Higher integrability results for porous medium-type equations

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Very very short summary

$$u_t - \Delta u^m = 0, \qquad u \geq 0, \qquad m > 0$$

$$\Downarrow$$

The “gradient” is more integrable than expected.

Section 1

Introduction

- E bounded, open set in $\mathbb{R}^N$ $(N \geq 2)$
- $(0, T)$ time interval
- $E_T \equiv E \times (0, T)$

We consider non-negative, weak solutions to

$$u_t - \Delta u^m = 0 \qquad \text{in } E_T.$$

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$$u_t - \operatorname{div}\left(mu^{m-1}Du\right) = 0 \qquad \text{in } E_T.$$

More generally

$$u_t - \operatorname{div}\mathbf{A}(x, t, u, Du) = 0 \quad \text{in } E_T,$$

with $\mathbf{A}$ of porous medium-type.

Weak solution:

$$\iint_{E_T} -u\phi_t + mu^{m-1} Du \cdot D\phi \, dz = 0, \qquad \forall \phi \in C_0^\infty(E_T).$$

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$$\iint_{E_T} -u\phi_t + m u^{m-1} Du \cdot D\phi \, dz = 0, \qquad \forall \phi \in C_0^\infty(E_T).$$

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Singular $(0 < m < 1)$: $\phi = u^m$ is *admissible*

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Degenerate $(m > 1)$: $\phi = u$ is *admissible*

$$u \in C^0_{\text{loc}}\left(0,T; L^2_{\text{loc}}(E)\right) \quad \text{with} \quad u^{\frac{m+1}{2}} \in L^2_{\text{loc}}\left(0,T,W^{1,2}_{\text{loc}}(E)\right)$$

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From the definition

$$\begin{aligned} \left|Du^{\frac{m+1}{2}}\right| &\in L^2_{\text{loc}}(E_T) && m > 1, \\ |Du^m| &\in L^2_{\text{loc}}(E_T) && 0 < m < 1. \end{aligned}$$

Higher integrability:

$$\text{“gradient”} \in L^{2+\varepsilon}_{\text{loc}}(E_T).$$

Section 2

A brief history

Elliptic p-Laplace equation ($p > 1$):

$$\Delta_p u = 0 \quad \text{in} \quad E \subset \mathbb{R}^N$$

$$\text{A priori:} \quad u \in W^{1,p}_{\text{loc}}(E)$$

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$$\text{Meyers–Elcrat (1975):} \quad u \in W^{1,p+\varepsilon}_{\text{loc}}(E)$$

- From energy estimates and Sobolev inequality:

$$\left(\fint_B |Du|^p \, dx\right)^{\frac{1}{p}} \leq C \left(\fint_{2B} |Du|^s \, dx\right)^{\frac{1}{s}} \qquad s < p.$$

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- Gehring's lemma:

Reverse Hölder inequality

$$\Downarrow$$

$$\left(\fint_B |Du|^{p+\varepsilon} \, dx\right)^{\frac{1}{p+\varepsilon}} \leq C \left(\fint_{2B} |Du|^p \, dx\right)^{\frac{1}{p}} \qquad \text{for some } \varepsilon > 0.$$

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From elliptic to parabolic

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Extensions:

- Higher order equations: Bögelein (2008)
- Global higher integrability: Parviainen (2009)
- Global, higher order equations: Bögelein–Parviainen (2010)
- Variable exponent: Bögelein–Duzaar (2011)

Section 3

Degenerate equations

Take the test function $\phi = u\eta^2$, $\eta = 1$ in $Q(\rho, \tau)$ and zero outside $Q(2\rho, 2\tau)$

$$\sup_t \int_{B_\rho} |u(\cdot, t)|^2 \eta^2 \, dx + \iint_{Q(\rho,\tau)} \left| D u^{\frac{m+1}{2}} \right|^2 \eta^2 \, dz$$
$$\le C \iint_{Q(2\rho,2\tau)} u^2 \eta |\eta_t| \, dz + C \iint_{Q(2\rho,2\tau)} u^{m+1} |D\eta|^2 \, dz.$$

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$$\text{Dimension } = \frac{[u]^2}{[\tau]} + \frac{[u]^{m+1}}{[\rho]^2}.$$

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$$\text{Dimension} \;\; = \frac{[u]^2}{[u]^{1-m}[\rho]^2} + \frac{[u]^{m+1}}{[\rho]^2}.$$

We need

$$[\tau] = [u]^{1-m}[\rho]^2.$$

- Properties of u $\quad \Rightarrow \quad \tau = \left(\frac{u(x_o,t_o)}{c}\right)^{1-m} \rho^2.$

We need

$$[\tau] = [u]^{1-m}[\rho]^2 = \left(\left[Du^{\frac{m+1}{2}}\right]^{\frac{2}{m+1}}\right)^{1-m} [\rho]^{\frac{2(1-m)}{m+1}} [\rho]^2.$$

- Properties of u $\quad\Rightarrow\quad$ $\tau = \left(\frac{u(x_o,t_o)}{c}\right)^{1-m} \rho^2.$
- Properties of $Du^{\frac{m+1}{2}}$:

$$\tau = \kappa^{1-m} \rho^{\frac{4}{m+1}}, \qquad [\kappa] = \left[Du^{\frac{m+1}{2}}\right]^{\frac{2}{m+1}}.$$

Intrinsic cylinders:

$$Q(\rho,\kappa) \equiv Q\left(\rho, \kappa^{1-m}\rho^{\frac{4}{m+1}}\right)$$

We choose

$$\kappa \le C_1 \left(\fint\!\!\fint_{Q(\rho,\kappa)} \left| D u^{\frac{m+1}{2}} \right|^2 dz \right)^{\frac{1}{m+1}}$$

$$\left(\fint\!\!\fint_{Q(20\rho,\kappa)} \left| D u^{\frac{m+1}{2}} \right|^2 dz \right)^{\frac{1}{m+1}} + \left(\fint\!\!\fint_{Q(20\rho,\kappa)} \left| \frac{u^{\frac{m+1}{2}}}{\rho} \right|^2 dz \right)^{\frac{1}{m+1}} \le C_1 \kappa.$$

Intrinsic reverse Hölder inequality

Energy estimates + intrinsic geometry:

$$\iint_{Q(\rho,\kappa)} \left|Du^{\frac{m+1}{2}}\right|^2 dz \leq \frac{C}{\kappa^{1-m}\rho^{\frac{4}{m+1}}} \iint_{Q(2\rho,\kappa)} u^2\, dz + \frac{C}{\rho^2} \iint_{Q(2\rho,\kappa)} u^{m+1}\, dz.$$

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Energy estimates + intrinsic geometry:

$$\iint_{Q(\rho,\kappa)} \left|Du^{\frac{m+1}{2}}\right|^2 dz \le \varepsilon\kappa^{m+1} + \frac{C(\varepsilon)}{\rho^2} \iint_{Q(2\rho,\kappa)} u^{m+1}\, dz.$$

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Sobolev + energy estimates + intrinsic geometry:

$$\begin{aligned}\frac{1}{\rho^2}&\iint_{Q(2\rho,\kappa)} u^{m+1}\, dz \\ &\le \frac{C}{\rho^2}\iint_{Q(4\rho,\kappa)} \left|Du^{\frac{m+1}{2}}\right|^s dz \left(\sup_t \int_{B_{4\rho}} |u(\cdot,t)|^2\, dx\right)^{\frac{s}{N}}.\end{aligned}$$

Energy estimates + intrinsic geometry:

$$\iint_{Q(\rho,\kappa)} \left|Du^{\frac{m+1}{2}}\right|^2 dz \le \varepsilon\kappa^{m+1} + \frac{C(\varepsilon)}{\rho^2} \iint_{Q(2\rho,\kappa)} u^{m+1}\, dz.$$

Sobolev + energy estimates + intrinsic geometry:

$$\frac{1}{\rho^2} \iint_{Q(2\rho,\kappa)} u^{m+1}\, dz$$
$$\le \delta\kappa^{m+1} + C(\delta)\left(\iint_{Q(4\rho,\kappa)} \left|Du^{\frac{m+1}{2}}\right|^s dz\right)^{\frac{2}{s}} \qquad (s < 2).$$

Idea of the proof:

- Subdivide a cylinder using a proper level set of the gradient
- When the gradient is bounded there are no problems
- The set where the gradient is large can be covered by intrinsic cylinders (they exist!)
  $\Rightarrow$ we have the reverse Hölder inequality
- Modifying the Gehring's lemma ("intrinsic version"), we deduce the higher integrability

Idea of the proof:

- Subdivide a cylinder using a proper level set of the gradient
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- The set where the gradient is large can be covered by intrinsic cylinders (they exist!)
  $\Rightarrow$ we have the reverse Hölder inequality
- Modifying the Gehring's lemma ("intrinsic version"), we deduce the higher integrability

key point: quantify "large" in order to build the intrinsic cylinders:

$$\kappa \approx (\text{average of the gradient})\,.$$

Section 4

Singular equations

Intrinsic geometry: similar calculations, but with the test function $\phi = u^m \eta^2$.

$$\sup_t \int_{B_\rho} |u(\,\cdot\,,t)|^{m+1}\eta^2\,dx + \iint_{Q(\rho,\tau)} \left|Du^m\right|^2 \eta^2\,dz$$
$$\leq C \iint_{Q(2\rho,2\tau)} u^{m+1}\eta|\eta_t|\,dz + C \iint_{Q(2\rho,2\tau)} u^{2m}|D\eta|^2\,dz.$$

Rescaled time is

$$\tau = \kappa^{1-m}\rho^{\frac{m+1}{m}}, \qquad [\kappa] = [Du^m]^{\frac{1}{m}}.$$

The proof of the reverse Hölder inequality is essentially the same, but...

Without additional assumptions, we are forced to assume

$$m > m_* \equiv \frac{(N-2)_+}{N+2} \qquad \text{super-critical range}.$$

Similar situation in the work of Kinnunen and Lewis:

$$p > p_* \equiv \frac{2N}{N+2}.$$

Parabolic Sobolev embedding:

$$\iint_Q \left|u^m - (u^m)_B\right|^q dz$$
$$\leq C \left(\iint_Q |Du^m|^s \, dz \right) \left(\sup_t \int_B \left|u^m(\,\cdot\,,t) - (u^m)_B\right|^k dx \right)^{\frac{s}{N}}$$

where

$$q = s\left(\frac{N+k}{N}\right).$$

In our setting

$$q = \frac{m+1}{m}, \qquad k = \frac{m+1}{m}.$$

Condition

$$s < 2 \qquad \text{(reverse Hölder inequality)}$$

is equivalent to ask

$$\frac{m+1}{m}\frac{Nm}{Nm+m+1} < 2 \qquad \Rightarrow \qquad m > m_*.$$

From

$$\frac{m+1}{m} = s\left(\frac{N+k}{N}\right):$$

if we want $0 < m \leq m_*$, we need k to be larger than $\frac{m+1}{m}$.

Solution: assume u locally bounded in E_T; equivalently

$$u \in L^r_{\mathrm{loc}}(E_T) \quad \text{for some } r \ge m+1 \text{ such that}$$
$$\lambda_r \equiv N(m-1) + 2r > 0.$$

The sub-critical range $0 < m \leq m_*$

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$$u \in L^r_{\text{loc}}(E_T) \quad \text{for some } r \geq m+1 \text{ such that}$$
$$\lambda_r \equiv N(m-1) + 2r > 0.$$

With this extra-integrability, we can use the test function

$$\phi = u^{r-1}\eta^2$$

and obtain a bound on

$$\sup_t \int_B |u^m(\cdot\,,t)|^{\frac{r}{m}}\,dx.$$

This inequality allows us to choose

$$k = \frac{r}{m} \geq \frac{m+1}{m}$$

in the Sobolev embedding and to obtain the result also when

$$0 < m \leq m_*.$$

Short bibliography

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