Qualitative behavior of global solutions to some nonlinear fourth order differential equations

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Madrid, September 19-20, 2011
Consider the equation

$$w'''(s) + kw''(s) + f(w(s)) = 0 \quad s \in \mathbb{R}$$

where $k \in \mathbb{R}$ and $f$ is a locally Lipschitz function

$\Rightarrow k > 0$ Swift-Hohenberg equation

$\Rightarrow k \leq 0$ extended Fisher-Kolmogorov equation

A connected biharmonic Gelfand-type equation

Let $u = u(r)$, $r := |x|$ be a radial solution to

$$
\Delta^2 u + e^u = \frac{1}{|x|^4} \quad \text{in } \mathbb{R}^4 \setminus \{0\}
$$

and with the change of variables

$$
s = \log r \quad w(s) := u(e^s) + 4s \quad s \in \mathbb{R}
$$

we get

$$
w'''(s) - 4w''(s) + e^{w(s)} - 1 = 0 \quad s \in \mathbb{R}.
$$

- C.S. Lin, (1998); S.Y.A. Chang, W. Chen (2001);
- J.Dávila, L.Dupaigne, I.Guerra, M.Montenegro, (2007);
A connected biharmonic Gelfand-type equation

Let $k \in \mathbb{R}$ and $u = u(r), r := |x|$ be a radial solution to

$$\Delta^2 u - 2(n-4)\frac{x \cdot \nabla \Delta u}{|x|^2} + (n^2 - 6n + 12 + k)\frac{\Delta u}{|x|^2}$$

$$- (n - 2) [(n - 2)^2 + k] \frac{x \cdot \nabla u}{|x|^4} + e^u = \frac{1}{|x|^4}$$

in $\mathbb{R}^n \setminus \{0\}$ $(n \geq 2)$, with the change of variables

$$s = \log r \quad w(s) := u(e^s) + 4s \quad s \in \mathbb{R}$$

$w$ solves

$$w''''(s) + kw''(s) + e^{w(s)} - 1 = 0 \quad s \in \mathbb{R}.$$
Physical models

\[ w''''(s) + kw''(s) + f(w(s)) = 0 \quad (s, k \in \mathbb{R}) \]

⇒ Suspension bridge model

\[ f(w) = (w + 1)^+ - 1; \quad f(w) = e^w - 1 \]

- P.J. McKenna, W. Walter (1990); Y. Chen, P.J. McKenna (1997)

⇒ Configuration of a nonlinearly supported elastic strut

\[ f(w) = w - w^2; \quad f(w) = w - w^3 + w^5 \]

- G.W. Hunt, H.M. Bolt, J.M.T. Thompson (1989);

⇒ Pattern formation in physical, chemical or biological systems

\[ f(w) = w^3 - w \]

Global continuation
Qualitative behavior
Homoclinics

Main assumption on \( f \)

\[
\begin{align*}
  w''''(s) + kw''(s) + f(w(s)) &= 0 & s \in \mathbb{R}
\end{align*}
\]

we assume

\((S)\) \quad f \in \text{Lip}_{\text{loc}}(\mathbb{R}), \quad f(t) \ t > 0 \quad \text{for every } t \in \mathbb{R} \setminus \{0\}

Theorem

Let \( k \in \mathbb{R} \) and assume that \( f \) satisfies (S).

(i) If a local solution to

\[
\frac{d^4 w}{ds^4}(s) + kw''(s) + f(w(s)) = 0 \quad s \in \mathbb{R}
\]

blows up at some finite \( R \in \mathbb{R} \), then

\[
\liminf_{s \to R} w(s) = -\infty \quad \text{and} \quad \limsup_{s \to R} w(s) = +\infty.
\]

(ii) If \( f \) also satisfies

\[
\limsup_{t \to +\infty} \frac{f(t)}{t} < +\infty \quad \text{or} \quad \limsup_{t \to -\infty} \frac{f(t)}{t} < +\infty.
\]

Then, any local solution exists for all \( s \in \mathbb{R} \).
Sketch of the proof of (i)

Key Lemma

Assume that $f$ satisfies (S) and let $w$ be a solution in a maximal interval of continuation $(0, R)$. The following implications hold

1) $\exists C \in \mathbb{R}, \ w(s) \leq C \ \forall s \in (0, R) \implies R = +\infty,$

2) $\exists C \in \mathbb{R}, \ w(s) \geq C \ \forall s \in (0, R) \implies R = +\infty.$

Plan of the proof of 1). By contradiction, let $R < +\infty$:

- $\exists C \in \mathbb{R}, \ w(s) \leq C \ \forall s \in (0, R) \implies w$ bounded in $(0, R)$;
- $w$ bounded in $(0, R) \implies w''$ bounded in $(0, R)$;
- $v := w'' + kw, \ v'' = -f(w) \implies v$ bdd in $(0, R) \implies w'' = v - kw$ bdd;
- $w, w''$ bounded in $(0, R) \implies w''' = -kw'' - f(w)$ bounded $(0, R)$;
- All the derivatives of $w$ bounded $\implies w$ can be continued beyond $R$. 

Global continuation
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Sketch of the proof of (i)

\[ \exists C \in \mathbb{R}, \quad w(s) \leq C \quad \forall s \in (0, R) \quad \implies \quad w \text{ bounded in } (0, R). \]

Set \( v(s) := w''(s) + kw(s) \), from the equation

\[ v''(s) = w''''(s) + kw''(s) = -f(w(s)) \geq C_1 \quad \forall s \in (0, R). \]

Integrating twice, \( v(s) \geq C_2 \) in \( (0, R) \).

If \( k \geq 0 \), this gives

\[ w''(s) = v(s) - kw(s) \geq C_2 - kC \quad \implies \quad w \text{ bounded from below}. \]

If \( k < 0 \), we exploit the fact that

\[ w(s) = w(0) \text{Ch}(\sqrt{|k|s}) + \frac{w'(0)}{\sqrt{|k|}} \text{Sh}(\sqrt{|k|s}) + \frac{1}{\sqrt{|k|}} \int_0^s \text{Sh}[\sqrt{|k|}(s-t)]v(t) \, dt. \]
Corollary

Let $B_R$ be the ball in $\mathbb{R}^n (n \geq 2)$ with radius $0 < R < +\infty$ and center the origin. Then, any radial solution to

$$
\Delta^2 u - 2(n-4) \frac{x \cdot \nabla \Delta u}{|x|^2} + (n^2 - 6n + 12 + k) \frac{\Delta u}{|x|^2}
$$

$$
-(n-2) \left[ (n-2)^2 + k \right] \frac{x \cdot \nabla u}{|x|^4} + e^u = \frac{1}{|x|^4}
$$

in $B_R \setminus \{0\}$ admits a radial extension to $\mathbb{R}^n \setminus \{0\}$. In particular, the equation subject to the boundary condition

$$
\lim_{|x| \to R} u(x) = \infty,
$$

admits no radial solution.

Explicit solution $u(x) = -4 \log |x| \iff w \equiv 0$
Qualitative behavior when $k \geq 0$

$$w''''(s) + kw''(s) + f(w(s)) = 0 \quad s \in \mathbb{R}$$

**Theorem**

Let $k \geq 0$ and $f$ satisfy (S). If $w$ is a global solution, then

$$\liminf_{s \to +\infty} w(s) \leq 0 \leq \limsup_{s \to +\infty} w(s),$$

so that if $\lim_{s \to +\infty} w(s)$ exists then

$$\lim_{s \to +\infty} w(s) = 0.$$

Furthermore, if $w \neq 0$ then $w(s)$ changes sign infinitely many times as $s \to +\infty$. Similar statements hold for $s \to -\infty$. 
Global solutions to some nonlinear fourth order differential equations

\[ w'''(s) + 4w''(s) + e^{w(s)} - 1 = 0 \quad s \in \mathbb{R} \]

\[ w(0) = 0 \quad w'(0) = 0 \quad w''(0) = -2 \quad w'''(0) = 0 \]
\[ w^{(\prime\prime\prime)}(s) - 4w^{(\prime\prime)}(s) + e^{w(s)} - 1 = 0 \quad s \in \mathbb{R} \]

\[ w(0) = 0 \quad w'(0) = 0 \quad w''(0) = -2 \quad w^{(\prime\prime\prime)}(0) = 0 \]
Qualitative behavior when $k < 0$

$$w'''(s) + kw''(s) + f(w(s)) = 0 \quad s \in \mathbb{R}$$

**Theorem**

Let $k < 0$ and assume that $f$ satisfies (S) and

$$\inf_{t \in \mathbb{R}} f(t) = -M > -\infty,$$  \hspace{1cm} (1)

then there exists a global solution $w$ which is **eventually negative**, decreasing, and concave as $s \to +\infty$ and

$$\lim_{s \to +\infty} w(s) = -\infty.$$  

Similar statement if

$$\sup_{t \in \mathbb{R}} f(t) = M < +\infty.$$
Let $k < 0$ and $\inf_{t \in \mathbb{R}} f(t) = -M > -\infty$. Consider a solution to

$$w''''(s) + kw''(s) + f(w(s)) = 0 \quad s \in \mathbb{R}$$

satisfying

$$w(0) = 0, \quad w'(0) = 0, \quad w''(0) < -\frac{M}{|k|} < 0, \quad w'''(0) = 0.$$ 

$$\implies w''''(0) = -kw''(0) < -M.$$ 

Set

$$\bar{s} := \sup \{s > 0 : w''''(\sigma) < 0 \text{ for all } \sigma \in (0, s) \} \in (0, +\infty].$$

Claim: $\bar{s} = +\infty$. 
Sketch of the proof

\[ \bar{s} := \sup \{ s > 0 : w'''(\sigma) < 0 \text{ for all } \sigma \in (0, s) \} \in (0, +\infty] . \]

Were \( \bar{s} < +\infty \) \( \implies \) \( w'''(\bar{s}) = 0 \).

\( \implies \) \( w'''(s) \downarrow \text{ in } (0, \bar{s}], \; w'''(0) = 0 \implies w'''(s) < 0 \text{ in } (0, \bar{s}] \)

\( \implies \) \( w''(s) \downarrow \text{ in } (0, \bar{s}], \; w''(0) < 0 \implies w''(s) < 0 \text{ in } (0, \bar{s}] \)

\( \implies \) \( w'(s) \downarrow \text{ in } (0, \bar{s}], \; w'(0) = 0 \implies w'(s) < 0 \text{ in } (0, \bar{s}] \)

\( \implies \) \( w(s) \downarrow \text{ in } (0, \bar{s}], \; w(0) = 0 \implies w(s) < 0 \text{ in } (0, \bar{s}] \)

\[ w'''(\bar{s}) = |k| w''(\bar{s}) - f(w(\bar{s})) \leq |k| w''(0) + M < 0 , \]

a contradiction \( \implies \) \( \bar{s} = +\infty \).
Qualitative behavior when \( k < 0 \)

\[
w''''(s) + kw''(s) + f(w(s)) = 0 \quad s \in \mathbb{R}
\]

**Theorem**

Let \( k < 0 \) and \( f \) satisfy (S), \( \sup_{t \in \mathbb{R}} f(t) = M < +\infty \) and 

\[
\lim_{t \to -\infty} \frac{f(t)}{t} = +\infty.
\]

Then any solution \( w \) is global and 

\[
\sup_{s \in \mathbb{R}} w(s) = +\infty \quad \text{and} \quad \inf_{s \in \mathbb{R}} w(s) > -\infty.
\]
Energy functions

\[ F(w) := \int_0^w f(\tau) \, d\tau, \]

\[ E(s) := \frac{1}{2} w''(s)^2 - \frac{k}{2} w'(s)^2 - F(w(s)) \quad s \in \mathbb{R}. \]

If \( w \) is a solution and \( w'(s_1) = w'(s_2) = 0 \implies E(s_1) = E(s_2). \)

\[ E(s) := E(s) - w'(s)w'''(s) \quad s \in \mathbb{R}. \]

If \( w \) is a solution \( \implies \varepsilon'(s) = 0 \implies \varepsilon(s) = C \), for some \( C \in \mathbb{R}. \)
Energy functions

\[ H(s) := w'(s)w''(s) - w(s)w'''(s) - kw(s)w'(s) \]

and its antiderivative

\[ G(s) := w'(s)^2 - w(s)w''(s) - \frac{k}{2}w(s)^2 \quad s \in \mathbb{R}. \]

w solution, \( k \leq 0 \implies H'(s) = w''(s)^2 - kw'(s)^2 + w(s)f(w(s)) \geq 0 \)

\[ \implies H \text{ is nondecreasing and } G \text{ is convex.} \]
An application

**Proposition**

Let $k \leq 0$ and $f$ satisfy (S). Then, the equation

$$w''''(s) + kw''(s) + f(w(s)) = 0 \quad s \in \mathbb{R}$$

has no nontrivial bounded solutions. In particular, the equation has no nontrivial homoclinic solutions.


**A simpler proof**

$w$ homoclinic $\iff \lim_{s \to \pm \infty} w(s) = 0 \Rightarrow \lim_{s \to \pm \infty} w^i(s) = 0$ for $i = 1, .., 4$

Recall the energy function

$$H(s) := w'(s)w''(s) - w(s)w'''(s) - kw(s)w'(s)$$

$H'(s) \geq 0$ for $k \leq 0 \Rightarrow \lim_{s \to \pm \infty} H(s) = 0 \Rightarrow H = H' \equiv 0 \Rightarrow w \equiv 0$
A connected biharmonic Gelfand-type equation

Corollary

Let $k \leq 0$ and $u$ be a radial solution to

$$\Delta^2 u - 2(n - 4)\frac{x \cdot \nabla \Delta u}{|x|^2} + (n^2 - 6n + 12 + k)\frac{\Delta u}{|x|^2}$$

$$-(n - 2) [(n - 2)^2 + k] \frac{x \cdot \nabla u}{|x|^4} + e^u = \frac{1}{|x|^4}$$

in $\mathbb{R}^n \setminus \{0\}$ ($n \geq 2$) (in $\mathbb{R}^4 \setminus \{0\}$). If

$$\lim_{|x| \to 0} (u(x) + 4 \log |x|) = 0 = \lim_{|x| \to +\infty} (u(x) + 4 \log |x|),$$

then $u(x) \equiv -4 \log |x|.$
Dynamical system

Put

\[ Y(s) = (y_1(s), y_2(s), y_3(s), y_4(s)) = (w(s), w'(s), w''(s), w'''(s)) \]

the equation

\[ w'''(s) + kw''(s) + f(w(s)) = 0 \quad s \in \mathbb{R} \]

may be rewritten as

\[
\begin{aligned}
    y'_1 &= y_2 \\
    y'_2 &= y_3 \\
    y'_3 &= y_4 \\
    y'_4 &= -ky_3 - f(y_1).
\end{aligned}
\]

\((S) \Rightarrow \text{unique stationary point } O = (0, 0, 0, 0) \text{ corresponding to } w \equiv 0\)
Proposition

Assume \( (S) \) and \( f \) differentiable near the origin: \( f'(0) = 1 \).

(i) If \( k < -2 \), \( O \) has a 2-dimensional **stable** manifold and a 2-dimensional **unstable** manifold, both not oscillating near \( O \);

(ii) if \( k = -2 \), \( O \) has a 2-dimensional **stable** manifold and a 2-dimensional **unstable** manifold;

(iii) if \( -2 < k < 2 \), \( O \) has a 2-dimensional **stable** manifold and a 2-dimensional **unstable** manifold, both having locally the form of a spiral near \( O \);

(iv) if \( k = 2 \), the linearized problem at \( O \) has 2 (opposite) double **purely imaginary** eigenvalues;

(v) if \( k > 2 \), the linearized problem at \( O \) has 4 **purely imaginary** eigenvalues.
Theorem

Assume that $f$ satisfies (S).

(i) If $k \leq -2$ and $f$ satisfies one of the following

$$f(t) \geq t \text{ near } t = 0 \quad \text{or} \quad f(t) \leq t \text{ near } t = 0,$$

any global solution: $\lim_{s \to +\infty} w(s) = 0$ is of one sign as $s \to +\infty$.

Similar statement with $+\infty$ replaced by $-\infty$.

(ii) If $-2 < k < 0$, $f'(0) = 1$ and

$$\liminf_{|t| \to +\infty} \frac{f(t)}{t} > k^2,$$

any global solution changes sign infinitely many times both as $s \to \pm\infty$. 

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Global solutions to some nonlinear fourth order differential equations
Sketch of the proof of (i)

\[
\lim_{s \to +\infty} w(s) = 0 \quad \Rightarrow \quad \lim_{s \to +\infty} (w(s), w'(s), w''(s), w'''(s)) = (0, 0, 0, 0)
\]

\[k \leq -2 \Rightarrow 4 \text{ real eigenvalues } \pm \lambda, \pm \mu: \lambda \geq \mu > 0.\]

\[
\implies (\partial_s - \lambda)(\partial_s - \mu)(\partial_s + \lambda)(\partial_s + \mu)w(s) = w(s) - f(w(s))
\]

\[f(t) \geq t \text{ near } t = 0 \quad \Rightarrow \quad (e^{-\lambda s}(\partial_s - \mu)(\partial_s + \lambda)(\partial_s + \mu)w(s))' \leq 0 \text{ for } s \text{ large}\]

\[e^{-\lambda s}(\partial_s - \mu)(\partial_s + \lambda)(\partial_s + \mu)w(s) \to 0 \text{ as } s \to +\infty\]

\[\implies e^{-\mu s} ((\partial_s + \lambda)(\partial_s + \mu)w(s)) \nearrow 0\]

\[(\partial_s + \lambda)(\partial_s + \mu)w(s) \leq 0 \text{ for } s \text{ large}\]

\[\Rightarrow s \mapsto e^{\lambda s}(w'(s) + \mu w(s)) \text{ decreasing.}\]
Sketch of the proof of \((i)\)

\[ s \mapsto e^{\lambda s}(w'(s) + \mu w(s)) \text{ decreasing.} \]

**Case 1** \( \exists s_0: w'(s_0) + \mu w(s_0) < 0 \)

\[ w'(s) + \mu w(s) < 0 \text{ for all } s \geq s_0 \Rightarrow e^{\mu s}w(s) \downarrow \ell \in \mathbb{R} \text{ as } s \to +\infty \]

If \( \ell > 0 \) or \( \ell < 0 \) \( \Rightarrow \) \( w \) is eventually of one sign.

If \( \ell = 0 \) \( \Rightarrow \) \( e^{\mu s}w(s) \downarrow 0 \Rightarrow w \) is eventually positive.

**Case 2** \( w' + \mu w \geq 0 \text{ for every } s \in R \)

\[ e^{\mu s}w(s) \uparrow \ell \text{ as } s \to +\infty \ldots. \]
Let $k \leq 0$ and $f$ satisfy (S). Then, the equation

$$w'''''(s) + kw''(s) + f(w(s)) = 0 \quad s \in \mathbb{R}$$

has no nontrivial bounded solutions. In particular, the equation has no nontrivial homoclinic solutions.

Theorem

Let $k > 0$.

(i) $k \leq 2$ and $\frac{f(t)}{t} \geq 1$ for all $t \neq 0$

$\implies$ no homoclinics

(ii) $f$ satisfies $(S)$ and $f'(0) = 1$, $\{s_m\}_{m \geq 1}$ increasing sequence of zeroes of an homoclinic (as $s \to +\infty$)

$\implies \liminf_{m \to +\infty} (s_{m+1} - s_m) \geq \frac{\pi \sqrt{k + \sqrt{k^2 + 12}}}{\sqrt{6}}$.

Similar statement as $s \to -\infty$.

(iii) $k < 2$, $f$ satisfies $(S)$ and $f'(0) = 1$, $w$ is a homoclinic

$\implies w \in H^2(\mathbb{R})$
Sketch of the proof of (i)

\( w \) homoclinic, \( k > 0 \) \( \Rightarrow \) \( w \) changes sign infinitely many times.

Let \( s_1 < s_2 \) any two of its roots

\[
2 \int_{s_1}^{s_2} w'(s)^2 \, ds = -2 \int_{s_1}^{s_2} w(s)w''(s) \, ds
\]

\[
= \int_{s_1}^{s_2} [w(s)^2 + w''(s)^2] \, ds - \int_{s_1}^{s_2} [w(s) + w''(s)]^2 \, ds
\]

\[
\leq \int_{s_1}^{s_2} [w(s)^2 + w''(s)^2] \, ds.
\]

\( \Rightarrow \) there exist two roots \( \bar{s}_1 < \bar{s}_2 \) such that \( < \) holds (otherwise \( w''(s) + w(s) = 0 \) for all \( s \in \mathbb{R} \), contradicting \( w \) homoclinic).

If \( s_1 \leq \bar{s}_1 < \bar{s}_2 \leq s_2 \) \( \Rightarrow \) \( \int_{s_1}^{s_2} [w(s)^2 + w''(s)^2] \, ds > 2 \int_{s_1}^{s_2} w'(s)^2 \, ds. \)
Sketch of the proof of (i)

\[ H(s) := w'(s)w''(s) - w(s)w''')(s) - kw(s)w'(s) \]

Since \( \frac{f(t)}{t} \geq 1 \)

\[ H(s_2) - H(s_1) = \int_{s_1}^{s_2} [w''(s)^2 - kw'(s)^2 + w(s)f(w(s))] \, ds \]

\[ \geq \int_{s_1}^{s_2} [w''(s)^2 - kw'(s)^2 + w(s)^2] \, ds > (2 - k) \int_{s_1}^{s_2} w'(s)^2 \, ds \geq 0, \]

whenever \( s_1 \leq \overline{s}_1 < \overline{s}_2 \leq s_2 \)

\( w \) homoclinic \( \Rightarrow H(s) \to 0 \) as \( s \to \pm \infty \), a contradiction.