A Moser inequality for the $1-$bilaplacian

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Limiting Sobolev's embeddings and the borderline case

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a smooth bounded domain. Then

$$W^{m,p}(\Omega) \hookrightarrow L^{p^*}(\Omega), \quad p^* = \frac{Np}{N - mp}, \quad \text{if } mp < N$$

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In the limiting case $mp = N$ and $\boxed{p > 1}$

$$W^{m,\frac{N}{m}}(\Omega) \hookrightarrow L^{\phi}(\Omega), \quad \phi(u) \sim e^{|u|^{\frac{N}{N-m}}}$$

(Pohožaev, Trudinger; Strichartz)

D.R. Adams (Annals of Math.'88) extends Moser's result ($m = 1$):

$$W_0^{m,\frac{N}{m}}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|u\|_{W^{m,\frac{N}{m}}}}, \qquad \boxed{p = \frac{N}{m} > 1}$$

Note that for

$$\left.\begin{array}{l} m \text{ even}: \ \|\nabla^m u\|_{L^{\frac{N}{m}}} := \|\Delta^{\frac{m}{2}} u\|_{L^{\frac{N}{m}}} \\ m \text{ odd}: \ \|\nabla^m u\|_{L^{\frac{N}{m}}} := \|\nabla \Delta^{\frac{m-1}{2}} u\|_{L^{\frac{N}{m}}} \end{array}\right\} \text{equivalent norms on } W_0^{m,\frac{N}{m}}(\Omega)$$

Then:

$$\sup_{\|\nabla^m u\|_{L^{\frac{N}{m}}} \leq 1} \int_\Omega e^{\beta |u|^{\frac{N}{N-m}}} \begin{cases} \leq c, & \text{if } \beta \leq \beta_{m,N} \\ = +\infty, & \text{if } \beta > \beta_{m,N} \end{cases}$$

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When $\boxed{p = 1}$ (hence $N = m$) we end up with the borderline case

$$W^{N,1}(\Omega) \hookrightarrow L^\infty(\Omega)$$

Maximal summability for solutions to PDE

Consider the following problem

$$(P)\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \qquad f \in L^p(\Omega) \quad \boxed{p > 1}$$

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- Sobolev embeddings (and improvements) give the sharp maximal degree of uniform summability for u

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$$\sup_{\|f\|=1} \int_\Omega |u|^q \begin{cases} < \infty, & \text{if } 1 \leq q < \frac{N}{N-2} \\ = \infty, & \text{if } q = \frac{N}{N-2} \end{cases} \quad \text{(Mazya, '61)}$$

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No sharp results $\Rightarrow$ interesting!

A "new" function space

We define the Banach space

$$W^{2,1}_{\Delta}(\Omega) := cl\left\{u \in \mathcal{C}^{\infty}(\Omega) \cap \mathcal{C}^{0}(\overline{\Omega}), u|_{\partial\Omega} = 0 : \|\Delta u\|_1 < \infty\right\}$$

endowed with the norm $\|\Delta \cdot \|_1$

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- for $p > 1 \Rightarrow W^{2,p}_{\Delta}(\Omega) \equiv W^{2,p}(\Omega) \cap W^{1,p}_{0}(\Omega)$
- for $p = 1 \Rightarrow W^{2,1}(\Omega) \subsetneq W^{2,1}_{\Delta}(\Omega)$ Why?

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  - Off diagonal derivatives can not be bounded by $\|\Delta \cdot \|_1$
  - Sobolev's representation formulas yield different spaces
  - Actually $W^{2,1}(\Omega) \subsetneq W^{2,1}_{\mathcal{O}D}(\Omega)$ (R. Adams '88)

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$W^{2,1}_{\Delta}(\Omega)$ as native space for optimal regularity!

The Zygmund space $L_{exp}(\Omega)$

Let u^* be the *decreasing rearrangement* of u:

$u^*(s) = |\{t \in [0,\infty) : \mu_u(t) > s\}|, \quad \mu_u(t)$ distribution function of u

and $u^\sharp$ be the *spherically symmetric rearrangement* of u,

$$u^\sharp(x) = u^*\left(\omega_N |x|^N\right), \quad \omega_N = |B_1|$$

The Zygmund space $Z^\alpha(\Omega)$ consists of all measurable functions s.t.

$$\int_\Omega e^{\lambda |u|^{1/\alpha}} dx < \infty, \quad \forall \lambda > 0$$

It turns out that

$u \in Z^\alpha(\Omega) \Longleftrightarrow u^*(s) \leq c\left(1 + \log\left(\frac{|\Omega|}{s}\right)\right)^\alpha, 0 < s < |\Omega|, c = c(u) > 0$

$Z^1(\Omega) := L_{exp}(\Omega)$ with $\|u\|_{L_{\exp}} = \sup_{s\in(0,|\Omega|)} \frac{u^*(s)}{\left(1+\log\left(\frac{|\Omega|}{s}\right)\right)}$

Complementing Adams' result in the borderline case

Theorem 1 (C.-Ruf-Tarsi '10)

Let $N = 2$ and Ω be a bounded domain in $\mathbb{R}^2$. Then

$$W^{2,1}_{\Delta}(\Omega) \hookrightarrow L_{\exp}(\Omega)$$

Moreover

$$\|u\|_{L_{\exp}} \leq \frac{1}{4\pi}\|\Delta u\|_1 \qquad (1)$$

The constant appearing in (1) is sharp for any bounded domain.

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Key ingredients:

- Talenti's comparison principle $\Rightarrow$ radial case
- Moser's change of variable and representation formulas
- sharp constant by means of a counterexample

Improvement of the Brezis-Merle result

Brezis-Merle, '91

Let u be a solution of

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \qquad f \in L^1(\Omega)$$

Then for any $\beta < 4\pi$

$$\sup_{\|f\|_1 \leq 1} \int_\Omega e^{\beta|u|} dx < C_\beta |\Omega|$$

The inequality is sharp.

Theorem 2 (C.-Ruf-Tarsi '10)

Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ continuous, s.t. $e^{4\pi t}\Phi(t) \uparrow$ near infinity. Then

$$\sup_{u \in W^{2,1}_{\Delta}, \|\Delta u\|_1 = 1} \int_\Omega e^{4\pi|u|}\Phi(|u|)\, dx < C(\Phi)|\Omega|$$

if and only if $\Phi(t)$ is integrable near infinity.

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Example:

$$\sup_{u\in W^{2,1}_\Delta, \|\Delta u\|_1=1} \int_\Omega \frac{e^{4\pi|u|}}{(1+|u|)^\alpha}\,dx < C|\Omega|, \quad \alpha > 1$$

The borderline case is sensitive to boundary conditions

Adams' result concerns compactly supported functions. In our case we require just Dirichlet boundary data: why?

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Adams' result concerns compactly supported functions. In our case we require just Dirichlet boundary data: why?

$$W^{2,1}_{\Delta,0}(\Omega) := cl\left\{u \in \mathcal{C}^\infty_c(\Omega) : \|\Delta u\|_1 < \infty\right\}$$

Theorem 3 (C.-Ruf-Tarsi '10)

Let $N = 2$ and Ω be a bounded domain in $\mathbb{R}^2$ containing the origin. Then, for all radially symmetric $u \in W^{2,1}_{\Delta,0}(\Omega)$

$$\|u\|_{L_{\exp}} \leq \frac{1}{8\pi}\|\Delta u\|_1 \tag{2}$$

The constant appearing in (2) is sharp.

Similarly to the non compact case we have

Theorem 4 (C.-Ruf-Tarsi '10)

Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ continuous, s.t. $e^{8\pi t}\Phi(t) \uparrow$ near infinity. Then

$$\sup_{u \in W^{2,1}_{\Delta,0}, \|\Delta u\|_1 = 1} \int_\Omega e^{8\pi|u|}\Phi(|u|)\,dx < C(\Phi)|\Omega|$$

if and only if $\Phi(t)$ is integrable near infinity.

- radial framework because of technical issues related to symmetrization and comparison principles
- the proof buys the line of the Dirichlet case
- sharpness is very delicate!

Remarks

- Adams' result which is based on potential estimates can be proved as well with previous techniques and extended allowing also different boundary conditions (the same best constants!)

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- In dimension $N = 2$, differently from the higher dimensional case, the knowledge of the best embedding constant is crucial to obtain optimal regularity.
- The limit of maximizing sequences $\notin W^{2,1}_{\Delta}(\Omega)$: however, it can be shown that the best constant is achieved in the larger space of functions whose laplacian is of bounded variation.

Further applications: Δ_1 $(\int |\nabla u|)$ vs Δ_1^2 $(\int |\Delta u|)$

- Higher order development in elasto–plastic problems with cracks and image restoration (De Giorgi);
- Cheeger sets (Kawohl);
- Game theory
  (Evans: *"Extreme cases reveal interesting structure"*).

The Trudinger-Moser inequality revisited

Consider the space $H_0^1(\Omega)$ endowed with the norm $\|\nabla u\|_2$. It is well known:

$$H_0^1(\Omega) \hookrightarrow \begin{cases} L^{2^*}(\Omega), & N \geq 3 \quad \text{(Sobolev)} \\ L_{e^{u^2}}(\Omega), & N = 2 \quad \text{(Pohožaev, Trudinger)} \end{cases}$$

where the critical Sobolev exponent $2^* := \frac{2N}{N-2}$ has important and so far well understood connections with:

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The Trudinger-Moser inequality revisited

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where the critical Sobolev exponent $2^* := \frac{2N}{N-2}$ has important and so far well understood connections with:

- compactness issues
- existence/non-existence of solutions to PDE
- the best constant is not achieved.

On the contrary the role of criticality when $N = 2$ is not yet completely clear:

- Moser (1970) has investigated criticality of the Pohožaev-Trudinger embedding in the following sense:

$$\sup_{\|\nabla u\|_2 \leq 1} \int_\Omega e^{\alpha |u|^2} dx \quad \begin{cases} \leq c|\Omega| \ , & \text{if } \alpha \leq 4\pi \\ = +\infty \ , & \text{if } \alpha > 4\pi \end{cases} \tag{3}$$

where the "critical" value 4π still confines compactness but here the best constant is attained!
(Carleson-Chang'86, Flucher'92, De Figueiredo-do Ó-Ruf'02)

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  (Carleson-Chang'86, Flucher'92, De Figueiredo-do Ó-Ruf'02)
- Exploiting the technique developed in the second order case we approach the problem from the point of view of the underlying embedding inequality in the Zygmund tuning scale

Theorem 5 (C.-Ruf-Tarsi '10)

Let Ω be a bounded domain in $\mathbb{R}^2$. Then, the following inequality holds

$$\|u\|_{Z_\varepsilon^{1/2}} \leq \frac{1}{\sqrt{4\pi}} \|\nabla u\|_2 \qquad (4)$$

for any $u \in H_0^1(\Omega)$ and where

$$\|u\|_{Z_\varepsilon^{1/2}} := \sup_{t\in(0,|\Omega|)} \frac{u^*(t)}{\left[\varepsilon + \log\left(\frac{|\Omega|}{t}\right)\right]^{\frac{1}{2}}}, \quad \varepsilon \geq 0$$

Moreover, the constant appearing in (4) is sharp and it is not attained as long as $\varepsilon > 0$.

For $\varepsilon = 0$ the best constant in (4) is attained when Ω is the ball by the Moser truncated functions

$$u_{r_0}(r) = \frac{1}{\sqrt{2\pi}} \begin{cases} \dfrac{-\log r}{(-\log r_0)^{1/2}}, & 0 \le r \le r_0 \\ (-\log r_0)^{\frac{1}{2}}, & r_0 \le r \le 1 \end{cases}$$

which are the unique solutions of

$$\begin{cases} -\dfrac{d}{dr}\,(y' r) = \dfrac{1}{\sqrt{\pi}}(-2\log r_0)^{-\frac{1}{2}}\delta_{r_0}, \quad r \in (0,1) \\ y(r_0) = \dfrac{(-\log r_0)^{\frac{1}{2}}}{(2\pi)^{1/2}} \\ y'(0) = y(1) = 0 \end{cases}$$

Main references:

- D.R. Adams, *A sharp inequality of J. Moser for higher order derivatives*, Ann. of Math. **128** (1988), 385–398.
- P. Bénilan, L. Boccardo, T. Gallouët, M. Pierre and J.L. Vazquez, *An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **22** (1995), 241–273.
- C. Bennett and K. Rudnick, *On Lorentz-Zygmund spaces*, Dissert. Math. **175** (1980), 1–72.
- H. Brezis and F. Merle, *Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions*, Comm. in P.D.E. **16** (1991), 1223–1253.
- D. Cassani, B. Ruf, C. Tarsi, *Best constants in a borderline case of second order Moser type inequalities*, Ann. Inst. H. Poincaré, **27** (2010), 73–93
- D. Cassani, B. Ruf, C. Tarsi, *Best constants for Moser type inequalities in Zygmund spaces*, Matemática Contemporânea **36** (2009), 79–90.
- D. Cassani, B. Ruf, C. Tarsi, *Group invariance and Pohozaev identities in limiting Moser type equations*, Preprint.
- J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1970/71), 1077–1092.
- D. Ornstein, *A non-equality for differential operators in the L_1 norm*, Arch. Rational Mech. Anal. **11** (1962), 40–49.
- G. Talenti, *Elliptic equations and rearrangements*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **3** (1976), 697–718.