega$ such that

$$(P_l) \qquad \begin{cases} \Delta_p\, u = 0 \ \text{ in } \ \Omega \setminus \overline{K}, \\ u = 0 \quad \text{in } \overline{K}, \\ u = 1 \quad \text{in } \partial\Omega, \\ \mathrm{dist}(x, \{u = l\}) = \lambda, \quad \text{for all } x \in \partial K, \end{cases}$$

Theorem C

$\exists$ a constant $\lambda_{\Omega,max}$ such that for any $\lambda \leq \lambda_{\Omega,max}$ problem (P_l) has a solution (u_K, K).

- $\lambda_{\Omega,max}$ is called the Bernoulli constant.
- No uniqueness.
- $\partial\Omega \in \mathcal{C}^{\alpha}$.

The Bernoulli constant

Consider the radial case $\Omega = B(0, R)$:

Lemma

If Ω is a ball in $\mathbb{R}^2$ or $\mathbb{R}^3$, there exists a constant λ_{max} such that problem (P_l) has a unique solution only for $\lambda = \lambda_{max}$, two solutions if $0 < \lambda < \lambda_{max}$ and no solutions if $\lambda > \lambda max$.

To Do

- Non-constant boundary condition.
- Star-shaped case.
- Variational formulation.
- Numerical analysis.