Behaviour near extinction for the fast diffusion equation in bounded domains

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(**Joint work with *G. Grillo* and *J. L. Vázquez***)

NONLINEAR PDES AND FUNCTIONAL INEQUALITIES
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The Dirichlet Problem for the Fast Diffusion Equation in $\Omega \subset \mathbb{R}^d$

We consider, in a bounded and smooth domain Ω, positive solutions to:

$$\begin{cases} \partial_\tau u = \Delta\left(u^m\right) = \nabla \cdot \left(u^{m-1}\nabla u\right), & \forall (\tau, y) \in (0, +\infty) \times \Omega \\ u(0, y) = u_0, & \forall y \in \Omega \\ u(\tau, y) = 0, & \forall (\tau, y) \in (0, +\infty) \times \partial\Omega \end{cases}$$

where $0 < m < 1$ (i.e. *Fast Diffusion*, FDE)

- Existence and uniqueness of weak solutions for the parabolic problem is well known for any $m > 0$. Recall that $0 < m < 1$ is the Fast Diffusion case, $m = 1$ is the Linear Heat Equation and $m > 1$ is the Porous Medium case.
- The initial datum is chosen to be

$$0 \le u_0 \in \mathrm{L}^r(\Omega) \quad \text{with} \quad r \ge 1 \quad \text{and} \quad r > \frac{d(1-m)}{2},$$

so that the corresponding solution is bounded and nonnegative for all $m > 0$.

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Some Properties of Solutions

- Since we deal with the Fast Diffusion case $m < 1$, *the mass $\int_\Omega u(y,\tau)\mathrm{d}y$ is not preserved*, and solutions extinguish in finite time

$$\exists\, T = T(u_0)\ :\ u(\tau,\cdot) \equiv 0 \quad \forall\, t \geq T$$

Consequence of Sobolev and Poincaré inequalities (sufficient condition).

- Under our hypothesis, solutions are indeed positive in $\Omega \times (0,T)$ and for all $0 < m < 1$, as a consequence of parabolic (intrinsic) Harnack inequalities:
  - For $\frac{d-2}{d} < m < 1$, DiBenedetto et al. (1992)
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  and they are at least $C^\alpha(\overline{\Omega})$ (DiBenedetto et al. 1988, 1992).
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Review of previous results

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$$u(\tau,x) = \left(\frac{T-\tau}{T}\right)^{\frac{1}{1-m}} v(t,x) \qquad \text{and} \qquad t = T\log\left(\frac{T}{T-\tau}\right).$$

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The crucial exponent is

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(First Pioneering Result) ***J. G. Berryman, C. J. Holland*** **ARMA (1980)**

Let $m_s < m < 1$. Then there exists a sequence of times $t_n \to \infty$ as $n \to \infty$ and one or several solutions S to the stationary problem such that

$$v(t_n) \xrightarrow[n\to\infty]{W_0^{1,2}(\Omega)} S\,.$$

(Uniqueness of asymptotic profile) ***E. Feiresl, F. Simondon*** **J. Dynamic Diff. Eq. (2000)**

Let v, S be as above and assume $m_s < m < 1$. Then there exists a **unique** stationary solution S such that

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E. DiBenedetto, Y. C. Kwong, V. Vespri **Indiana Univ. Math. J. (1991)**

Let w be the solution to the rescaled Dirichlet problem with $m_s < m < 1$. Then, for any $\sigma > 0$ there exist positive constants $\lambda, \mu > 0$ depending on $d, m, \|u_0\|_{m+1}, \|\nabla u_0^m\|_2, \partial\Omega$ and σ, such that for any $t \geq \sigma$ and for any $x \in \Omega$

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$$\lambda \,\mathrm{dist}\,(x, \partial\Omega)^{1/m} (T-\tau)^{1/(1-m)} \leq u(\tau,x) \leq \mu \,\mathrm{dist}\,(x, \partial\Omega)^{1/m} (T-\tau)^{1/(1-m)}.$$

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(Convergence in Relative Error) ***M.B., G. Grillo, J.L. Vázquez***, **JMPA (2011)**

Let u be the solution to the Dirichlet problem and $T = T(m, d, u_0)$ be its extinction time. Then we have that

$$\lim_{\tau \to T^-} \left\| \frac{u(\tau, \cdot)}{\mathcal{U}(\tau, \cdot)} - 1 \right\|_{L^\infty(\Omega)} = 0$$

where the special solution $\mathcal{U}$ is defined as

$$\mathcal{U}(\tau, x) = S(x) \left[(T - \tau)/T\right]^{1/(1-m)} \qquad \left[\text{one has } S(x) \sim \mathrm{dist}(x, \partial\Omega)^{\frac{1}{m}} \right]$$

and S is a suitable positive classical solution to the stationary problem. Equivalently, the following improved Global Harnack Principle

$$c_0(\tau)\, S(x)\, (T - \tau)^{1/(1-m)} \le u(\tau, x) \le c_1(\tau)\, S(x)\, (T - \tau)^{1/(1-m)}.$$

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Steps of the proof.

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$$\frac{1}{m}(1+\phi)^{\frac{1}{m}-1}\,\phi_t = S^{m-1}\Delta\phi + 2\frac{\nabla(S^m)}{S}\cdot\nabla\phi + F(\phi)$$

  where F is given by $F(\phi) = \mathbf{c}\left[(1+\phi)^{1/m} - (1+\phi)\right]$.

- Convergence far away from the boundary is easy.
- One can choose positive constants A, B, C and t_0, so that the function

$$\Phi(t,x) = C - B\,d(x) - A(t-t_0)$$

  is a supersolution to the differential equation satisfied by ϕ, in a small neighborhood of the spatial boundary $\Omega_\delta =: \{x \in \Omega\ :\ \mathrm{dist}(x,\partial\Omega) < \delta\}$. Technical.

- Use parabolic comparison to compare ϕ and Φ in $t \in (t_0, T] \times \Omega_\delta$. $\square$

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Define the **relative error function**

$$\theta(t,x) = \frac{v(t,x)}{S(x)} - 1.$$

It satisfies the equation

$$\theta_t = \frac{1}{S^{1+m}} \nabla\cdot\left(S^{2m}\nabla(1+\theta)^m\right) + \mathbf{c} f(\theta)$$

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$$\theta_t = \frac{1}{S^{1+m}} \nabla\cdot\left(S^{2m}\nabla(1+\theta)^m\right) + \mathbf{c} f(\theta)$$

where

$$f(\theta) := (1+\theta) - (1+\theta)^m$$

In the sequel, the constants $m_\sharp, \gamma_0$ are *explicit*. They depend on m and on the geometry of the domain.

(Decay Rates, Rescaled Version) *M.B., G.Grillo, J.L. Vázquez*, **JMPA (2011)**

Let $m_\sharp < m < 1$. Let v be the rescaled solution corresponding to an initial datum u_0, and let S be the stationary profile to which the solution converges. Let $0 < \gamma < \gamma_0$. Then for all $t > t_0$:

$$\mathcal{E}[\theta(t)] := \frac{1}{2}\int_\Omega \left|\theta(t) - \overline{\theta}(t)\right|^2 S^{m+1}\,\mathrm{d}x \le \mathrm{e}^{-\gamma(t-t_0)}\mathcal{E}[\theta(t_0)]\,,$$

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- The constants $k_i(m)$ have an explicit expression and indeed $k_i(m) \to 1$ as $m \to 1^-$. In the limit $m \to 1^-$ we have that $f_\Omega(m) \to 2\lambda_1/(\lambda_1 + \lambda_2) < 1$, hence the range of $m < 1$ for which (1) holds is nonempty. Note that $m_\sharp$ changes with m and with the geometry of the domain.
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Similar consideration also work for the porous media case ($m > 1$), which has been studied long ago by completely different methods (Aronson-Peletier, JDE (1981)).

(Decay Rates, Porous Medium)

Let $m > 1$, let v be a the rescaled solution, that converges to its *unique* stationary state S, and let $\theta = v/S$. Then, for all $0 < \beta < 2 + \frac{Km}{m-1}$ there exists a time t_1 depending on m, d, β and on the constant $K > 0$ of the weighted Poincaré inequality, such that the entropy decays as

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$$\|v(t,\cdot) - S(\cdot)\|_{L^q(\Omega)} \le \kappa_1\, e^{-(t-t_1)} \qquad \text{for all } t \ge t_1.$$

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- Rescale $v(x,t) = e^{\lambda_1 t} u(x,t)$to get the equation $v_t = \Delta v + \lambda_1 v$.
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Sketch of the proof.

Step 1: an "entropy functional" and its derivative. Recall that

$$\mathcal{E}[\theta(t)] = \frac{1}{2}\int_\Omega \left|\theta(t) - \overline{\theta}(t)\right|^2 S^{1+m}\,\mathrm{d}x\,,$$

where S is a (positive) solution to the elliptic problem

$$\begin{cases} -\Delta S^m = \mathbf{c}\, S & \text{in } \Omega \\ S = 0 & \text{on } \partial\Omega \end{cases}$$

whenever $m_s < m < 1$. We then have:

(Entropy/Entropy-production)

Let $m_s < m < 1$ and θ be the solution to the equation

$$\theta_t = \frac{1}{S^{m+1}}\nabla\cdot\left(S^{2m}\nabla(1+\theta)^m\right) + \mathbf{c}\, f(\theta), \qquad \text{with} \qquad f(\theta) = (1+\theta) - (1+\theta)^m\,.$$

Then the following inequality holds

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Sketch of the proof.

Step 1: an “entropy functional" and its derivative. Recall that

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Let $f \in W_0^{1,2}(\Omega)$, ϕ_1 the ground state eigenfunction of the Dirichlet Laplacian, $g = f/\phi_1$ and S as above. Then the following inequality holds

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Step 3 (conclusion).

The difficult issue is to estimate the ratio $\frac{k_0(m)^2}{k_1(m)^2}$.

On the one hand, one can prove results about quantitative elliptic Harnack inequalities for the equation $-\Delta u = u^p$. This is the topic of M.B., G. Grillo, J.L. Vázquez (2011, in preparation).

The resulting bounds give explicit constants in the Harnack inequality. It is then possible to use them to compare solutions with different values of p, which then yield the required bounds on $\frac{k_0(m)^2}{k_1(m)^2}$.

Such bounds then yield explicit $m_\sharp$ and γ_0, but it has to be remarked that unfortunately

$$\lim_{m\uparrow 1} \frac{k_0(m)^2}{k_1(m)^2} < 1$$

which is not what is expected.

But we can also prove the following purely elliptic result (see also Grossi (Annali Pisa, 2009) for related results):

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Nonlinear elliptic problems near $p = 1$

Let $p = 1/m$. Let U_p be a family of solutions of the problem

$$\begin{cases} -\Delta U = \lambda_p\, U^p & \text{in } \Omega \\ U > 0 & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega \end{cases} \tag{4}$$

with $p \in [1, p_s)$, $p_s = \frac{d+2}{d-2}$, $\|U_p\|_{p+1} = 1$, so that $\|\nabla U_p\|_2^2 = \lambda_p$. Then as $p \to 1$, one has $\lambda_p \to \lambda_1$, $U_p \to \Phi_1$ in $\mathrm{L}^\infty(\Omega)$, $\nabla U_p \to \nabla\Phi_1$ in $\left(\mathrm{L}^2(\Omega)\right)^d$. Besides, there exist two explicit constants $0 < c_0 < c_1$ such that

$$c_0^{p-1}\lambda_1 \le \lambda_p \le c_1^{p-1}\lambda_1 \,. \tag{5}$$

Moreover, there exists constants $0 < \widetilde{k}_0(p) \le \widetilde{k}_1(p)$ such that $\widetilde{k}_i(p) \to 1$ as $p \to 1^+$, such that

$$\widetilde{k}_0(p) \le \frac{U_p(x)}{\Phi_1(x)} \le \widetilde{k}_1(p), \qquad \text{for all } x \in \Omega \,. \tag{6}$$

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with $p \in [1, p_s)$, $p_s = \frac{d+2}{d-2}$, $\|U_p\|_{p+1} = 1$, so that $\|\nabla U_p\|_2^2 = \lambda_p$. Then as $p \to 1$, one has $\lambda_p \to \lambda_1$, $U_p \to \Phi_1$ in $\mathrm{L}^\infty(\Omega)$, $\nabla U_p \to \nabla \Phi_1$ in $\left(\mathrm{L}^2(\Omega)\right)^d$. Besides, there exist two explicit constants $0 < c_0 < c_1$ such that

$$c_0^{p-1} \lambda_1 \leq \lambda_p \leq c_1^{p-1} \lambda_1 \,. \tag{5}$$

Moreover, there exists constants $0 < \widetilde{k}_0(p) \leq \widetilde{k}_1(p)$ such that $\widetilde{k}_i(p) \to 1$ as $p \to 1^+$, such that

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Nonlinear elliptic problems near $p = 1$

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The End

Thank you!!!

The kind of argument outlined here can be used to study other related problems. For example consider positive solutions to the fast diffusion equation

$$\dot{u} = \Delta u^m, \quad \text{on } \mathbb{H}^n$$

where

- $m \in (m_s, 1]$
- $\mathbb{H}^n$ is the hyperbolic space and Δ the corresponding Riemannian Laplacian.

Recall that, on the hyperbolic space, both the Sobolev inequality and the L^2-Poincaré inequality hold, so that the L^2-spectrum of $-\Delta$ is $\left[\frac{(n-1)^2}{4}, +\infty\right)$.
Certain classes of positive solutions do vanish in a finite time (Bonforte, G. Vazquez, JEE 2008). There are some rough estimates on the extinction time there.

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It can be expected that the asymptotics of solutions is related to solutions, if any, of the elliptic problem $-\Delta u = u^{1/m}$ (up to rescalings). No results on this till recently, but:

- Mancini-Sandeep (Annali Pisa, 2008) have shown that there exist exactly one solution U to the elliptic problem. It is radial, and it has finite energy, namely it belongs to $W^{1,2}(\mathbb{H}^n)$. It decays at infinity as $ce^{-(n-1)r}$, r being the Riamannian distance from the given point. There are infinitely many other radial positive solutions, but they have infinite energy. Notice that $U^{1/m} \in \mathrm{L}^1$.
- M.B., F. Gazzola, G. Grillo and J. L. Vázquez have just proved that there is no other positive radial solution apart the ones found above, and that all of them apart U decay polynomially at infinity, hence they do not belong to L^q for any $q \neq \infty$ (recall that the Riemannian measure has a density whose radial part is $e^{(n-1)r}$).

Hence the asymptotics of solutions to the fast diffusions should be related to the separate variable solution $\mathcal{U}(t,x) = U(x)[(T-t)/T]^{1/(1-m)}$. Presently under investigation using the above methods, as a part of a more general study of nonlinear diffusions on manifolds.

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