Qualitative behavior of global solutions to some nonlinear fourth order differential equations

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The equation

Consider the equation

$$w''''(s) + kw''(s) + f(w(s)) = 0 \qquad s \in \mathbb{R}$$

where $k \in \mathbb{R}$ and f is a locally Lipschitz function

$\Rightarrow$ $k > 0$ Swift-Hohenberg equation

$\Rightarrow$ $k \leq 0$ extended Fisher-Kolmogorov equation

- L.A. Peletier, W.C. Troy, Birkhäuser Boston Inc. (2001)

A connected biharmonic Gelfand-type equation

Let $u = u(r)$, $r := |x|$ be a radial solution to

$$\Delta^2 u + e^u = \frac{1}{|x|^4} \qquad \text{in } \mathbb{R}^4 \setminus \{0\}$$

and with the change of variables

$$s = \log r \qquad w(s) := u(e^s) + 4s \qquad s \in \mathbb{R}$$

we get

$$w''''(s) - 4w''(s) + e^{w(s)} - 1 = 0 \qquad s \in \mathbb{R}\,.$$

- C.S. Lin, (1998); S.Y.A. Chang, W. Chen (2001);
- G.Arioli, F.Gazzola, H.-Ch.Grunau, (2006); G.Arioli, F.Gazzola, H.-Ch.Grunau, E.Mitidieri, (2005); E.Berchio, D.Cassani, F.Gazzola, (2010); E.Berchio, F.Gazzola, (2005); J.Dávila, L.Dupaigne, I.Guerra, M.Montenegro, (2007); J.Dávila, I.Flores, I.Guerra, (2009)

A connected biharmonic Gelfand-type equation

Let $k \in \mathbb{R}$ and $u = u(r)$, $r := |x|$ be a radial solution to

$$\Delta^2 u - 2(n-4)\frac{x \cdot \nabla \Delta u}{|x|^2} + (n^2 - 6n + 12 + k)\frac{\Delta u}{|x|^2}$$

$$-(n-2)\left[(n-2)^2 + k\right]\frac{x \cdot \nabla u}{|x|^4} + e^u = \frac{1}{|x|^4}$$

in $\mathbb{R}^n \setminus \{0\}$ $(n \geq 2)$, with the change of variables

$$s = \log r \qquad w(s) := u(e^s) + 4s \qquad s \in \mathbb{R}$$

w solves

$$w''''(s) + kw''(s) + e^{w(s)} - 1 = 0 \qquad s \in \mathbb{R}\,.$$

Physical models

$$w''''(s) + kw''(s) + f(w(s)) = 0 \qquad (s, k \in \mathbb{R})$$

$\Rightarrow$ Suspension bridge model

$$f(w) = (w+1)^{+} - 1\,; \quad f(w) = e^{w} - 1$$

• P.J. McKenna, W. Walter (1990); Y. Chen, P.J. McKenna (1997)

$\Rightarrow$ Configuration of a nonlinearly supported elastic strut

$$f(w) = w - w^2\,; \quad f(w) = w - w^3 + w^5$$

• G.W. Hunt, H.M. Bolt, J.M.T. Thompson (1989);
C.J. Amick, J.F. Toland (1992); L.A. Peletier (2001)

$\Rightarrow$ Pattern formation in physical, chemical or biological systems

$$f(w) = w^3 - w$$

• L.A. Peletier, W.C. Troy (2001); D. Bonheure, L. Sanchez(2006)

Main assumption on *f*

$$w''''(s) + kw''(s) + f(w(s)) = 0 \qquad s \in \mathbb{R}$$

we assume

$$\textbf{(S)} \quad f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})\,, \quad f(t)\,t > 0 \quad \text{for every } t \in \mathbb{R} \setminus \{0\}$$

- E.B., A. Ferrero, F. Gazzola, P. Karageorgis, J. Diff. Eq. 2011

Global continuation

Theorem

Let $k \in \mathbb{R}$ and assume that f satisfies **(S)**.

(*i*) If a local solution to

$$w''''(s) + kw''(s) + f(w(s)) = 0 \qquad s \in \mathbb{R}$$

blows up at some finite $R \in \mathbb{R}$, then

$$\liminf_{s \to R} w(s) = -\infty \qquad \text{and} \qquad \limsup_{s \to R} w(s) = +\infty\,.$$

(*ii*) If f also satisfies

$$\textbf{(L)} \quad \limsup_{t \to +\infty} \frac{f(t)}{t} < +\infty \qquad \text{or} \qquad \limsup_{t \to -\infty} \frac{f(t)}{t} < +\infty.$$

Then, **any local solution exists for all** $s \in \mathbb{R}$.

Sketch of the proof of (i)

Key Lemma

Assume that f satisfies **(S)** and let w be a solution in a maximal interval of continuation $(0, R)$. The following implications hold

$$\mathbf{1)} \quad \exists C \in \mathbb{R}\,, \quad w(s) \leq C \quad \forall s \in (0, R) \implies R = +\infty,$$

$$\mathbf{2)} \quad \exists C \in \mathbb{R}\,, \quad w(s) \geq C \quad \forall s \in (0, R) \implies R = +\infty.$$

Plan of the proof of 1). By contradiction, let $R < +\infty$:

- $\exists C \in \mathbb{R}\,, \quad w(s) \leq C \quad \forall s \in (0, R) \quad \implies \quad w$ **bounded** in $(0, R)$;
- w **bounded** in $(0, R) \quad \implies \quad w''$ **bounded** in $(0, R)$;

$v := w'' + kw$, $v'' = -f(w) \implies v$ bdd in $(0, R) \implies w'' = v - kw$ bdd

- w, w'' **bounded** in $(0, R) \implies w'''' = -kw'' - f(w)$ **bounded** $(0, R)$;
- All the derivatives of w bounded $\implies w$ can be continued beyond R.

Sketch of the proof of (i)

$$\exists C \in \mathbb{R}\,, \quad w(s) \le C \quad \forall s \in (0,R) \quad \Longrightarrow \quad w \text{ **bounded** in } (0,R).$$

Set $v(s) := w''(s) + kw(s)$, from the equation

$$v''(s) = w''''(s) + kw''(s) = -f(w(s)) \ge C_1 \qquad \forall s \in (0,R).$$

Integrating twice, $v(s) \ge C_2$ in $(0,R)$.

If $\mathbf{k} \ge 0$, this gives

$$w''(s) = v(s) - kw(s) \ge C_2 - kC \quad \Longrightarrow \quad w \text{ bounded from below.}$$

If $\mathbf{k} < 0$, we exploit the fact that

$$w(s) = w(0)\mathrm{Ch}(\sqrt{|k|}s) + \frac{w'(0)}{\sqrt{|k|}}\,\mathrm{Sh}(\sqrt{|k|}s) + \frac{1}{\sqrt{|k|}}\int_0^s \mathrm{Sh}[\sqrt{|k|}(s-t)]v(t)\,dt.$$

A connected biharmonic Gelfand-type equation

Corollary

Let B_R be the ball in $\mathbb{R}^n (n \geq 2)$ with radius $0 < R < +\infty$ and center the origin. Then, any radial solution to

$$\Delta^2 u - 2(n-4)\frac{x \cdot \nabla \Delta u}{|x|^2} + (n^2 - 6n + 12 + k)\frac{\Delta u}{|x|^2}$$

$$-(n-2)\left[(n-2)^2 + k\right]\frac{x \cdot \nabla u}{|x|^4} + e^u = \frac{1}{|x|^4}$$

in $B_R \setminus \{0\}$ **admits a radial extension** to $\mathbb{R}^n \setminus \{0\}$.
In particular, the equation subject to the boundary condition

$$\lim_{|x| \to R} u(x) = \infty \,,$$

admits **no** radial solution.

Explicit solution $u(x) = -4 \log |x| \iff w \equiv 0$

Qualitative behavior when $k \geq 0$

$$w''''(s) + kw''(s) + f(w(s)) = 0 \qquad s \in \mathbb{R}$$

Theorem

Let $k \geq 0$ and f satisfy **(S)**. If w is a global solution, then

$$\liminf_{s\to+\infty} w(s) \leq 0 \leq \limsup_{s\to+\infty} w(s) \ ,$$

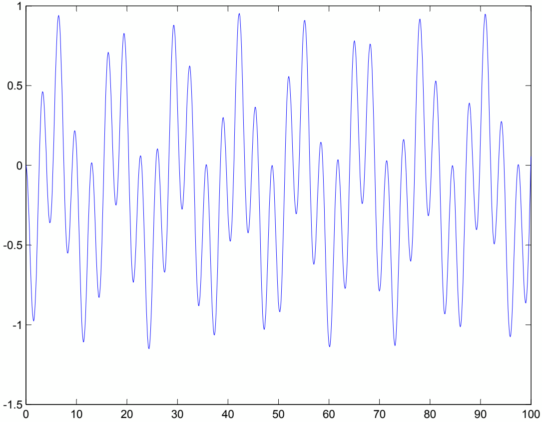
so that if $\lim_{s\to+\infty} w(s)$ exists then

$$\lim_{s\to+\infty} w(s) = 0 \ .$$

Furthermore, if $w \not\equiv 0$ then $w(s)$ **changes sign infinitely many times** as $s \to +\infty$. Similar statements hold for $s \to -\infty$.

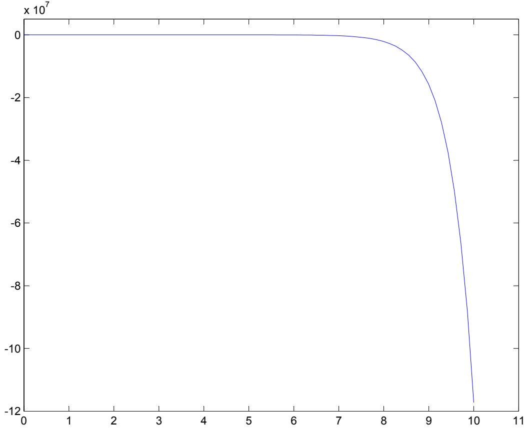
$$w''''(s) + 4w''(s) + e^{w(s)} - 1 = 0 \qquad s \in \mathbb{R}$$

$$w(0) = 0 \qquad w'(0) = 0 \qquad w''(0) = -2 \qquad w'''(0) = 0$$



$$w''''(s) - 4w''(s) + e^{w(s)} - 1 = 0 \qquad s \in \mathbb{R}$$

$$w(0) = 0 \qquad w'(0) = 0 \qquad w''(0) = -2 \qquad w'''(0) = 0$$



Qualitative behavior when $k < 0$

$$w''''(s) + kw''(s) + f(w(s)) = 0 \qquad s \in \mathbb{R}$$

Theorem

Let $k < 0$ and assume that f satisfies **(S)** and

$$\inf_{t \in \mathbb{R}} f(t) = -M > -\infty, \tag{1}$$

then there exists a global solution w which is **eventually negative**, decreasing, and concave as $s \to +\infty$ and

$$\lim_{s \to +\infty} w(s) = -\infty \ .$$

Similar statement if

$$\sup_{t \in \mathbb{R}} f(t) = M < +\infty.$$

Sketch of the proof

Let $k < 0$ and $\inf_{t\in\mathbb{R}} \ f(t) = -M > -\infty$. Consider a solution to

$$w''''(s) + kw''(s) + f(w(s)) = 0 \qquad s \in \mathbb{R}$$

satisfying

$$w(0) = 0, \qquad w'(0) = 0, \qquad w''(0) < -\frac{M}{|k|} < 0, \qquad w'''(0) = 0\,.$$

$$\Longrightarrow \quad w''''(0) = -kw''(0) < -M\,.$$

Set

$$\overline{s} := \sup\{s > 0 : w''''(\sigma) < 0 \ \text{ for all } \ \sigma \in (0, s)\} \in (0, +\infty]\,.$$

Claim: $\overline{s} = +\infty$.

Sketch of the proof

$$\overline{s} := \sup\{s > 0 : w''''(\sigma) < 0 \ \text{ for all } \ \sigma \in (0, s)\} \in (0, +\infty]\,.$$

$$\text{Were } \overline{s} < +\infty \quad \Longrightarrow \quad w''''(\overline{s}) = 0\,.$$

$\Longrightarrow w'''(s) \searrow$ in $(0, \overline{s}]$, $w'''(0) = 0 \Longrightarrow w'''(s) < 0$ in $(0, \overline{s}]$

$\Longrightarrow w''(s) \searrow$ in $(0, \overline{s}]$, $w''(0) < 0 \Longrightarrow w''(s) < 0$ in $(0, \overline{s}]$

$\Longrightarrow w'(s) \searrow$ in $(0, \overline{s}]$, $w'(0) = 0 \Longrightarrow w'(s) < 0$ in $(0, \overline{s}]$

$\Longrightarrow w(s) \searrow$ in $(0, \overline{s}]$, $w(0) = 0 \Longrightarrow w(s) < 0$ in $(0, \overline{s}]$

$$w''''(\overline{s}) = |k|w''(\overline{s}) - f(w(\overline{s})) \leq |k|w''(0) + M < 0\,,$$

a contradiction $\Longrightarrow \overline{s} = +\infty$.

Qualitative behavior when $k < 0$

$$w''''(s) + kw''(s) + f(w(s)) = 0 \qquad s \in \mathbb{R}$$

Theorem

Let $k < 0$ and f satisfy **(S)**, $\sup_{t\in\mathbb{R}} f(t) = M < +\infty$ and

$$\lim_{t\to-\infty} \frac{f(t)}{t} = +\infty\,.$$

Then any solution w is global and

$$\sup_{s\in\mathbb{R}} w(s) = +\infty \quad \text{and} \quad \inf_{s\in\mathbb{R}} w(s) > -\infty\,.$$

Energy functions

$F(w) := \int_0^w f(\tau)\, d\tau$,

$$E(s) := \frac{1}{2}\, w''(s)^2 - \frac{k}{2}\, w'(s)^2 - F(w(s)) \qquad s \in \mathbb{R}\,.$$

If w is a solution and $w'(s_1) = w'(s_2) = 0 \Longrightarrow E(s_1) = E(s_2)$.

$$\mathcal{E}(s) := E(s) - w'(s)w'''(s) \qquad s \in \mathbb{R}\,.$$

If w is a solution $\Longrightarrow \mathcal{E}'(s) = 0 \Longrightarrow \mathcal{E}(s) = C$, for some $C \in \mathbb{R}$.

Energy functions

$$H(s) := w'(s)w''(s) - w(s)w'''(s) - kw(s)w'(s)$$

and its antiderivative

$$G(s) := w'(s)^2 - w(s)w''(s) - \frac{k}{2}w(s)^2 \quad s \in \mathbb{R}\,.$$

w solution, $k \leq 0 \Longrightarrow H'(s) = w''(s)^2 - kw'(s)^2 + w(s)f(w(s)) \geq 0$

$\Longrightarrow$ H is nondecreasing and G is convex.

An application

Proposition

Let $k \leq 0$ and f satisfy **(S)**. Then, the equation

$$w''''(s) + kw''(s) + f(w(s)) = 0 \qquad s \in \mathbb{R}$$

has no nontrivial bounded solutions. In particular, the equation has **no nontrivial homoclinic solutions**.

- L.A. Peletier, W.C. Troy, Birkhäuser Boston Inc. (2001)

A simpler proof

w homoclinic $\Leftrightarrow \lim_{s\to\pm\infty} w(s) = 0 \Rightarrow \lim_{s\to\pm\infty} w^i(s) = 0$ for $i = 1, .., 4$

Recall the energy function

$$H(s) := w'(s)w''(s) - w(s)w'''(s) - kw(s)w'(s)$$

$H'(s) \geq 0$ for $k \leq 0 \Longrightarrow \lim_{s\to\pm\infty} H(s) = 0 \Longrightarrow H = H' \equiv 0 \Longrightarrow w \equiv 0$

A connected biharmonic Gelfand-type equation

Corollary

Let $k \leq 0$ and u be a radial solution to

$$\Delta^2 u - 2(n-4)\frac{x\cdot\nabla\Delta u}{|x|^2} + (n^2 - 6n + 12 + k)\frac{\Delta u}{|x|^2}$$

$$-(n-2)\left[(n-2)^2 + k\right]\frac{x\cdot\nabla u}{|x|^4} + e^u = \frac{1}{|x|^4}$$

in $\mathbb{R}^n \setminus \{0\}$ $(n \geq 2)$ (in $\mathbb{R}^4 \setminus \{0\}$). If

$$\lim_{|x|\to 0}(u(x) + 4\log|x|) = 0 = \lim_{|x|\to+\infty}(u(x) + 4\log|x|)\,,$$

then $u(x) \equiv -4\log|x|$.

Dynamical system

Put

$$Y(s) = (y_1(s), y_2(s), y_3(s), y_4(s)) = (w(s), w'(s), w''(s), w'''(s))$$

the equation

$$w''''(s) + kw''(s) + f(w(s)) = 0 \qquad s \in \mathbb{R}$$

may be rewritten as

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y_4 \\ y_4' = -ky_3 - f(y_1). \end{cases}$$

(S) $\Rightarrow$ unique stationary point $O = (0, 0, 0, 0)$ corresponding to $w \equiv 0$

Proposition

Assume **(S)** and f differentiable near the origin: $f'(0) = 1$.

(i) If $k < -2$, O has a 2-dimensional **stable** manifold and a 2-dimensional **unstable** manifold, both not oscillating near O;

(ii) if $k = -2$, O has a 2-dimensional **stable** manifold and a 2-dimensional **unstable** manifold;

(iii) if $-2 < k < 2$, O has a 2-dimensional **stable** manifold and a 2-dimensional **unstable** manifold, both having locally the form of a spiral near O;

(iv) if $k = 2$, the linearized problem at O has 2 (opposite) double **purely imaginary** eigenvalues;

(v) if $k > 2$, the linearized problem at O has 4 **purely imaginary** eigenvalues.

Theorem

Assume that f satisfies **(S)**.

(i) If $k \leq -2$ and f satisfies one of the following

$$f(t) \geq t \quad \text{near } t = 0 \quad \text{or} \quad f(t) \leq t \quad \text{near } t = 0,$$

any global solution : $\lim_{s \to +\infty} w(s) = 0$ is of **one sign** as $s \to +\infty$. Similar statement with $+\infty$ replaced by $-\infty$.

(ii) If $-2 < k < 0$, $f'(0) = 1$ and

$$\liminf_{|t| \to +\infty} \frac{f(t)}{t} > k^2 \,,$$

any global solution **changes sign infinitely many times** both as $s \to \pm\infty$.

Sketch of the proof of (i)

$$\lim_{s\to+\infty} w(s) = 0 \quad \Rightarrow \quad \lim_{s\to+\infty} (w(s), w'(s), w''(s), w'''(s)) = (0,0,0,0)$$

$k \leq -2 \Rightarrow$ 4 real eigenvalues $\pm\lambda$, $\pm\mu$: $\lambda \geq \mu > 0$.

$$\Longrightarrow \quad (\partial_s - \lambda)(\partial_s - \mu)(\partial_s + \lambda)(\partial_s + \mu)w(s) = w(s) - f(w(s))$$

$f(t) \geq t$ near $t = 0 \implies \quad \left(e^{-\lambda s}(\partial_s - \mu)(\partial_s + \lambda)(\partial_s + \mu)w(s)\right)' \leq 0$ s large

$$e^{-\lambda s}(\partial_s - \mu)(\partial_s + \lambda)(\partial_s + \mu)w(s) \to 0 \text{ as } s \to +\infty$$

$\Longrightarrow e^{-\mu s}\left((\partial_s + \lambda)(\partial_s + \mu)w(s)\right) \nearrow 0$

$$(\partial_s + \lambda)(\partial_s + \mu)w(s) \leq 0 \text{ for } s \text{ large}$$

$\Longrightarrow s \mapsto e^{\lambda s}(w'(s) + \mu w(s))$ decreasing.

Sketch of the proof of (i)

$\Rightarrow s \mapsto e^{\lambda s}(w'(s) + \mu w(s))$ decreasing.

Case 1 $\exists s_0$: $w'(s_0) + \mu w(s_0) < 0$

$\Rightarrow w'(s) + \mu w(s) < 0$ for all $s \geq s_0 \Rightarrow e^{\mu s} w(s) \searrow \ell \in \mathbb{R}$ as $s \to +\infty$

If $\ell > 0$ or $\ell < 0 \Rightarrow w$ is eventually of one sign.

If $\ell = 0 \Rightarrow e^{\mu s} w(s) \searrow 0 \Rightarrow w$ is eventually positive.

Case 2 $w' + \mu w \geq 0$ for every $s \in R$

$\Rightarrow e^{\mu s} w(s) \nearrow \ell$ as $s \to +\infty$.....

Homoclinics

$$w \text{ homoclinic} \quad \Longleftrightarrow \quad \lim_{s\to\pm\infty} w(s) = 0$$

Proposition

Let $k \leq 0$ and f satisfy **(S)**. Then, the equation

$$w''''(s) + kw''(s) + f(w(s)) = 0 \qquad s \in \mathbb{R}$$

has no nontrivial bounded solutions. In particular, the equation has **no nontrivial homoclinic** solutions.

- L.A. Peletier, W.C. Troy, Birkhäuser Boston Inc. (2001)

Homoclinics

Theorem

Let $k > 0$.

(i) $k \leq 2$ and $\frac{f(t)}{t} \geq 1$ for all $t \neq 0$

$$\Longrightarrow \quad \textbf{no homoclinics}$$

(ii) f satisfies **(S)** and $f'(0) = 1$, $\{s_m\}_{m \geq 1}$ increasing sequence of zeroes of an homoclinic (as $s \to +\infty$)

$$\Longrightarrow \quad \liminf_{m \to +\infty} (s_{m+1} - s_m) \geq \frac{\pi \sqrt{k + \sqrt{k^2 + 12}}}{\sqrt{6}}.$$

Similar statement as $s \to -\infty$.

(iii) $k < 2$, f satisfies **(S)** and $f'(0) = 1$, w is a homoclinic

$$\Longrightarrow \quad w \in H^2(\mathbb{R})$$

Sketch of the proof of (i)

w homoclinic, $k > 0 \Rightarrow w$ changes sign infinitely many times.
Let $s_1 < s_2$ any two of its roots

$$2\int_{s_1}^{s_2} w'(s)^2\, ds = -2\int_{s_1}^{s_2} w(s)w''(s)\, ds$$

$$= \int_{s_1}^{s_2} [w(s)^2 + w''(s)^2]\, ds - \int_{s_1}^{s_2} [w(s) + w''(s)]^2\, ds$$

$$\leq \int_{s_1}^{s_2} [w(s)^2 + w''(s)^2]\, ds\, .$$

$\Longrightarrow$ there exist two roots $\overline{s}_1 < \overline{s}_2$ such that $<$ holds (otherwise $w''(s) + w(s) = 0$ for all $s \in \mathbb{R}$, contradicting w homoclinic).

If $s_1 \leq \overline{s}_1 < \overline{s}_2 \leq s_2 \Longrightarrow \int_{s_1}^{s_2} [w(s)^2 + w''(s)^2]\, ds > 2\int_{s_1}^{s_2} w'(s)^2\, ds$.

Sketch of the proof of (i)

$$H(s) := w'(s)w''(s) - w(s)w'''(s) - kw(s)w'(s)$$

Since $\frac{f(t)}{t} \geq 1$

$$H(s_2) - H(s_1) = \int_{s_1}^{s_2} [w''(s)^2 - kw'(s)^2 + w(s)f(w(s))]\, ds$$

$$\geq \int_{s_1}^{s_2} [w''(s)^2 - kw'(s)^2 + w(s)^2]\, ds > (2-k) \int_{s_1}^{s_2} w'(s)^2\, ds \geq 0\,,$$

whenever $s_1 \leq \overline{s}_1 < \overline{s}_2 \leq s_2$

w homoclinic $\Longrightarrow H(s) \to 0$ as $s \to \pm\infty$, a contradiction.