Existence Results For p-Superlinear Neumann Problems With A Nonhomogeneous Differential Operator

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$$-\,div(a(z, Du(z)) = f(z, u(z)) \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\ . \qquad (1)$$

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$a:\overline{\Omega}\times\mathbb{R}^N\to\mathbb{R}^N$.
The reaction term $f(z,x)$ is a Carathéodory function that exhibits a $(p-1)$-superlinear growth with respect to $x\in\mathbb{R}$ near to $\pm\infty$

$$\lim_{|x|\to\infty}\frac{f(z,x)}{|x|^{p-2}x}=+\infty \text{ uniformly for a.a. } z\in\Omega.$$

Superlinear Problems with Dirichlet boundary conditions

[4] T. Bartsch-Z. Liu-T. Weth: *Nodals solutions of a p-Laplacian equation*, Proc. London Math. Soc. **91** (2005), 129-152.

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[6] J. P. Garcia Azorero-J. Manfredi-I. Peral Alonso: *Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations*, Comm. Contemp. Math., **2** (2000), 385-404.

$$-\triangle_p u = |u|^{r-2}u + \lambda|u|^{q-2}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega .$$

Superlinear Problems with Dirichlet boundary conditions

[8] Z. Guo-Z. Zhang: *$W^{1,p}$ versus C^1 local minimizers and multiplicity results for quasilinear elliptic equations*, J. Math. Anal. Appl. **286** (2003), 32-50.

$$-\triangle_p u = \lambda u^q + u^\omega,\ u > 0 \text{ in } \ \Omega, \quad u = 0 \text{ on } \ \partial\Omega \ .$$

The authors show the existence of at least two positive solutions for $\lambda \in]0, \Lambda[$ and of at least one positive solution for $\lambda = \Lambda$.

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$$-\triangle_p u = \lambda |u|^{q-1} u + g(u), \; \text{in } \; \Omega, \quad u = 0 \text{ on } \; \partial\Omega \; .$$

This problem admits at least two positive solutions for $\lambda \in]0, \Lambda^+[$, two negative solutions for $\lambda \in]0, \Lambda^-[$ and a nodal solution for $\lambda \in]0, min\{\Lambda^-, \Lambda^+\}[$.

Superlinear Problems with Neumann boundary conditions

[2] S. Aizicovici-N. S. Papageorgiou-V. Staicu: *Existence of multiple solutions with precise sign informations for superlinear Neumann problems*, Annali di Mat. Pura ed Applicata **188** (2009), 679-719.

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AR-condition

There exists $\mu > p$ and $M > 0$ such that

$$0 < \mu F(z, x) \leq f(z, x)x \text{ for a.a. } z \in \Omega, \text{ all } |x| \geq M \ . \tag{2}$$

Neumann Problems that exclude a $(p-1)-$superlinear reaction

[7] L. Gasinski-N. S. Papageorgiou: *Nonlinear Analysis*, Chapman Hall/ CRC Press, Boca Raton, FL. (2006).
[9] D. Motreanu- N. S. Papageorgiou: *Multiple solutions for nonlinear Neumann problems driven by a nonhomogeneous differential operator*, Proc. Amer. Math. Soc. **139** (2011), no. 10, 35273535.

$$C_n^1(\overline{\Omega}) = \left\{ u \in C^1(\overline{\Omega}) : \ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\} ,$$

and $W_n^{1,p}(\Omega) = \overline{C_n^1(\overline{\Omega})}^{\|\cdot\|}$ where $\|\cdot\|$ is the usual norm on $W^{1,p}(\Omega)$. $C_n^1(\overline{\Omega})$ is a Banach space with ordered positive cone

$$C_+ = \left\{ u \in C_n^1(\overline{\Omega}) : \ u(z) \geq 0 \text{ for all } z \in \overline{\Omega} \right\} .$$

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$$-\, div(|Du(z)|^{p-2} Du(z)) = \widehat{\lambda} |u(z)|^{p-2} u(z) \text{ in } \Omega, \ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega. \quad (3)$$

A number $\widehat{\lambda} \in \mathbb{R}$ for which problem (3) has a nontrivial solution $\widehat{u}$, is an eigenvalue of $(-\Delta_p, W_n^{1,p}(\Omega))$ and $\widehat{u}$ is a corresponding eigenfunction.
$\widehat{\lambda} \geq 0$, and $\widehat{\lambda}_0 = 0$ is an eigenvalue with corresponding eigenspace $\mathbb{R}$.
By $\widehat{u}_0$ we denote the corresponding L^p-normalized eigenfunction, i.e. $\widehat{u}_0(z) = \frac{1}{|\Omega|_N^{\frac{1}{p}}}$. If $\sigma(p)$ is the set of all eigenvalues of (3), then the increasing sequence $\{\widehat{\lambda}_n\}_{n \geq 0}$ of the "LS-eigenvalues" is contained in $\sigma(p)$.

Assumptions on a

$\underline{H(a)}$: $a(z,y) = h(z, \|y\|)y$ for all $(z,y) \in \overline{\Omega} \times \mathbb{R}^N$ with $h(z,t) > 0$ for all $(z,t) \in \overline{\Omega} \times (0,+\infty)$ and

(i) $a \in C^{0,\alpha}(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^N) \cap C^1(\overline{\Omega} \times \mathbb{R}^N \setminus \{0\}, \mathbb{R}^N)$ with $0 < \alpha < 1$;

(ii) for all $(z,y) \in \overline{\Omega} \times \mathbb{R}^N \setminus \{0\}$, we have $\|D_y a(z,y)\| \leq c_1 \|y\|^{p-2}$ for some $c_1 > 0$, $1 < p < \infty$;

(iii) for all $(z,y) \in \overline{\Omega} \times \mathbb{R}^N \setminus \{0\}$ and all $\xi \in \mathbb{R}^N$, we have

$$(D_y a(z,y)\xi, \xi)_{\mathbb{R}^N} \geq c_0 \|y\|^{p-2} \|\xi\|^2 \text{ for some } c_0 > 0\,;$$

(iv) the $\mathbb{R}$-valued function $G(z,y)$ defined by $D_y G(z,y) = a(z,y)$ and $G(z,0) = 0$ for all $(z,y) \in \overline{\Omega} \times \mathbb{R}^N$, satisfies

$$\beta(z) \leq pG(z,y) - (a(z,y), y)_{\mathbb{R}^N} \text{ for a.a. } z \in \Omega \text{ with } \beta \in L^1(\Omega)\,.$$

The map V

Let $V : W_n^{1,p}(\Omega) \to W_n^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle V(u), y \rangle = \int_\Omega (a(z, Du), Dy)_{\mathbb{R}^N} \, dz \text{ for all } u, y \in W_n^{1,p}(\Omega). \tag{4}$$

If hypotheses $H(a)$ hold, then V defined by (4) is maximal monotone and of type $(S)_+$, that is
for every sequence $\{x_n\}_{n \geq 1} \subseteq W_n^{1,p}(\Omega)$ such that $x_n \rightharpoonup x$ in $W_n^{1,p}(\Omega)$ and $\limsup_{n \to +\infty} \langle V(x_n), x_n - x \rangle \leq 0$, one has $x_n \to x$ in $W_n^{1,p}(\Omega)$.

Examples

In what follows $\theta \in C^1(\overline{\Omega})$ *and* $\theta(z) > 0$ *for all* $z \in \overline{\Omega}$.

$$a(z, y) = \theta(z)\|y\|^{p-2}y \text{ with } 1 < p < \infty\,.$$

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$$a(z,y) = \begin{cases} \theta(z)\left(\|y\|^{p-2}y + \|y\|^{q-2}y\right) & \text{if } \|y\| \le 1 \\ \theta(z)\left(\|y\|^{p-2}y + c\|y\|^{\tau-2}y - (c-1)y\right) & \text{if } \|y\| > 1\,. \end{cases}$$

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$$a(z,y) = \theta(z)\left(\|y\|^{p-2}y + c\frac{\|y\|^{p-2}y}{1+\|y\|^p}\right),$$

$$0 < c < 4p(p-1) \quad \text{if } 1 \le p < 2, \quad 0 < c < \frac{4p}{(p-1)^2} \quad \text{if } \; p \ge 2\,.$$

Assumptions on f

$H(f)$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.a. $z \in \Omega$ $f(z,0) = 0$ and

(i) $|f(z,x)| \leq a(z) + c|x|^{r-1}$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$ with $a \in L^{\infty}(\Omega)_{+}$, $c > 0$, $1 < p < r < p^{*}$;

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(ii) if $F(z,x) = \int_0^x f(z,s)ds$ and $\xi(z,x) = f(z,x)x - pF(z,x)$, then

$$\lim_{|x|\to\infty} \frac{F(z,x)}{|x|^p} = +\infty \text{ uniformly for a.a. } z \in \Omega \tag{5}$$

and there exists $\beta^* \in L^1(\Omega)_+$ such that

$$\xi(z,x) \leq \xi(z,y) + \beta^*(z) \text{ for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq y \text{ or } y \leq x \leq 0; \tag{6}$$

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(iii) there exists $\lambda^* > \frac{c_1}{p-1}\widehat{\lambda}_1$ such that

$$\lambda^* \leq \liminf_{x\to 0} \frac{pF(z,x)}{|x|^p} \text{ uniformly for a.a. } z \in \Omega\,;$$

(iv) there exist functions $w_+, w_- \in C^1(\overline{\Omega})$ such that

$$w_-(z) \leq c_- < 0 < c_+ \leq w_+(z) \text{ for all } z \in \overline{\Omega}\,,$$

$$V(w_-) \leq 0 \leq V(w_+) \text{ in } W_n^{1,p}(\Omega)^*\,,$$

and

$$esssup_\Omega f(\cdot, w_+(\cdot)) \leq \beta_+ < 0 < \beta_- \leq essinf_\Omega f(\cdot, w_-(\cdot))\,;$$

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(v) for every $\rho > 0$ there exists $\theta_\rho > 0$ such that for a.a. $z \in \Omega$, $x \to f(z,x) + \theta_\rho |x|^{p-2}$ is nondecreasing on $[-\rho,\, \rho]$.

Example

The following function satisfies hypotheses $H(f)$:

$$f(x) = \begin{cases} \eta\left(|x|^{p-2}x - 2|x|^{r-2}x\right) & \textit{if } |x| \leq 1 \\ \left(|x|^{p-2}x \ln|x| - \eta|x|^{\tau-2}x\right) & \textit{if } |x| > 1\,. \end{cases}$$

with $\eta > \frac{c_1}{p-1}\widehat{\lambda}_1$, $1 < \tau < p < q < +\infty$. *Note that* $f(\cdot)$ *does not satisfy the AR-condition (2).*

First result

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Theorem 1

If hypotheses $H(a)$ and $H(f)\,(i)$, $(iv)\,,(v)$ hold, then problem (1) has at least two nontrivial, constant sign smooth solutions

$$u_0 \in int\, C_+,\ v_0 \in -int\, C_+ \ \text{and}$$

$$w_-(z) < v_0(z) < 0 < u_0(z) < w_+(z) \text{ for all } z \in \overline{\Omega}\,.$$

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$$w_-(z) < v_0(z) < 0 < u_0(z) < w_+(z) \text{ for all } z \in \overline{\Omega}\,.$$

We introduce the following Carathéodory truncations-perturbations of $f(z,\cdot)$:

$$\widehat{f}_+(z,x) = \begin{cases} 0 & \text{if } x < 0 \\ f(z,x) + x^{p-1} & \text{if } 0 \le x \le w_+(z) \\ f(z,w_+(z)) + w_+(z)^{p-1} & \text{if } w_+(z) < x \end{cases} \quad \text{and}$$

$$\widehat{f}_-(z,x) = \begin{cases} f(z,w_-(z)) + |w_-(z)|^{p-2}w_-(z) & \text{if } x < w_-(z) \\ f(z,x) + |x|^{p-2}x & \text{if } w_-(z) \le x \le 0 \\ 0 & \text{if } 0 < x\,. \end{cases} \tag{7}$$

We set $\widehat{F}_{\pm}(z,x)=\int_0^x \widehat{f}_{\pm}(z,s)ds$ and consider the C^1-functionals $\widehat{\varphi}_{\pm}: W_n^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\widehat{\varphi}_{\pm}(u)=\int_{\Omega}G(z,Du)dz+\frac{1}{p}\|u\|_p^p-\int_{\Omega}\widehat{F}_{\pm}(z,u)dz \quad \text{for all } u\in W_n^{1,p}(\Omega)\,.$$

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- $\widehat{\varphi}_{+}$ is coercive and sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_0 \in W_n^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_{+}(u_0) = \inf\left[\widehat{\varphi}_{+}(u) : \ u \in W_n^{1,p}(\Omega)\right] = \widehat{m}_{+}\,. \tag{8}$$

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$$0 = \widehat{\varphi}'_+(u_0) = V(u_0) + |u_0|^{p-2}u_0 - N_{\widehat{f}_+}(u_0)\,, \tag{9}$$

  where $N_{\widehat{f}_+}(\cdot) = \widehat{f}_+(\cdot, u(\cdot))$ for all $u \in W_n^{1,p}(\Omega)$, with suitable test functions we deduce
- $u_0 \in [0\,, w_+] = \{u \in W_n^{1,p}(\Omega) : \; 0 \le u(z) \le w_+(z) \text{ a.e. in}\, \Omega\}$, that is

$$-\,div\, a(z, Du_0(z)) = f(z, u_0(z)) \text{ a.e. in } \Omega, \; \frac{\partial u_0}{\partial n} = 0 \text{ on } \partial\Omega\,. \tag{10}$$

- $u_0 \in C_+ \setminus \{0\}$.

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- Using $H(f)(v)$ and $H(f)(iv)$, we obtain $0 < u_0 < w_+$.
- Working with $\widehat{\varphi}_-$, we obtain a nontrivial negative solution $v_0 \in -int\, C_+$, with $w_-(z) < v_0(z) < 0$ for all $z \in \overline{\Omega}$.

Mathematical background for the second existence result

Let X be a Banach space and X^* its topological dual. Let $\varphi \in C^1(X)$. We say that φ satisfies the "C-condition at the level $c \in \mathbb{R}$" (the C_c-condition for short), if the following is true:

"Every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that

$$\varphi(x_n) \to c \text{ and } (1 + \|x_n\|)\varphi'(x_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence".

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Theorem (MPT, [3])

If $\varphi \in C^1(X)$ and $r > 0$ satisfies
$\max\{\varphi(x_0), \varphi(x_1)\} \leq \inf[\varphi(x) : \|x - x_0\| = r] = \eta_r, \ \|x_1 - x_0\| > r$
$c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t))$, *with*
$\Gamma = \{\gamma \in C([0,1], X) : \ \gamma(0) = x_0, \ \gamma(1) = x_1\}$, *and*
φ satisfies the C_c-condition,
then $c \geq \eta_r$ and c is a critical value of φ. Moreover, if $c = \eta_r$, then φ has a critical point $x \in X$ such that $\varphi(x) = c$ and $\|x - x_0\| = r$.

If $\varphi : W_n^{1,p}(\Omega) \to \mathbb{R}$ is the energy functional for problem (1) defined by

$$\varphi(u) = \int_\Omega G(z, Du)dz - \int_\Omega F(z, u)dz \quad \text{for all } u \in W_n^{1,p}(\Omega)\,,$$

then from (7) it follows that $\varphi_{|[0,\,w_+]} = \widehat{\varphi}_{+|[0,\,w_+]}$ and $\varphi_{|[w_-,\,0]} = \widehat{\varphi}_{-|[w_-,\,0]}$, and so from the proof of Theorem 1 it follows that u_0, v_0 are both local $C_n^1(\overline{\Omega})$-minimizers of φ, hence the result below guarantees that u_0, v_0 are also local $W_n^{1,p}(\Omega)$-minimizers of φ.

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Theorem (Motreanu-Papageorgiou, [9])

If $u_0 \in W_n^{1,p}(\Omega)$ is a local $C_n^1(\overline{\Omega})$-minimizer of φ, i.e. there exists $\rho_0 > 0$ such that

$$\varphi(u_0) \le \varphi(u_0 + h) \text{ for all } h \in C_n^1(\overline{\Omega}),\ \|h\|_{C_n^1(\overline{\Omega})} \le \rho_0,$$

then $u_0 \in C_n^1(\overline{\Omega})$ and u_0 is a local $W_n^{1,p}(\Omega)$-minimizer of φ, i.e. there exists $\rho_1 > 0$ such that

$$\varphi(u_0) \le \varphi(u_0 + h) \text{ for all } h \in W_n^{1,p}(\Omega),\ \|h\| \le \rho_1.$$

Second Existence Result

Theorem 2

If hypotheses $H(a)$ and $H(f)$ hold, then problem (1) has two more nontrivial, constant sign smooth solutions

$$\widehat{u} \in int\, C_+,\ u_0 \leq \widehat{u},\ u_0 \neq \widehat{u},\ \widehat{v} \in -int\, C_+,\ \widehat{v} \leq v_0,\ \widehat{v} \neq v_0\,.$$

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We consider the following Carathéodory truncation-perturbation of the reaction $f(z,x)$:

$$\widehat{h}+(z,x) = \begin{cases} f(z,u_0(z)) + u_0(z)^{p-1} & \text{if } x \leq u_0(z) \\ f(z,x) + x^{p-1} & \text{if } u_0(z) < x \end{cases} \quad \text{and}$$

$$\widehat{h}_-(z,x) = \begin{cases} f(z,x) + |x|^{p-2}x & \text{if } x < v_0(z) \\ f(z,v_0(z)) + |v_0(z)|^{p-2}v_0(z) & \text{if } v_0(z) \leq x\,. \end{cases} \tag{11}$$

We set $\widehat{H}_\pm(z,x) = \int_0^x \widehat{h}_\pm(z,s)ds$ and consider the C^1-functionals $\widehat{\psi}_\pm : W_n^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\psi}_{\pm}(u) = \int_{\Omega} G(z, Du)dz + \frac{1}{p}\|u\|_p^p - \int_{\Omega} \widehat{H}_{\pm}(z, u)dz \quad \text{for all } u \in W_n^{1,p}(\Omega)\,.$$

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- $\widehat{\psi}_{+}$ satisfies the C-condition.

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- $\widehat{\psi}_+$ satisfies the C-condition.
- We obtain a nontrivial positive smooth solution of (1), $\tilde{u}_0 \in int\, C_+$, $u_0 \leq \tilde{u}_0$.

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- If $\tilde{u}_0 \neq u_0$, then we are done.
- If $u_0 = \tilde{u}_0$ then it is a local $W_n^{1,p}(\Omega)$-minimizer of $\widehat{\psi}_+$. If it is not an isolated critical point of $\widehat{\psi}_+$, then we have a whole sequence of distinct positive smooth solutions $u_n \in int\, C_+$ of (1) such that $u_n \geq u_0$ for all $n \geq 1$ and so we are done.

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- If $u_0 = \tilde{u}_0$ is an isolated critical point of $\widehat{\psi}_+$ then we can apply the MPT and we obtain $\widehat{u} \in W_n^{1,p}(\Omega)$ such that

$$\widehat{u} \geq u_0, \; \widehat{u} \neq u_0 \; \widehat{u} \in int\, C_+, \text{ and solves } (1)\,. \qquad (12)$$

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$$\widehat{u} \geq u_0, \; \widehat{u} \neq u_0 \; \widehat{u} \in int\, C_+, \text{ and solves } (1)\,. \tag{12}$$

- Similarly, working this time with $\widehat{\psi}_-$, we obtain a second negative smooth solution $\widehat{v} \in -int\, C_+$, $\widehat{v} \leq v_0$, $\widehat{v} \neq v_0$.

A variational characterization of $\widehat{\lambda}_1$

Proposition (Aizicovici-Papageorgiou-Staicu [1])

Let $\partial B_1^{L^p} = \{u \in L^p(\Omega) : \|u\|_p = 1\}$ *and* $S = W_n^{1,p}(\Omega) \cap \partial B_1^{L^p}$. *We have:*
$\widehat{\lambda}_1 = \inf_{\widehat{\gamma} \in \widehat{\Gamma}} \max_{-1 \leq t \leq 1} \|D\widehat{\gamma}(t)\|_p^p$, *where*
$\widehat{\Gamma} = \{\widehat{\gamma} \in C([-1,1], S) : \widehat{\gamma}(-1) = -\widehat{u}_0,\ \widehat{\gamma}(1) = \widehat{u}_0\}$.

Third Existence Result

Theorem 3.1

If hypothesis $H(a)$ *and* $H(f)$ *hold, then problem (1) has a nontrivial smooth solution* $y \in C^1_n(\overline{\Omega})$ *such that* $v_0 \leq y \leq u_0,\ y \neq v_0,\ y \neq u_0$.

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- Let $\rho = \max\{\|u_0\|_\infty,\ \|v_0\|_\infty\}$ and take $\theta_\rho > 0$ as in hypothesis $H(f)(v)$.

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- Let $\rho = \max\{\|u_0\|_\infty, \ \|v_0\|_\infty\}$ and take $\theta_\rho > 0$ as in hypothesis $H(f)(v)$.
- We introduce the following Carathéodory truncation-perturbation of $f(z,x)$:

$$l(z,x) = \begin{cases} f(z, v_0(z)) + \theta_\rho |v_0(z)|^{p-2} v_0(z) & \text{if } x < v_0(z) \\ f(z,x) + \theta_\rho |x|^{p-2} x & \text{if } v_0(z) \leq x \leq u_0(z) \\ f(z, u_0(z)) + \theta_\rho u_0(z)^{p-1} & \text{if } u_0(z) < x\,. \end{cases} \quad (13)$$

$L(z,x) = \int_0^x l(z,s)ds.$

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  $L(z,x) = \int_0^x l(z,s)ds.$
- The corresponding energy functional $\tau : W_n^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\tau(u) = \int_\Omega G(z,Du)dz + \frac{\theta_\rho}{p}\|u\|_p^p - \int_\Omega L(z,u)dz \quad \text{for all } u \in W_n^{1,p}(\Omega)$$

  is C^1.

- $l_{\pm}(z,x) = l(z, \pm x^{\pm})$ and $L_{\pm}(z,x) = \int_0^x l_{\pm}(z,s)ds$

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- $K_\tau = \{u \in W_n^{1,p}(\Omega) : \tau'(u) = 0\} \subseteq [v_0,\, u_0]$,
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- If $\tilde{u} \in K_{\tau_+}$ and $\tilde{u} \notin \{0, u_0\}$, then $\tilde{u} \in int\, C_+$, $\tilde{u} \leq u_0 \leq \widehat{u}$ is a third positive solution of (1) so we are done. Similarly for K_{τ_-}.

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- $K_{\tau_+} = \{0, u_0\}$, $K_{\tau_-} = \{v_0, 0\}$.
- u_0 and v_0 are local $W_n^{1,p}(\Omega)$-minimizer of τ.
- As before, we may assume that u_0 is an isolated critical point of τ. Since τ satisfies the C-condition we can use the MPT. So, we can find $y \in W_n^{1,p}(\Omega)$ such that

$$\tau(v_0) \leq \tau(u_0) < \eta_{\rho} \leq \tau(y) = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \tau(\gamma(t)), \tag{14}$$

where $\Gamma = \{\gamma \in C\left([0, 1], W_n^{1,p}(\Omega)\right) : \ \gamma(0) = v_0,\ \gamma(1) = u_0\}$ and

$$\tau'(y) = 0\,. \tag{15}$$

- From (14) we see that $y \notin \{u_0, v_0\}$, while from (15) it follows that $y \in [v_0, u_0]$.

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- We can find $s^* \in (0,1)$ and a continuous path $\widehat{\gamma}$, such that $\widehat{\gamma}_0 = s^*\widehat{\gamma}$ connects $-s^*\widehat{u}_0$ and $s^*\widehat{u}_0$ and satisfies

$$\tau_{|\widehat{\gamma}_0} < 0\,. \tag{16}$$

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- We can find two continuous path $\widehat{\gamma}_+$ and $\widehat{\gamma}_-$, connecting respectively $s^*\widehat{u}_0$ and u_0 and $-s^*\widehat{u}_0$ and v_0. These paths satisfies

$$\tau_{|\widehat{\gamma}_+} < 0\,,\ \tau_{|\widehat{\gamma}_-} < 0\,. \tag{17}$$

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- Concatenating $\widehat{\gamma}_-$, $\widehat{\gamma}_0$ and $\widehat{\gamma}_+$, we produce a path $\widehat{\gamma}_* \in \Gamma$ such that

$$\tau_{|\widehat{\gamma}_*} < 0\,,$$

hence, owing to (14) we deduce $y \neq 0$.

Theorem 3.2

If hypotheses $H(a)$ and $H(f)$ hold, then problem (1) has at least five nontrivial smooth solutions

$$u_0,\, \widehat{u} \in int\, C_+,\, u_0 \leq \widehat{u},\, u_0 \neq \widehat{u},\ v_0,\, \widehat{v} \in -int\, C_+,\, \widehat{v} \leq v_0,\, v_0 \neq \widehat{v}$$

$$and\ y \in C^1_n(\overline{\Omega}) \setminus \{0\},\, v_0 \leq y \leq u_0,\ y \neq u_0,\ y \neq v_0\,.$$

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