Uniqueness and stability of saddle-shaped solutions to the Allen-Cahn equation

Xavier Cabré

ICREA and UPC, Barcelona

- 1. Minimal surfaces. The Simons cone
- 2. The analogue for Allen-Cahn: a conjecture of E. De Giorgi
- 3. Saddle-shaped solutions to Allen-Cahn

Minimal surfaces

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If $n \leq 7$ then $\partial E =$ hyperplane.

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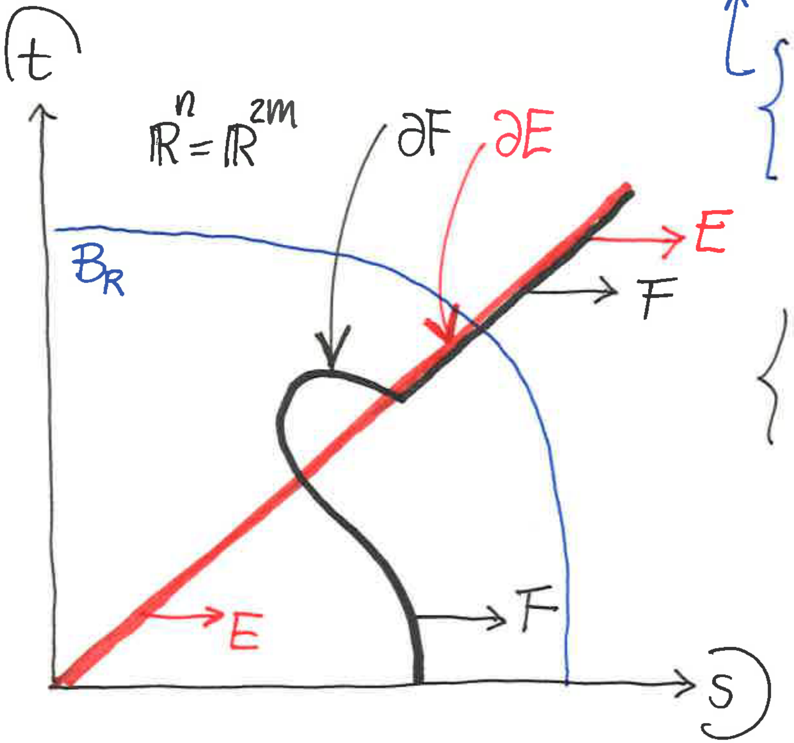

$\text{Area}(B_R \cap \partial E) \leq \text{Area}(B_R \cap \partial F)$

$s = \sqrt{x_1^2 + \cdots + x_m^2}$

$t = \sqrt{x_{m+1}^2 + \cdots + x_{2m}^2}$

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$(E = \{s > t\})$

$\forall n = 2m$ stationary
(mean curv $= 0$)

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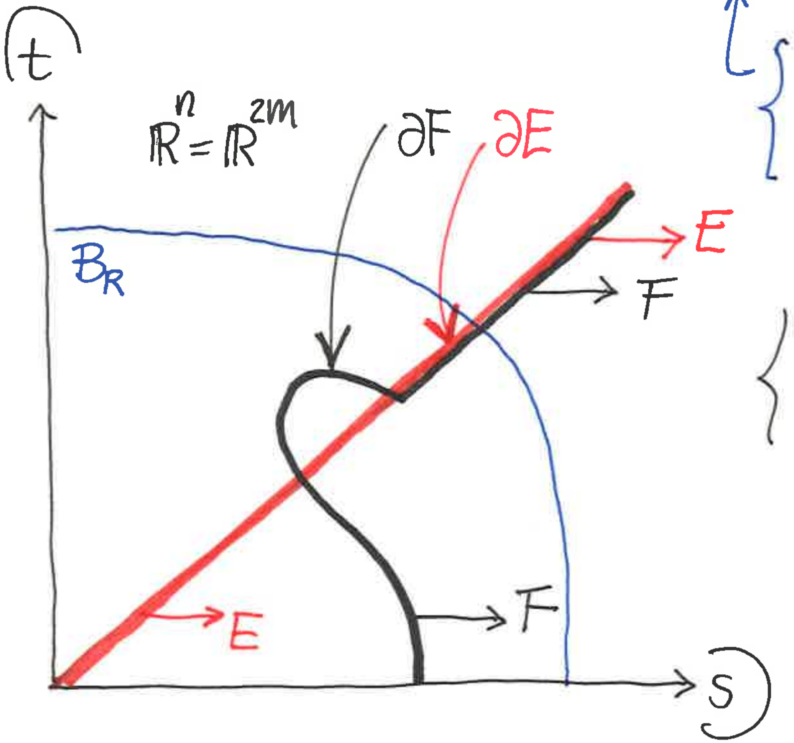

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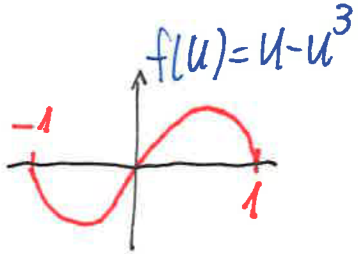
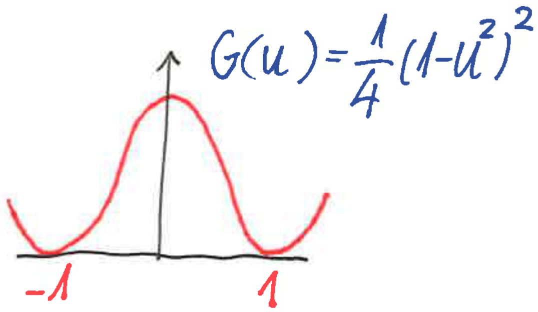
Thm [Bombieri-De Giorgi-Giusti]
Simons cone $\mathcal{C} \subset \mathbb{R}^{2m}$ minimal $\Leftrightarrow$ $\mathbf{2m \geq 8}$.

Allen-Cahn equation. A conjecture of De Giorgi

(AC) $\quad -\Delta u = \underline{u - u^3} \quad \text{in } \mathbb{R}^n$





$\hookrightarrow \quad E_{B_R}(u) = \int_{B_R} \frac{1}{2}|\nabla u|^2 + \underline{G(u)}$

Thm [Savin '03]

u global minimizer of (AC) in $\mathbb{R}^n$. If $\boldsymbol{n \leq 7}$, then u is **1-D**, i.e., $\{u = \lambda\}$ = hyperplanes.

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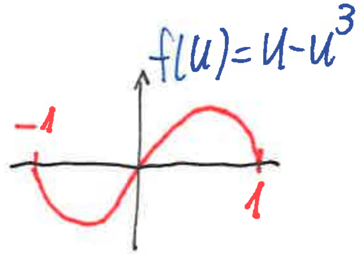

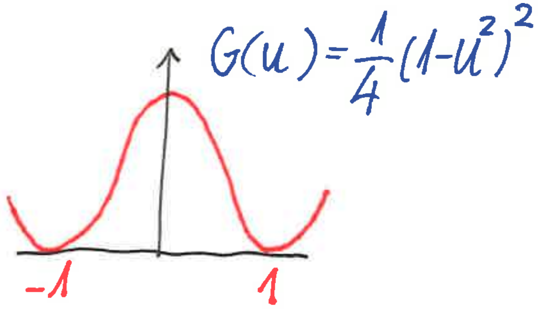

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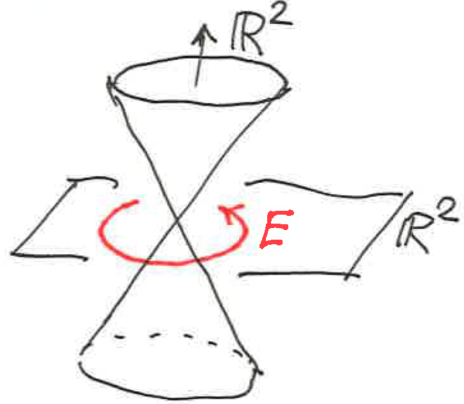
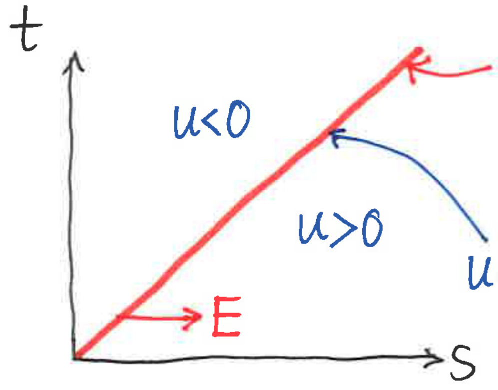
Thm [del Pino-Kowalczyk-Wei '08]

$\exists\, u$ global minimizer of (AC) in $\mathbb{R}^9$, u not 1D, with $u_{x_9} > 0$.

Open pb $\quad n = 8$? $\longrightarrow$ Saddle-shaped solutions of (AC):

Saddle-shaped sol'ns of (AC)





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$E = \{s>t\}$

$$u = u(s,t) \qquad \begin{cases} |u| < 1 \\ u(t,s) = -u(s,t) \end{cases}$$

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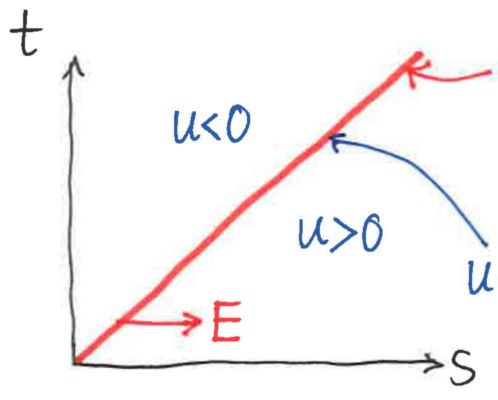

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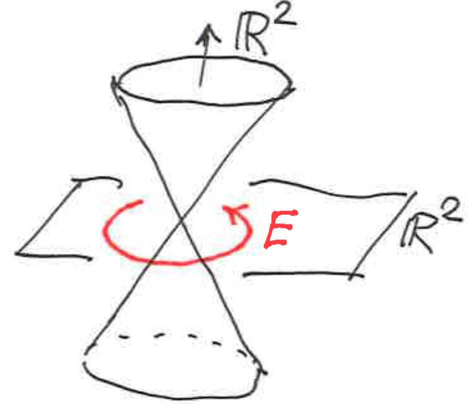

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$\exists$ saddle sol'n in $\mathbb{R}^{2m}$ $\forall m \geq 1$. It is **unstable** if $2m = 2, 4, 6$.

Its Morse index $= 1$ in $\mathbb{R}^2$ $\leftarrow$ [Schatzman]

$= \infty$ in $\mathbb{R}^4, \mathbb{R}^6$.

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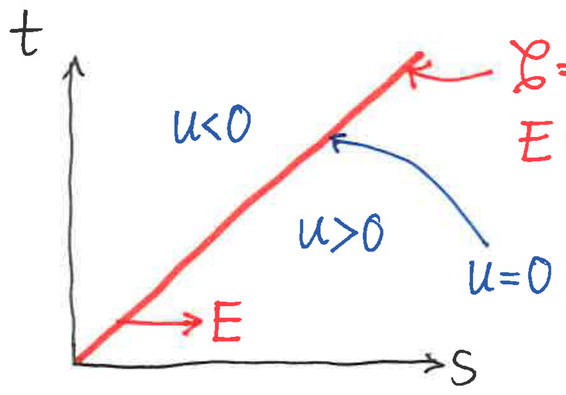

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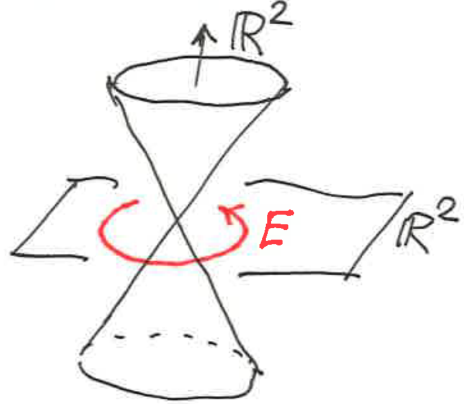

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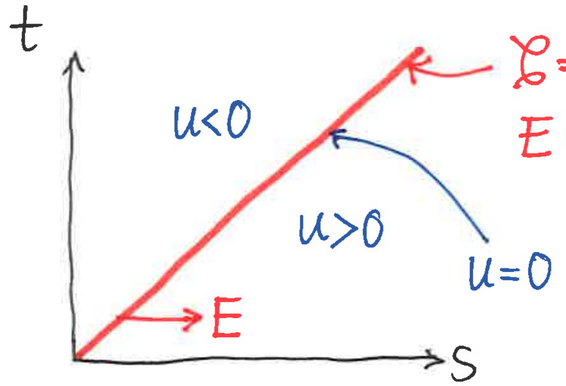

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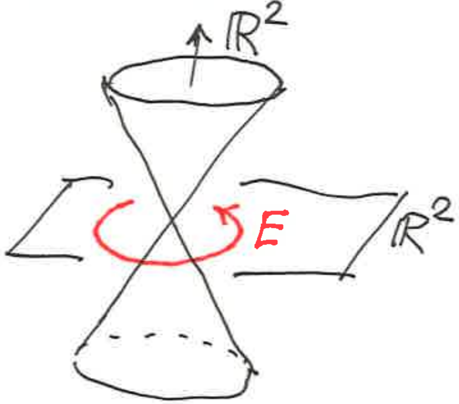

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Thm [C. ’10] In $\mathbb{R}^{14}$ saddle sol'n is stable.

Thm [Pacard-Wei ’11] In $\mathbb{R}^8$ $\exists$ stable sol'n not 1D.

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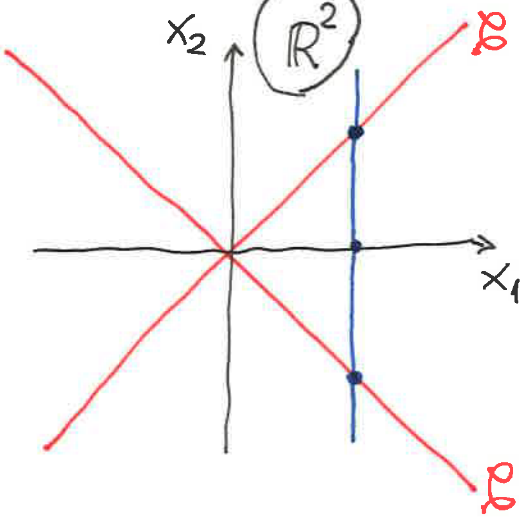
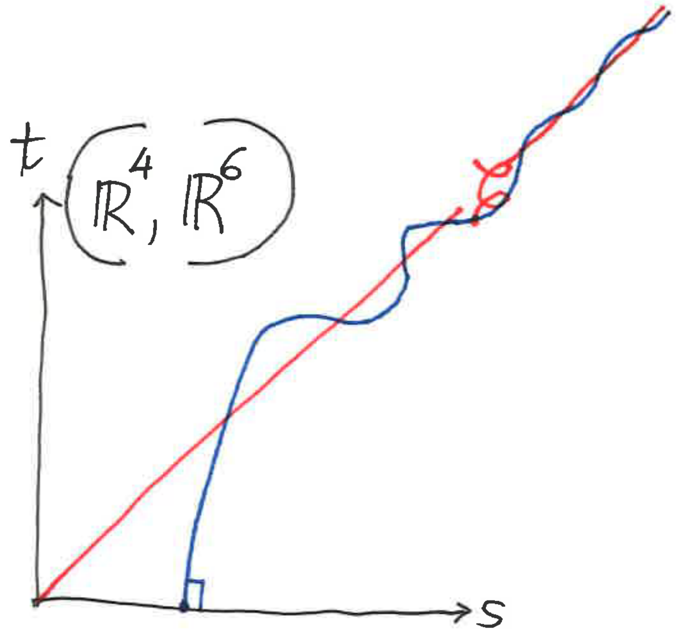
Simons cone. Foliations.





Minimizers are not singular
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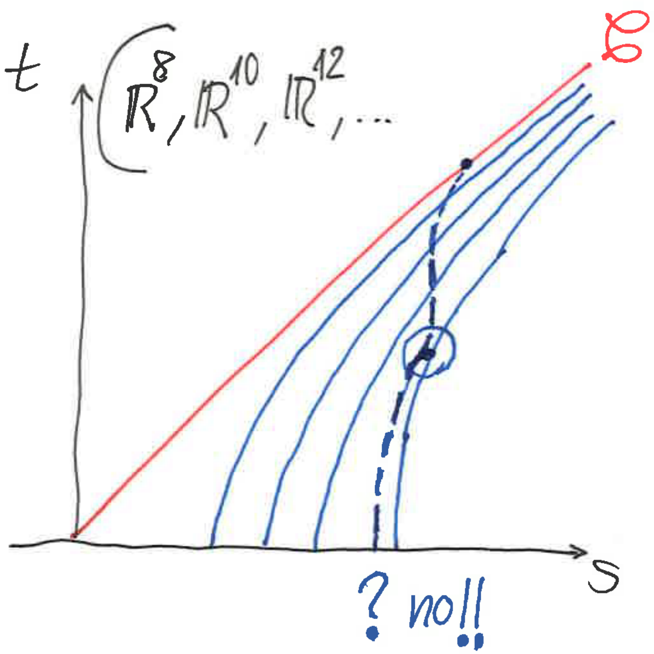
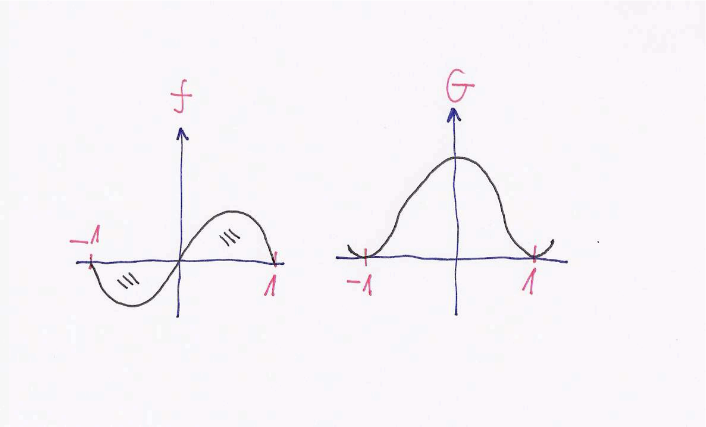


Phase transitions: interior reactions

$u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}, \quad -1 < u < 1, \quad \varepsilon > 0 :$

$$-\Delta u = \frac{1}{\varepsilon^2}\, f(u) \quad \text{in } \Omega$$

$$I_\varepsilon(u) = \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon} G(u), \qquad G' = -f$$

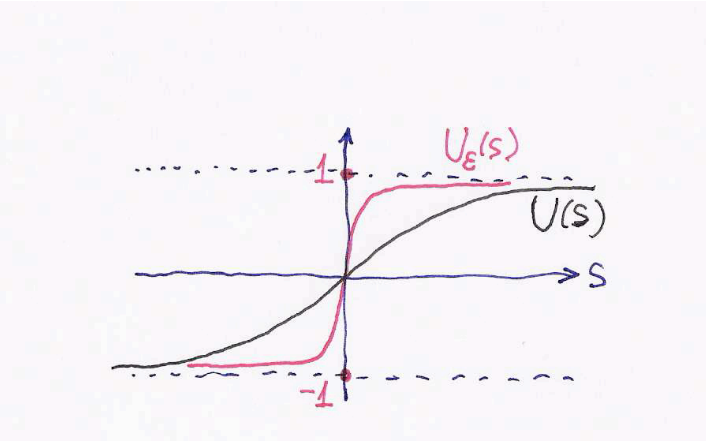


- Ginzburg-Landau model:

$f(u) = u - u^3$ **(bistable, balanced)**

$G(u) = \frac{1}{4}(1-u^2)^2$ **(double-well potential)**

- **1D solution:** $U_\varepsilon(s) = U(s/\varepsilon)$

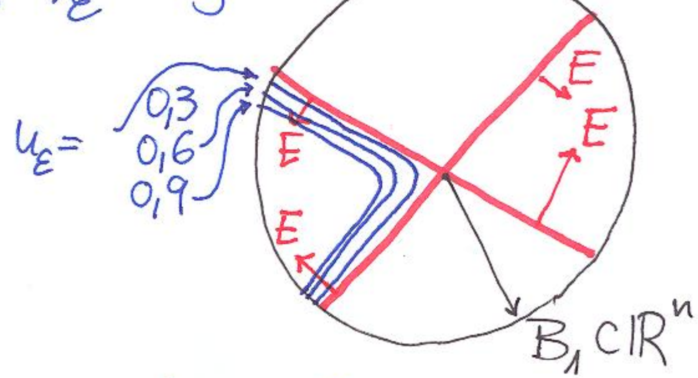


Modica-Mortola thm

$$-\Delta u = u - u^3 = f(u) \quad \text{in } \mathbb{R}^n$$

$$u_\varepsilon(x) = u(x/\varepsilon) = u(Rx) \quad , x \in B_1 \subset \mathbb{R}^n . \quad R = 1/\varepsilon \text{ large}$$

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2} f(u_\varepsilon)$$



Thm [MM]

Minimizers u_ε in $B_1 \subset \mathbb{R}^n$.

$$u_\varepsilon \xrightarrow[\varepsilon \downarrow 0]{} \begin{cases} 1 & \text{in } E \\ -1 & \text{in } B_1 \setminus E \end{cases} \quad \& \quad E \text{ is of minimal perimeter in } B_1 .$$

"Pf"

$$E_{B_1}(u_\varepsilon) = \int_{B_1} \varepsilon \frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{1}{2} 2G(u_\varepsilon) \frac{1}{\varepsilon}$$

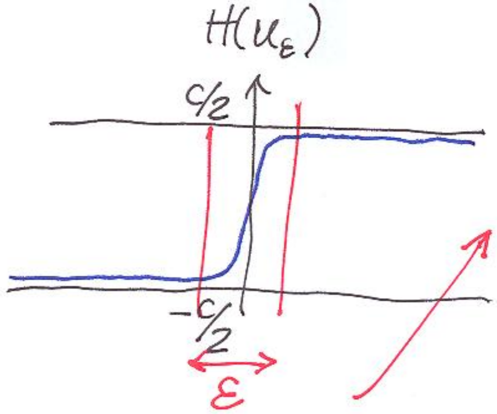
$$\rotatebox{90}{$\leq$}$$

$$\int_{B_1} \sqrt{\varepsilon} |\nabla u_\varepsilon| \cdot \sqrt{2G(u_\varepsilon)} \frac{1}{\sqrt{\varepsilon}}$$

(=) if parallel level sets

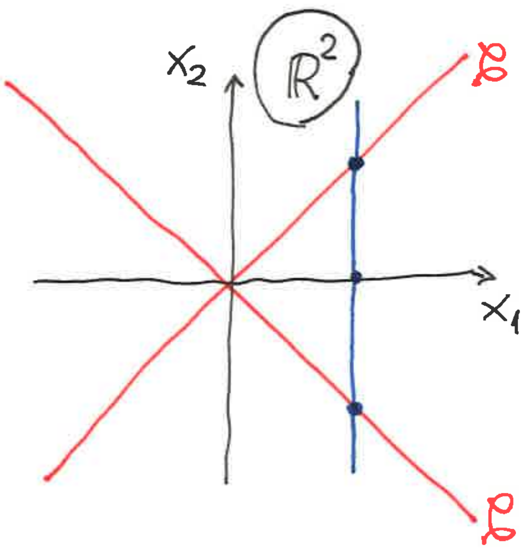
$$\|$$

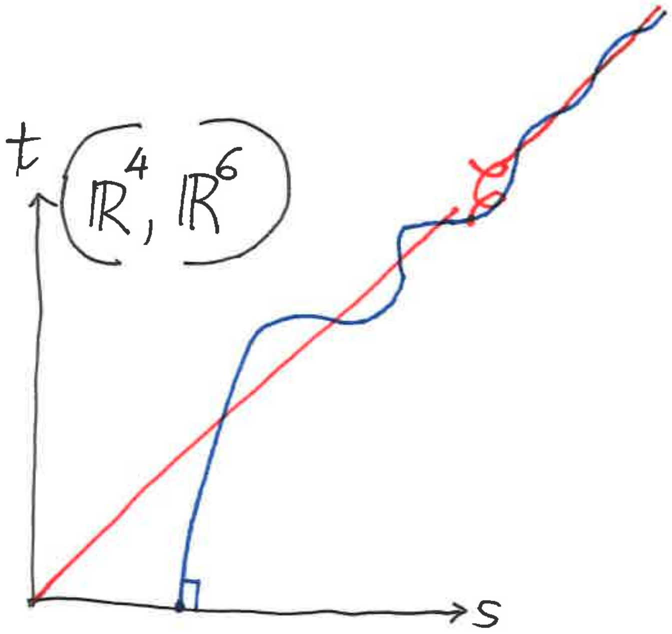
$$\int_{B_1} |\nabla H(u_\varepsilon)| . \xrightarrow[\varepsilon \downarrow 0]{} \int_{B_1} c \cdot |\nabla 1\!\!1_E| = c \cdot \text{perimeter}_{B_1}(E)$$



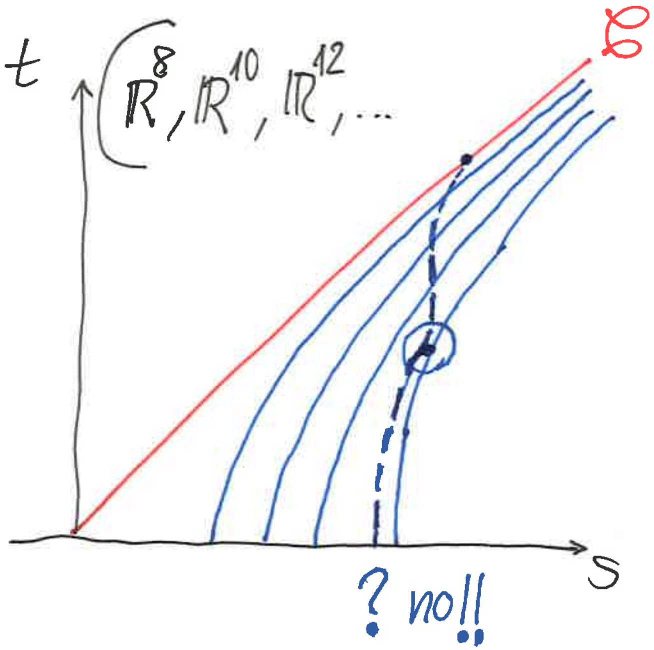
$$\int |\partial_x H(u_\varepsilon)| = \\ = \int \partial_x H(u_\varepsilon) = \text{jump} \\ = C$$

Simons cone. Foliations.





Minimizers $\Big\langle$ are not / are $\Big\rangle$ singular

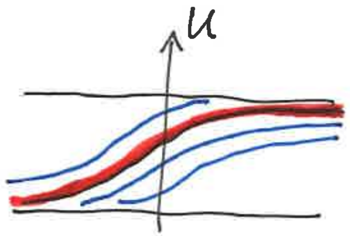


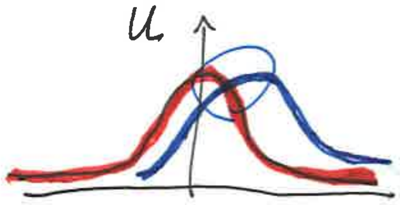
Foliation by stationary

$\Downarrow$

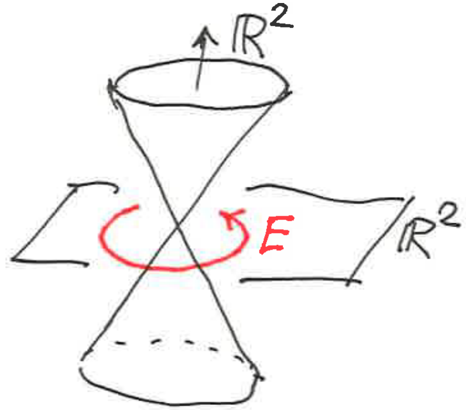
Minimizers

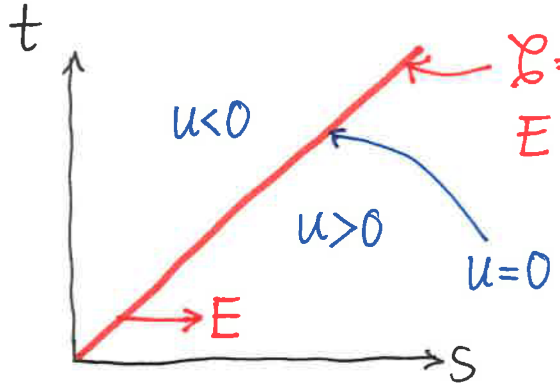
[Weierstrass - Caratheodory]





Saddle-shaped sol'ns of (AC)





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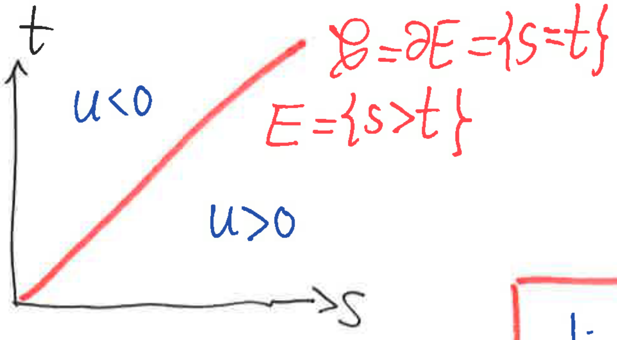

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$$u_{ss} + u_{tt} + (m-1)\left\{ \frac{u_s}{s} + \frac{u_t}{t} \right\} + u - u^3 = 0$$

$$\boxed{\operatorname{dist}_{\mathbb{R}^{2m}}(x, \mathcal{C}) = \frac{s-t}{\sqrt{2}}}$$

Asymptotic behaviour at ∞:

Let $\boxed{V(x) := u_0\left(\frac{s-t}{\sqrt{2}}\right) = \tanh\left(\frac{s-t}{2}\right).}$

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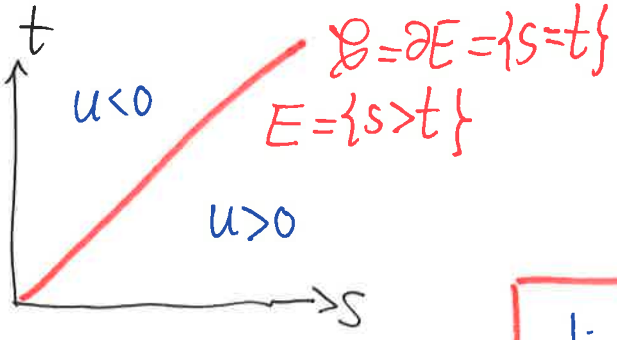

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$+$

LIOUVILLE THMS in $\mathbb{R}^{2m}$ & $\mathbb{R}^{2m}_+$ for (AC)

Asymptotic behaviour at ∞:

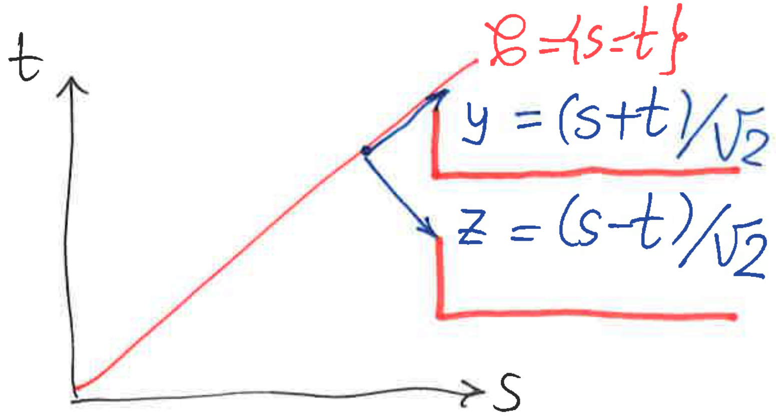
Thm [C-Terra '09] Let $\boxed{U(x) := u_0\left(\frac{s-t}{\sqrt{2}}\right) = \tanh\left(\frac{s-t}{2}\right).}$

u saddle sol'n in $\mathbb{R}^{2m}$, $\forall m \implies$

$$\big\| \, |u - U| + |\nabla u - \nabla U| \, \big\|_{L^\infty(\mathbb{R}^{2m} \setminus B_R(0))} \longrightarrow 0 \quad \text{as } R \to \infty.$$

Instability in $\mathbb{R}^4$ & $\mathbb{R}^6$: (C-Terra '09)

(in $\mathbb{R}^2$: [Dang-Fife-Peletier '92] [Schatzman '95])



(AC):

$$u_{yy}+u_{zz}+\frac{2(m-1)}{y^2-z^2}(yu_y-zu_z)+f(u)=0$$

$$\Downarrow$$

$$0=\{\Delta_{2m}+f'(u)\}u_z-\frac{2(m-1)}{y^2-z^2}u_z+\frac{4(m-1)z}{(y^2-z^2)^2}(yu_y-zu_z).$$

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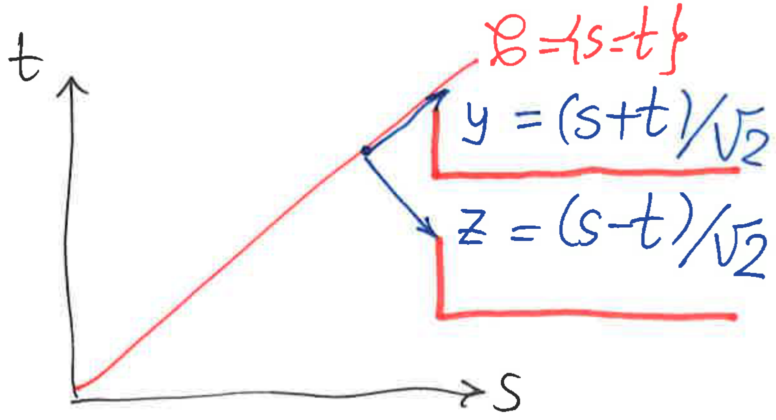

(AC):

$$u_{yy} + u_{zz} + \frac{2(m-1)}{y^2 - z^2}(y u_y - z u_z) + f(u) = 0$$

$$0 = \{\Delta_{2m} + f'(u)\} u_z - \frac{2(m-1)}{y^2 - z^2} u_z + \frac{4(m-1)z}{(y^2-z^2)^2}(y u_y - z u_z).$$

$$\boxed{D^2 E(u)(\xi, \xi) = \int_{\mathbb{R}^{2m}} |\nabla \xi|^2 - f'(u)\xi^2 < 0} \quad \text{for}$$

$$\xi(y,z) = \eta\left(\frac{y}{a}\right) u_z(y,z)$$

& let $a \to +\infty$: HARDY ineq.

Towards uniqueness in $\mathbb{R}^{2m}$ & stability in $\mathbb{R}^{14}$

Prop'n [C'10] u saddle sol'n in $\mathbb{R}^{2m}$ $\Longrightarrow$

$L_u := \Delta + f'(u(x))$ satisfies the maximum principle in $\mathcal{O} = \{s > t\}$.

(i.e., $L_u v \geq 0$ in $\mathcal{O}$, $v \leq 0$ on $\partial\mathcal{O}$ & $\limsup\limits_{x\in\mathcal{O}, |x|\to\infty} v(x) \leq 0$

$\Longrightarrow v \leq 0$ in $\mathcal{O}$)

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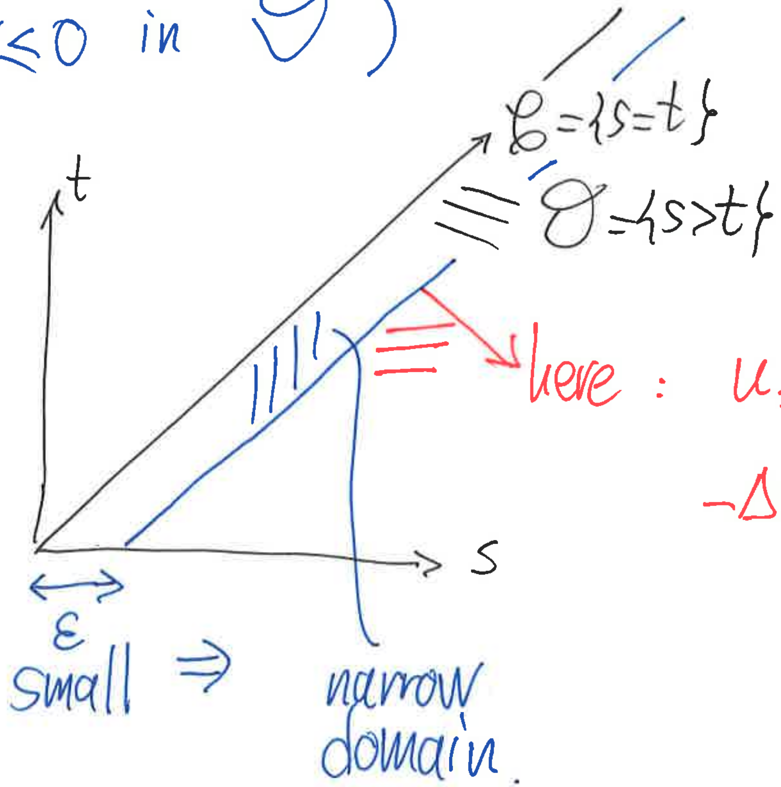
Proof. uses



here: $u \geq \delta > 0$ &

$-\Delta u = f(u) \geq f'(u)u$

$-L_u u \geq 0$

(supersol'n, $\geq \delta > 0$) $\square$

$$\left.\begin{array}{l}\text{Maximum principle in } \mathcal{O} \text{ for } L_u \\ \qquad\qquad \oplus \\ \text{Asymptotics of saddle sol'ns at } \infty \\ \qquad\qquad \oplus \\ \exists \text{ of smallest saddle in } \mathcal{O}\end{array}\right\} \Longrightarrow$$

Thm [C'10] (**Uniqueness**) The saddle sol'n in $\mathbb{R}^{2m}$ is unique, $\forall 2m \geq 2$.

Maximum principle in $\mathcal{O}$ for L_u

$\oplus$

Asymptotics of saddle sol'ns at ∞

$\oplus$

$\exists$ of smallest saddle in $\mathcal{O}$

$\Bigg\} \Longrightarrow$

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Pf $\quad \underline{u} \leq u$ in $\mathcal{O}$

$\underline{u}$: smallest saddle in $\mathcal{O}$

$$\Rightarrow \quad -\Delta(u-\underline{u}) = f(u) - f(\underline{u}) \leq f'(u)(u-\underline{u}) \text{ in } \mathcal{O}$$

$$\Downarrow \text{ Asymptotics + Max. Pr.}$$

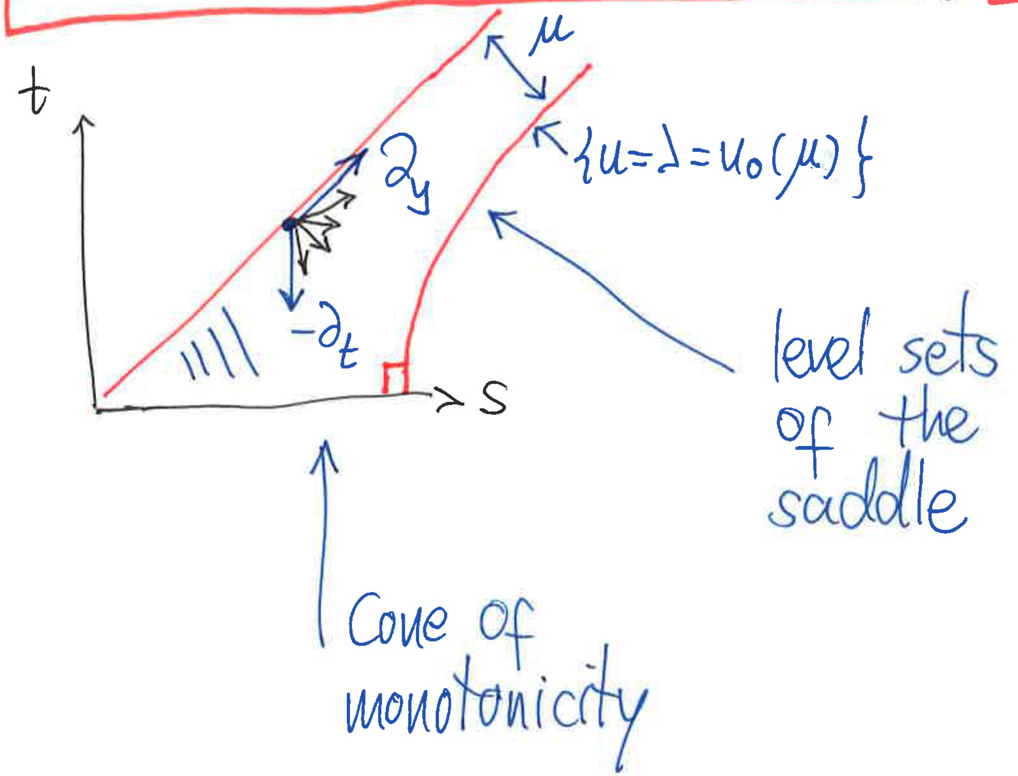
$$u - \underline{u} \leq 0 \text{ in } \mathcal{O}. \qquad \square$$

Maximum principle in $\mathcal{O}$ $\oplus$ Asymptotics at ∞ $\Downarrow$

$\hookrightarrow$ Monotonicity & convexity properties of saddles.

Thm [C'10] u saddle sol'n in $\mathbb{R}^{2m}$, $2m \geq 2$. Then:

in $\mathcal{O} \setminus \{t=0\} = \{s > t > 0\}$: $u_y > 0$, $-u_t > 0$, $u_{st} > 0$.



Maximum principle in $\mathcal{O}$ ⊕ Asymptotics at ∞ ⇓

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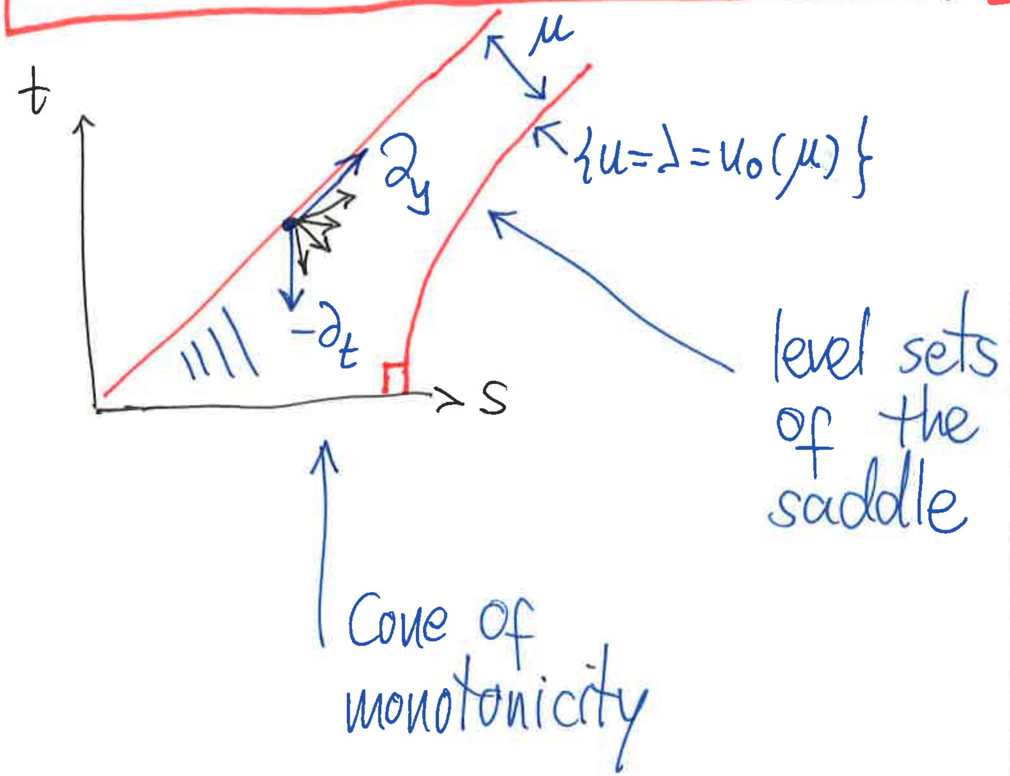

Pf: MPrinciple ⊕ asympt. ∞

⊕

$$\{\Delta + f'(u)\} u_y = \frac{m-1}{s^2} u_y + \frac{(m-1)(s^2-t^2)}{\sqrt{2}\, s^2 t^2} u_t$$

$$\{\Delta + f'(u)\} u_t - \frac{m-1}{t^2} u_t = 0$$

$$\{\Delta + f'(u)\} u_{st} - (m-1)\left(\frac{1}{s^2} + \frac{1}{t^2}\right) u_{st} \leq 0$$

□

Thm [C'10] (stability in $\mathbb{R}^{14}$, $\mathbb{R}^{2m}$ for $2m \geq 14$)

$2m \geq 14 \iff \exists b \in \mathbb{R}$ s.t. $b(b-m+2)+m-1 \leq 0$. Then:

$\varphi = \varphi(s,t) := t^{-b} u_s - s^{-b} u_t$ $(b>0)$

satisfies
$$\left.\begin{array}{c} \varphi > 0 \\ \{\Delta + f'(u)\}\varphi \leq 0 \end{array}\right\} \text{ in } \mathbb{R}^{2m} \setminus \{st=0\}.$$

$\Rightarrow$ Stability of the saddle in $\mathbb{R}^{2m}$, $2m \geq 14$.

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$\Longrightarrow$ Stability of the saddle in $\mathbb{R}^{2m}$, $2m \geq 14$.

Pf: φ is even w.r.t. $\mathcal{C}$ $\Longrightarrow$

$\varphi > 0$ in $\mathcal{O}$ $\quad (\Rightarrow$ in $\mathbb{R}^{2m} \setminus \{st=0\})$

$$\Delta u_s + f'(u)u_s - \frac{m-1}{s^2}u_s = 0 \quad ; \quad \Delta u_t + f'(u)u_t - \frac{m-1}{t^2}u_t = 0$$

$$\Delta t^{-b} = b(b-m+2)t^{-b-2} \quad ; \quad \Delta s^{-b} = b(b-m+2)s^{-b-2}$$

$$\varphi = t^{-b}u_s - s^{-b}u_t$$

$$\longrightarrow \underline{\Delta\varphi + f'(u)\varphi} = u_s t^{-b}\{(m-1)s^{-2} + b(b-m+2)t^{-2}\}$$
$$+ (-u_t)s^{-b}\{(m-1)t^{-2} + b(b-m+2)s^{-2}\}$$
$$+ u_{st}\, 2b\{s^{-b-1} - t^{-b-1}\}$$

$$\Delta u_s + f'(u) u_s - \frac{m-1}{s^2} u_s = 0 \quad ; \quad \Delta u_t + f'(u) u_t - \frac{m-1}{t^2} u_t = 0$$

$$\Delta t^{-b} = b(b-m+2)\, t^{-b-2} \quad ; \quad \Delta s^{-b} = b(b-m+2)\, s^{-b-2}$$

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$$\longrightarrow \quad \underline{\Delta\varphi + f'(u)\varphi} = u_s\, t^{-b}\{(m-1)s^{-2} + b(b-m+2)t^{-2}\}$$
$$+ (-u_t)\, s^{-b}\{(m-1)t^{-2} + b(b-m+2)s^{-2}\}$$
$$+ u_{st}\, 2b\{s^{-b-1} - t^{-b-1}\}$$

$u_{st} > 0$ in $\mathcal{O}$

$$\leq u_s\, t^{-b}\{(m-1)s^{-2} + b(b-m+2)t^{-2}\}$$
$$+ (-u_t)\, s^{-b}\{(m-1)t^{-2} + b(b-m+2)s^{-2}\} \quad \text{in } \mathcal{O}$$

$$\downarrow$$

$$
\begin{aligned}
\{\Delta + f'(u)\}\varphi \ &\leq\ t^{-b}(u_s + u_t)\{(m-1)s^{-2} + b(b-m+2)t^{-2}\} \\
&\quad -s^{-b}u_t\{(m-1)t^{-2} + b(b-m+2)s^{-2}\} \\
&\quad -t^{-b}u_t\{(m-1)s^{-2} + b(b-m+2)t^{-2}\} \\
&=\ u_y\,\sqrt{2}t^{-b}\{(m-1)s^{-2} + b(b-m+2)t^{-2}\} \\
&\quad +(-u_t)\ (m-1)(s^{-b}t^{-2} + t^{-b}s^{-2}) \\
&\quad +(-u_t)\ b(b-m+2)(s^{-2-b} + t^{-2-b}) \\
&\leq\ u_y\,\sqrt{2}t^{-b}(m-1)\{s^{-2} - t^{-2}\} \\
&\quad +(-u_t)\ (m-1)(s^{-b}t^{-2} + t^{-b}s^{-2} - s^{-2-b} - t^{-2-b}) \\
&\leq\ (-u_t)\ (m-1)(s^{-b}t^{-2} + t^{-b}s^{-2} - s^{-2-b} - t^{-2-b})
\end{aligned}
$$