Applications of two new functional inequalities to fractional diffusion II

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Nonlinear PDEs and Functional Inequalities
Madrid (Spain), September 19-20, 2011

Parabolic smoothing effect for a nonlocal transport equation with critical dissipativity

Joint work with F. Quirós, A. Rodríguez, J.L. Vázquez

The problem

We study the 1D transport equation, with nonlocal velocity and fractional viscosity

$$v_t - H(v)v_y = -\Lambda(v) \qquad y \in \mathbb{R},\ t > 0$$

$H =$ Hilbert transform, $\Lambda = (-\Delta)^{1/2}$, with initial value $v_0 \geq 0$, $v_0 \in L^1(\mathbb{R})$,

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- AIM: existence and uniqueness of a classical solution
- IDEA: relate to a fractional Porous Medium type equation

$$H(v)(y) = \frac{1}{\pi}\text{P.V.}\int_{\mathbb{R}} \frac{v(s)}{y-s}\, ds$$

- $\widehat{H(v)}\,(\xi) = -i\,\text{sign}(\xi)\widehat{v}\,(\xi)$; $H^2 = -I$
- $H\ :\ L^p(\mathbb{R}) \to L^p(\mathbb{R}),\ \ 1 < p < \infty$
- $H\ :\ L^p(\mathbb{R}) \cap C^\alpha(\mathbb{R}) \to L^p(\mathbb{R}) \cap C^\alpha(\mathbb{R}),\ \ 1 < p < \infty$
- $(-\Delta)^{1/2}(v)(y) = \frac{1}{\pi}\text{P.V.}\int_{\mathbb{R}} \frac{v(y)-v(s)}{|y-s|^2}\, ds = H(v_y)(y)$
- $H(v_y) = H(v)_y$, whenever $v,\ v_y \in L^p(\mathbb{R})$. Thus

$$\boxed{\Lambda(v) = H(v)_y}$$

- 3D incompressible Navier equation

$$\omega_t + V \cdot \nabla\omega = \omega D(\omega), \qquad \mathrm{div}(V) = 0$$

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- Constantin-Lax-Majda'85 proposed the scalar equation

$$\omega_t + V\omega_x = \omega H(\omega)$$

If $V = 0$ there exist finite time singularities. Several authors added a viscosity $\varepsilon\Delta\omega$ to avoid the singularities.

- De Gregorio'90 considers a velocity given by an integral operator of ω, like $V = -H(\omega)$. His equation is related to ours by a differentiation.

- 2D Quasi-geostrophic equation

$$\begin{cases} \theta_t + V \cdot \nabla\theta = 0 \\ V = \nabla^{\perp}\psi, \quad \theta = -\Lambda\psi \end{cases}$$

$V = R^{\perp}(\theta) = (-R_2(\theta), R_1(\theta))$, $R_j =$ Riesz transforms $\rightsquigarrow$

$$\theta_t + \mathrm{div}(R^{\perp}(\theta)\theta) = 0$$

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- Morlet'98 proposed the equation

$$\theta_t + (H(\theta)\theta)_x = 0$$

and the family of equations

$$\theta_t + \lambda(H(\theta)\theta)_x + (1-\lambda)H(\theta)\theta_x = 0$$

She showed the existence of singularities for $0 < \lambda < 1/3$, $\lambda = 1/2$ and $\lambda = 1$. See Córdoba-Córdoba-Fontelos'05 for the case $\lambda = 0$.

- Viscous 2D QGE

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- It was studied by Constantin-Wu'99 if $\alpha > 1$, Constantin-Córdoba-Wu'01 if $\alpha = 1$ and small data, and Caffarelli-Vasseur'07 for $\alpha = 1$ (general data and any dimensions)

- The 1D analogs ($\lambda = 0$ and $\lambda = 1$ in Morlet's family but with viscosity)

$$\theta_t + \lambda(H(\theta)\theta)_x + (1-\lambda)H(\theta)\theta_x = -\Lambda^\alpha \theta$$

were studied by Chae-Córdoba-Córdoba-Fontelos'05 and Córdoba-Córdoba-Fontelos'05. They showed, in the critical case $\alpha = 1$, that smooth small data remain smooth.

- Kiselev-Nazarov-Volberg'07 proved that, for the critical ($\alpha = 1$) viscous 2D QGE, periodic C^∞ data remain C^∞.

- Li-Rodrigo'08 studied blow-up for the viscous 1D transport equation if $0 < \alpha \leq 1/2$, and mentioned that in the critical case the Kiselev-Nazarov-Volberg's technique for 2D would also work for the 1D model.

- Hyperbolic point of view $\rightsquigarrow$ viscosity may avoid appearance of singularities

- Parabolic point of view $\rightsquigarrow$ singularities may disappear, smoothing effect

We consider strong L^1-energy solutions:

$$v \in C([0,\infty) : L^1(\mathbb{R}))$$
$$v_y \in L^\infty((\tau,\infty) : L^2(\mathbb{R}))$$
$$v_t \in L^1_{loc}(\mathbb{R} \times (0,\infty))$$

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Theorem

For every $v_0 \in L^1(\mathbb{R})$ there exists a unique solution which is moreover a classical solution.

Properties

- L^1-L^∞ smoothing effect (for t small):

$$\|v(\cdot,t)\|_\infty \le \log[1 + C\,t^{-3}\|v_0\|_1^2]$$

- Decay (for t large):

$$\|v(\cdot,t)\|_\infty \le \log[1 + C\,t^{-3/2}\|v_0\|_1]$$

- Regularity: $v \in C^{1,\alpha}(\mathbb{R} \times (0,\infty))$.
- Positivity: $v(y,t) > 0$ for every $y \in \mathbb{R}$, $t > 0$.
- Conservation law: $\displaystyle\int_{\mathbb{R}} [1 - e^{-v(y,t)}]\,dy = const.$

We first write the equation as a conservation law

$$(e^{-v})_t - (e^{-v}H(v))_y = 0$$

Then put $(y, t, v) \mapsto (x, t, u)$ given by the Backlund type transform

$$x(y,t) = \int_0^y e^{-v(s,t)}\, ds + c(t)\,, \qquad u(x,t) = e^{v(y,t)} - 1$$

with $c'(t) = e^{-v(0,t)}H(v)(0,t)$.

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$$x_y = e^{-v}, \quad x_t = e^{-v}H(v), \quad u_x = e^{2v}v_y$$

$$u_t + H(\log(1+u))_x = 0\,, \qquad x \in \mathbb{R}, \quad t > 0$$

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$$u_t + H(\log(1+u))_x = 0\,, \qquad x \in \mathbb{R}, \quad t > 0$$

$$\boxed{u_t + \Lambda\Phi(u) = 0, \qquad \Phi(u) = \log(1+u)}$$

Relation of the variables $u(x,t) = e^{v(y,t)} - 1$:

$$u(x,t) \geq 0 \;\Leftrightarrow\; v(y,t) \geq 0$$

$$\int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} (1 - e^{-v(y,t)})\,dy$$

$$\int_{\mathbb{R}} v(y,t)\,dy = \int_{\mathbb{R}} (1 + u(x,t)) \log(1 + u(x,t))\,dx$$

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We define the Orlicz space

$$\Theta = \{\varphi \in L^1(\mathbb{R}) \,:\, \int_{\mathbb{R}} (1 + |\varphi|) \log(1 + |\varphi|) < \infty\}$$

with the associated Luxembourg norm with N-function $\Psi(\varphi)$, $\Psi' = \Phi$

Existence

We put $w = \Phi(u) = \log(1+u)$, and following the harmonic extension procedure, dP-Quirós-Rodríguez-Vazquez'10, we study the problem

$$\begin{cases} \Delta w = 0, & (x,z) \in \mathbb{R}^2_+,\ t > 0, \\ w_z - (e^w)_t = 0, & x \in \mathbb{R}, z = 0,\ t > 0, \\ w = \Phi(u_0), & x \in \mathbb{R}, z = 0,\ t = 0. \end{cases}$$

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We obtain for every $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, exactly as in the PME case, existence of a weak solution $u \in C([0,\infty) : L^1(\mathbb{R}))$, $\Phi(u) \in L^\infty((\tau,\infty) : H^{1/2}(\mathbb{R}))$, and also uniqueness, contractivity, conservation of mass and C^α regularity. Higher regularity requires some extra work.

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The general case $u_0 \in \Theta$ will follow from the L^∞ estimates in terms only of the Θ norm.

Existence

First observe that the equation gives, for $w = \Phi(u)$.

$$\int_{\mathbb{R}} |(-\Delta)^{1/4} w|^2 \leq \frac{1}{t} \|u_0\|_{\Theta}$$

Also Nash-Gagliardo-Nirenberg inequality gives, for every $p \geq 1$

$$\|w\|_{p+2}^{p+2} \leq C \|(-\Delta)^{1/4} w\|_2^2 \, \|w\|_p^p$$

Therefore $w \in L^p(\mathbb{R})$ for every $p \geq 1$, but we do not get $p = \infty$. Also, this does not mean integrability for u, this will require more work.

We have that L^∞ implies C^α, Athanasopoulos-Caffarelli'10. To get higher regularity we follow the technique of Caffarelli-Vasseur'10 and write the solution as

$$u(x,t) = P_t * u_0(x) - g(x,t),$$

where $P_t(x) = P(x,t)$ is the Poisson kernel (gives the solution to the linear part) and

$$g(x,t) = \int_0^t \int_{\mathbb{R}} P(x-s, t-\tau)_x \, H(\log(1+u) - u) \, ds d\tau$$

Regularity depends on properties of the kernel. Recall that it is crucial that the problem is of divergence form. Therefore this technique cannot be applied to the original transport equation

Theorem

Assume $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then, for every $p > 1$ it holds

$$\|u(\cdot,t)\|_\infty \leq \max\{C\, t^{-\frac{1}{p-1}} \|u_0\|_p^{\frac{p}{p-1}},\ C\, t^{-\frac{1}{p}} \|u_0\|_p\}$$

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Formally put $m = 0$ for u large in the formula of the PME,

$$\|u(\cdot,t)\|_\infty \leq C\, t^{-\gamma} \|u_0\|_p^{\gamma p}, \qquad \gamma = (m-1+p)^{-1}$$

and $m = 1$ for u small.

Multiply the equation by $\varphi(u) = \dfrac{u^{p_k-1}}{p_k - 1} + \dfrac{u^{p_k}}{p_k}$, $p_k > p > 1$. Using Stroock-Varopoulos, Nash-Gagliardo-Nirenberg, and the decay of the L^p norms, we obtain a recurrence relation

$$U_{k+1} \leq ct^{-\frac{1}{p_{k+1}}} U_k$$

where $U_k = \max\{\|u(\cdot, t_k)\|_{p_k},\ \|u(\cdot, t_k)\|_{p_k+1}^{\frac{p_k+1}{p_k}}\}$, $p_k = 2^k p$.

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where $U_k = \max\{\|u(\cdot, t_k)\|_{p_k}, \|u(\cdot, t_k)\|_{p_k+1}^{\frac{p_k+1}{p_k}}\}$, $p_k = 2^k p$.
This will imply

$$\|u(\cdot, t)\|_\infty = \lim_{k\to\infty} U_k \le Ct^{-\frac{1}{p}} U_0 = Ct^{-\frac{1}{p}} \max\{\|u_0\|_p, \|u_0\|_{p+1}^{\frac{p+1}{p}}\}$$

An interpolation argument gives the estimate.

We use the following Trudinger inequality

Theorem

Let $w \in H^{1/2}(\mathbb{R})$. Then there exist constants c_1 and c_2 such that

$$\int_{\mathbb{R}} \left[\exp\left(\frac{w}{c_1 \|w\|_{H^{1/2}}} \right)^2 - 1 \right] \leq c_2$$

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- For compactly supported functions (in $I_0 \subset I$) see Strichartz'72.
- Prove an equivalence, in that case, of the norms

$$\|w\|^2_{L^2(I)} + \|(-\Delta)^{1/4} w\|^2_{L^2(I)} \sim \|w\|^2_{L^2(I)} + \iint_{I \times I} \frac{|w(x) - w(y)|^2}{|x-y|^2} \, dxdy$$

- Extend to the general case by summation, as in Adams, but take care of the cut-off.

Application to our case:

Trudinger inequality for u is

$$\int_{\mathbb{R}} \left[(1+u)^{\frac{\log(1+u)}{c_1 \|w\|_{H^{1/2}}}} - 1 \right] \leq c_2$$

But $(1+x)^{k\log(1+x)} > 1 + ckx^2$, i.e.

$$\|u\|_2 \leq c\|w\|_{H^{1/2}} \leq C(t, \|u_0\|_\Theta)$$

Thanks!!!