Regularity of stable solutions of p-Laplace equations through geometric Sobolev type inequalities [1]

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[1] Joint work with Manel Sanchón (Universitat de Barcelona).

Motivations

The purpose of this paper is twofold. First, we prove geometric type inequalities involving the functionals

$$I_{p,q}(v;\Omega) := \left(\int_\Omega \left(\frac{1}{p'} |\nabla_T |\nabla v|^{\frac{p}{q}}| \right)^q + |H_v|^q |\nabla v|^p \, dx \right)^{1/p}, \qquad (1)$$

where Ω is a smooth bounded domain of $\mathbb{R}^n$ with $n \geq 2$, $v \in C_0^\infty(\overline{\Omega})$, $H_v(x)$ denotes the mean curvature at x of the hypersurface $\{y \in \Omega : |v(y)| = |v(x)|\}$ and ∇_T is the tangential gradient along a level set of $|v|$.

We will prove a Morrey's type inequality when $n < p + q$ and a Sobolev inequality when $n > p + q$.

Motivations

Then, as an application of these inequalities, we obtain L^r and $W^{1,r}$ *a priori* estimates for semi-stable solutions of the reaction-diffusion problem

$$\begin{cases} -\Delta_p u = g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where g is any positive C^1 nonlinearity, Δ_p denotes the p-Laplace operator $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, and $p > 1$.

Schwarz symmetrization

Given a Lipschitz continuous function v and its Schwarz symmetrization v^*, our first result establishes that the functional $I_{p,q}$ is decreased (up to a universal multiplicative constant) by Schwarz symmetrization.

Let Ω be a smooth bounded domain of $\mathbb{R}^n$ with $n \geq 2$ and B_R the ball centered at the origin and with radius $R = (|\Omega|/|B_1|)^{1/n}$. Let $v \in C_0^\infty(\overline{\Omega})$ and $v^\star$ its Schwarz symmetrization. Let $I_{p,q}$ be the functional defined in (1) with $p, q \geq 1$. If $n > q + 1$ then there exists a universal constant C depending only on n, p, and q, such that

$$I_{p,q}(v^*; B_R) \leq C I_{p,q}(v; \Omega). \tag{3}$$

Mean convex functions

A related result was proved by Trudinger when $q = 1$ and by Cabré and Sanchón for the class of mean convex functions. More precisely, they proved the result replacing the functional $I_{p,q}$ by

$$\tilde{I}_{p,q}(v;\Omega) := \left(\int_\Omega |H_v|^q |\nabla v|^p \, dx\right)^{1/p} \tag{4}$$

and considering the Schwarz symmetrization with respect to the perimeter instead of the classical one like us (it is essential that the mean curvature H_v of the level sets of $|v|$ is nonnegative). Then using an Aleksandrov-Fenchel inequality for mean convex hypersurfaces they proved the result with constant $C = 1$ for the class of mean convex functions.

Ingredients of the proof

The first one is the classical isoperimetric inequality:

$$n|B_1|^{1/n}|D|^{(n-1)/n} \leq |\partial D| \tag{5}$$

for any smooth bounded domain of $\mathbb{R}^n$.

The second one is a geometric Sobolev inequality, due to Allard and to Michael and Simons, on compact $(n-1)$-hypersurfaces M without boundary: for every $q \in [1, n-1)$, there exists $A = A(n, q)$ such that

$$\left(\int_M |\phi|^{q^\star} \, d\sigma \right)^{1/q^\star} \leq A \left(\int_M |\nabla \phi|^q + |H\phi|^q \, d\sigma \right)^{1/q} \tag{6}$$

for every $\phi \in C^\infty(M)$, where $q^\star = (n-1)q/(n-1-q)$ and $d\sigma$ denotes the area element in M. We use (5) and (6) with $M = \{|v| = t\}$ and $\phi = |\nabla v|^{(p-1)/q}$.

Idea of the proof

Noting that $v^*(x) = v^*(|x|)$, and hence its level sets are spheres one obtains that the mean curvature $H_{v^*}(x) = 1/|x|$ and the tangential gradient of $|\nabla v^*|^{p/q}$ along a level set of v^* is identically zero. In particular we obtain

$$I_{p,q}(v^*; B_R) = \tilde{I}_{p,q}(v^*; B_R) = \left(\int_{B_R} \frac{1}{|x|^q} |\nabla v^*|^p \, dx \right)^{1/p} \leq C I_{p,q}(v; \Omega).$$

On the other hand, remember that Schwarz symmetrization preserves the L^r norm. Therefore, inequality

$$\|v\|_{L^r(\Omega)} \leq C I_{p,q}(v; \Omega)$$

is reduced to a Morrey or Sobolev inequality (for radial functions) with weight $|x|^{-q}$.

Geometric inequalities

Let Ω be a smooth bounded domain of $\mathbb{R}^n$ with $n \geq 2$ and $v \in C_0^\infty(\overline{\Omega})$. Let $I_{p,q}$ be the functional defined in (1) with $p, q \geq 1$ and

$$p_q^\star := \frac{np}{n-(p+q)}. \tag{7}$$

Assume $n > 1+q$. The following assertions hold:

(a) If $n < p+q$ then

$$\|v\|_{L^\infty(\Omega)} \leq C_1 |\Omega|^{\frac{p+q-n}{np}} I_{p,q}(v;\Omega) \tag{8}$$

for some constant C_1 depending only on n, p, and q.

Geometric inequalities

(b)If $n > p + q$, then

$$\|v\|_{L^r(\Omega)} \leq C_2 |\Omega|^{\frac{1}{r} - \frac{1}{p_q^\star}} I_{p,q}(v;\Omega) \quad \text{for every } r \leq p_q^\star, \tag{9}$$

where C_2 is a constant depending only on n, p, q, and r.

(c) If $n = p + q$, then

$$\int_\Omega \exp\left\{ \left(\frac{|v|}{C_3 I_{p,q}(v;\Omega)} \right)^{p'} \right\} dx \leq C_4 |\Omega|, \quad \text{where } p' = p/(p-1), \tag{10}$$

for some constants C_3 and C_4 depending only on n and p.

Cabré - Sanchón 2011

Cabré and Sanchón proved recently similar inequalities under the assumption $q \geq p$ using a different method (without the use of Schwarz symmetrization).

More precisely, they proved the theorem replacing the functional $I_{p,q}(v;\Omega)$ by the one defined in (4), $\tilde{I}_{p,q}(v;\Omega)$.

However, since $\tilde{I}_{p,q}(v;\Omega) \leq I_{p,q}(v;\Omega)$, if the inequalities hold for $\tilde{I}_{p,q}$, they also hold for $I_{p,q}$.

Hence, our geometric inequalities are only new in the range $1 \leq q < p$.

A-priori estimates for semistable solutions

The second part of the paper deals with *a priori* estimates for semi-stable solutions of problem (2).
Remember that a regular solution $u \in C^1_0(\overline{\Omega})$ of (2) is said to be *semi-stable* if the second variation of the associated energy functional at u is nonnegative definite, *i.e.*,

$$\int_\Omega |\nabla u|^{p-2} \left\{ |\nabla \phi|^2 + (p-2) \left(\nabla \phi \cdot \frac{\nabla u}{|\nabla u|} \right)^2 \right\} - g'(u)\phi^2 \, dx \geq 0, \tag{11}$$

for every $\phi \in H_0$. Here, H_0 denotes the space of admissible functions.
The class of semi-stable solutions includes local minimizers of the energy functional as well as minimal solutions and extremal solutions of (2).

A-priori estimates for semistable solutions

The next result extends the ones by Cabré '09 and Cabré-Sanchón '11 for the Laplacian case ($p = 2$).

Let g be any positive C^1 function and $\Omega \subset \mathbb{R}^n$ any smooth bounded domain. Let $u \in C^1_0(\overline{\Omega})$ be a semi-stable solution of (2), *i.e.*, a solution satisfying (11). The following assertions hold:

If $n \leq p + 2$ then there exists a constant C depending only on n and p such that

$$\|u\|_{L^\infty(\Omega)} \leq s + \frac{C}{s^{2/p}} |\Omega|^{\frac{p+2-n}{np}} \left(\int_{\{u<s\}} |\nabla u|^{p+2} \, dx \right)^{1/p} \quad \text{for all } s > 0. \tag{12}$$

A-priori estimates for semistable solutions

If $n > p+2$ then there exists a constant C depending only on n and p such that

$$\left(\int_{\{u>s\}} \left(|u| - s \right)^{\frac{np}{n-(p+2)}} dx \right)^{\frac{n-(p+2)}{np}} \leq \frac{C}{s^{2/p}} \left(\int_{\{u \leq s\}} |\nabla u|^{p+2} dx \right)^{1/p} \tag{13}$$

for all $s > 0$. Moreover, there exists a constant C depending only on n, p, and r such that

$$\int_{\Omega} |\nabla u|^r dx \leq C \left(|\Omega| + \int_{\Omega} |u|^q dx + \|g(u)\|_{L^1(\Omega)} \right) \tag{14}$$

for all $1 \leq r < r_1 := \frac{2np}{(1+p)n-p-2}$.

Idea of the proof

To prove (12) and (13) we use the semi-stability condition (11) with the test function $\phi = |\nabla u|\eta$ to obtain

$$\int_\Omega \left(\frac{4}{p^2} |\nabla_T |\nabla u|^{p/2}|^2 + \frac{n-1}{p-1} H_u^2 |\nabla u|^p \right) \eta^2 \, dx \leq \int_\Omega |\nabla u|^p |\nabla \eta|^2 \, dx \tag{15}$$

for every Lipschitz function η in $\overline{\Omega}$ with $\eta|_{\partial\Omega} = 0$. Then, taking $\eta = T_s u = \min\{s, u\}$, we obtain the estimates by using the Morrey and Sobolev inequalities when $n \neq p + 2$. The critical case $n = p + 2$ is more involved.

The gradient estimate established in (14) follows by using a technique introduced by Bénilan *et al.* '95 to get the regularity of entropy solutions for $p-$Laplace equations with L^1 data.

Extremal solution

Let us consider now the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (1.16)_\lambda$$

where λ is a positive parameter and f is a C^1 positive increasing function satisfying

$$\lim_{t \to +\infty} \frac{f(t)}{t^{p-1}} = +\infty. \qquad (17)$$

Let u_λ be the minimal (in the pointwise sense) solution of the above problem.

Extremal solution

Cabré and Sanchón recently proved that there exists an extremal parameter $\lambda^\star \in (0, \infty)$ such that problem $(P)_\lambda$ admits a regular minimal solution $u_\lambda \in C^1_0(\overline{\Omega})$ for $\lambda \in (0, \lambda^\star)$ and admits no regular solution for $\lambda > \lambda^\star$. Moreover, every minimal solution u_λ is a semistable for $\lambda \in (0, \lambda^\star)$.

For the Laplacian case ($p = 2$) the limit of minimal solutions

$$u^\star := \lim_{\lambda \uparrow \lambda^\star} u_\lambda$$

is a weak solution of the extremal problem $(P)_{\lambda^\star}$ and it is known as extremal solution. However, in the general case ($p > 1$) it is unknown if the limit of minimal solutions $u^\star$ is a (weak or entropy) solution or not. In the affirmative case we call it *extremal solution of* $(P)_{\lambda^\star}$.

Regularity for the extremal solution

Our next result improves the L^q estimates by Nedev and Sanchón for convex domains. We also prove that $u^\star$ belongs to the energy class $W_0^{1,p}(\Omega)$ independently of the dimension extending a result of Nedev for $p = 2$ to every $p \geq 2$.

Let f be an increasing positive C^1 function satisfying (17). Assume that Ω is a smooth convex domain of $\mathbb{R}^n$. Let $u_\lambda \in C_0^1(\overline{\Omega})$ be the minimal solution of $(P)_\lambda$. There exists a constant C independent of λ such that:

(a) If $n \leq p + 2$ then

$$\|u_\lambda\|_{L^\infty(\Omega)} \leq C\|f(u_\lambda)\|_{L^1(\Omega)}^{1/(p-1)}.$$

Regularity for the extremal solution

(b) If $n > p + 2$ then

$$\|u_\lambda\|_{L^{\frac{np}{n-p-2}}(\Omega)} \leq C\|f(u_\lambda)\|_{L^1(\Omega)}^{1/(p-1)}$$

and

$$\|u_\lambda\|_{W_0^{1,p}(\Omega)} \leq C\|f(u_\lambda)\|_{L^1(\Omega)}^{1/(p-1)}.$$

Assume, in addition, $p \geq 2$ and that f is p-convex. Then

(i) If $n \leq p + 2$ then $u^\star \in L^\infty(\Omega)$. In particular, $u^\star \in C_0^1(\overline{\Omega})$.

(ii) If $n > p + 2$ then $u^\star \in L^{\frac{np}{n-p-2}}(\Omega) \cap W_0^{1,p}(\Omega)$.

Idea of the proof

If $f(u_\lambda)$ is bounded in $L^1(\Omega)$ independently of λ then the limit of minimal solutions is an entropy solution of extremal problem $(P)_\lambda$. However, as we said before this estimate on $\|f(u_\lambda)\|_{L^1(\Omega)}$ is an open problem for $p \neq 2$.

To prove the L^r *a priori* estimates of we proceed as for the semistable solutions. First, we take as η in (15) a function related to $\mathrm{dist}(x, \partial\Omega)$. Then, we use the convexity of the domain to prove that

$$\{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \varepsilon\} \subset \{x \in \Omega : u_\lambda(x) < s\}$$

for a suitable s.

Idea of the proof

Then the energy estimate follows by extending the arguments of Nedev '01. Using a Pohozaev identity we obtain

$$\int_\Omega |\nabla u_\lambda|^p \, dx \le \frac{1}{p'} \int_{\partial\Omega} |\nabla u_\lambda|^p \, x \cdot \nu \, d\sigma, \qquad \text{for all } p > 1, \qquad (18)$$

where $d\sigma$ denotes the area element in $\partial\Omega$ and ν is the outward unit normal to Ω. Then using the convexity of the domain we control the right hand side of (18) by $\|f(u_\lambda)\|_{L^1(\Omega)}$.

Then, the estimates in parts (i) and (ii) follow from previous work by Sanchón '07. Indeed, since $f(u^\star) \in L^r(\Omega)$ for all $1 \le r < n/(n - p')$, the estimates follow directly from parts (a) and (b).