Talks

Uniqueness and stability of saddle-shaped solutions to the Allen-Cahn equation

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We establish the uniqueness of a saddle-shaped solution to the diffusion equation $-\Delta u = f(u)$ in all of $\mathbb{R}^{2m}$, where f is of bistable type, in every even dimension $2m \geq 2$. In addition, we prove its stability whenever $2m \geq 14$.

Saddle-shaped solutions are odd with respect to the Simons cone $\mathcal{C} = \{(x^1, x^2) \in \mathbb{R}^m \times \mathbb{R}^m : |x^1| = |x^2|\}$ and exist in all even dimensions. Their uniqueness was only known when $2m = 2$. On the other hand, they are known to be unstable in dimensions 2, 4, and 6. Their stability in dimensions 8, 10, and 12 remains an open question. In addition, since the Simons cone minimizes area when $2m \geq 8$, saddle-shaped solutions are expected to be global minimizers when $2m \geq 8$, or at least in higher dimensions. This is a property stronger than stability which is not yet established in any dimension.

Blow up oscillating solutions to some nonlinear fourth order differential equations

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We give strong theoretical and numerical evidence that solutions to some nonlinear fourth order ordinary differential equations blow up in finite time with infinitely many wild oscillations. We exhibit an explicit example where this phenomenon occurs. We discuss applications to biharmonic partial differential equations and to the suspension bridges model. In particular, we give a possible new explanation of the collapse of bridges.

Classification of radial solutions to the Emden-Fowler equation on the hyperbolic space

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We study the Emden-Fowler equation $-\Delta u = |u|^{p-1}u$ on the hyperbolic space $\mathbb{H}^n$. We are interested in radial solutions, namely solutions depending only on the geodesic distance from a given point. The critical exponent for such equation is $p = (n+2)/(n-2)$ as in the Euclidean setting, but the properties of the solutions show striking differences with the Euclidean case. We shall deal both with finite and with infinite energy radial solutions and we classify their exact asymptotic behavior.

The role of Wolff potentials in the analysis of degenerate parabolic equations

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(Joint work with Tuomo Kuusi)

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In this talk I will present recent results obtained in [6, 7], aimed at showing that, provided a natural intrinsic formulation is considered, *Wolff potentials play a fundamental role also in the regularity analysis of non-homogeneous degenerate parabolic equations of* p-Laplacean type, i.e., those modeled by

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = \mu\,. \tag{0.1}$$

Wolff potentials ([4]) play a fundamental role in the analysis of nonlinear elliptic equations and in the fine properties of solutions to boundary value problems. In particular, basic results allow [5, 9, 2, 3], allow to pointwise bound solutions (and their derivatives) to equations as $-\operatorname{div}(|Du|^{p-2}Du) = \mu$ with p-growth. In particular, the gradient estimates in [2, 3], whose proof allow to recover the pointwise estimates for u, allow to recover, for basic model problems, several of the integrability results know for measure data problems. Apart from the case $p = 2$, the problem of finding potential estimates for the parabolic case was still open, *even from the point of view of the definition of the nonlinear potentials used.* In particular, as crystalized in [1], it is impossible to analyze the behavior of equations as (0.1) without using the concept of intrinsic geometry, that is, studying the behavior of u on "intrinsic cylinders" of the type $Q = Q_r^\lambda(x_0,t_0) \equiv B(x_0,r) \times (t_0 - \lambda^{2-p}r^2, t_0)$ whose size depends on the solution itself in the following intrinsic way:

$$\left(\frac{1}{|Q_r^\lambda|}\int_{Q_r^\lambda} |Du|^{p-1}\,dx\,dt\right)^{1/(p-1)} = \left(\fint_{Q_r^\lambda} |Du|^{p-1}\,dx\,dt\right)^{1/(p-1)} \approx \lambda\,.$$

This, in turn, makes the usual definition of nonlinear Wolff potentials (constructed starting from the standard parabolic cylinders)

$$\tilde{\mathbf{W}}^\mu_{\beta,p}(x,t;r) := \int_0^r \left(\frac{|\mu|(Q_\varrho(x_0,t_0))}{\varrho^{N-\beta p}}\right)^{1/(p-1)} \frac{d\varrho}{\varrho}\,, \qquad \beta \in (0,N/p]\,,$$

of no immediate use in this setting (here $Q = Q_r(x_0,t_0) \equiv B(x_0,r) \times (t_0 - r^2, t_0)$ is the standard parabolic cylinder).

The approach of [6, 7] proposes *to adopt the intrinsic geometry approach in the context of nonlinear potential estimates.* This provides a class of *intrinsic Wolff potentials* that reveal to be the natural objects to be considered, as their structure allows to recast the behavior of the Barenblatt solution - the so-called nonlinear fundamental solution. For this reason we introduce the following *intrinsic Wolff potential*:

$$\mathbf{W}^\mu_\lambda(x,t;r) := \int_0^r \left(\frac{|\mu|(Q_\varrho^\lambda(x_0,t_0))}{\lambda^{2-p}\varrho^{N-1}}\right)^{1/(p-1)} \frac{d\varrho}{\varrho}\,, \qquad N := n+2$$

defined starting by intrinsic cylinders $Q_\varrho^\lambda(x_0,t_0)$, where N is the usual parabolic dimension. The key result is the following, and holds for properly defined solutions to measure data problems:

Theorem 0.1 *Let u be a solution to* (0.1) *with $p \geq 2$. For almost every $(x_0,t_0) \in \Omega_T = \Omega \times (0,T)$ there exists a constant $c \geq 1$, depending only on n, p, ν, L, such that whenever $Q_r^\lambda \equiv Q_r^\lambda(x_0,t_0) \equiv B(x_0,r) \times (t_0 - \lambda^{2-p}r^2, t_0) \subset \Omega_T$ is an intrinsic cylinder with vertex at (x_0,t_0), such that*

$$c\mathbf{W}^\mu_\lambda(x_0,t_0;r) + c\left(\fint_{Q_r^\lambda} (|Du|+s)^{p-1}\,dx\,dt\right)^{1/(p-1)} \leq \lambda\,, \tag{0.2}$$

holds, then

$$|Du(x_0,t_0)| \leq \lambda\,.$$

Theorem 0.1, *which in fact extends to general quasilinear parabolic equations*, in turn, gives back the classical L^∞-bound of DiBenedetto [1] who indeed proved that

$$c\left(\fint_{Q_r^\lambda}(|Du|+s)^{p-1}\,dx\,dt\right)^{1/(p-1)} \leq \lambda \Longrightarrow |Du(x_0,t_0)| \leq \lambda\,.$$

Moreover, Theorem 0.1 is in a way universal in that it allows

- To recast in a sharp way the asymptotic behavior of the Barenblatt (fundamental) solution when applied to the equation $u_t - \operatorname{div}(|Du|^{p-2}Du) = \delta$, where δ is the Dirac measure charging the origin; such an estimate is then found to hold for every quasilinear equation of the type $u_t - \operatorname{div} a(Du) = \delta$

- To formulate a non-intrinsic a priori estimate on standard parabolic cylinders, which in fact exhibits the natural anisotropic structure typical of parabolic problems

$$|Du(x_0,t_0)| \leq c\tilde{\mathbf{W}}^\mu_{1/p,p}(x_0,t_0;r) + c\fint_{Q_r}(|Du|+s+1)^{p-1}\,dx\,dt$$

  holds whenever $Q_{2r} \equiv Q_{2r}(x_0,t_0) \equiv B(x_0,2r)\times(t_0-4r^2,t_0) \subset \Omega_T$

- To recast the known elliptic gradient Wolff potential estimates in the stationary case

- To have an a priori estimate which involves standard elliptic Wolff potentials in those cases when μ is time independent or admits a favourable space/time decomposition

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Applications of two new functional inequalities to fractional diffusion II.

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We will discuss two new functional inequalities. The first one, of Nash-Gagliardo-Nirenberg type, will be applied to a nonlinear diffusion model involving fractional powers of the Laplacian. The second one, of Trudinger type, is used to develop a theory for integrable initial data for a viscous transport equation with non-local velocity.

Joint work with Arturo de Pablo, Ana Rodríguez and Juan Luis Vázquez.

Applications of two new functional inequalities to fractional diffusion I.

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We will discuss two new functional inequalities. The first one, of Nash-Gagliardo-Nirenberg type, will be applied to a nonlinear diffusion model involving fractional powers of the Laplacian. The second one, of Trudinger type, is used to develop a theory for integrable initial data for a viscous transport equation with non-local velocity.

Joint work with Arturo de Pablo, Ana Rodríguez and Juan Luis Vázquez.

Asympotic profile for solution of a class of singular parabolic equations

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We consider a class of quasilinear parabolic equations. We study the large time behaviour of the fundamental solution. We prove that the fundamental solution of these equation with time dependent coefficient has the "same" behaviour of the fundamental solution of equations with constant coefficients This result is obtained in collaboration with Ragnedda and Vernier. A fundamental tool to prove this result are recent Harnack inequalities proved by DiBenedetto , Gianazza and myself.

Behaviour near extinction for the fast diffusion equation in bounded domains

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We consider the Fast Diffusion Equation $u_t = \Delta u^m$ posed in a bounded smooth domain $\Omega \subset \mathbb{R}^d$ with homogeneous Dirichlet conditions, when $m_s = (d-2)_+/(d+2) < m < 1$. Solutions $u(t,x)$ of such problem extinguish in a finite time T, and approach a separate variable solution $u(t,x) \sim (T-t)^{1/(1-m)}S(x)$, as $t \to T^-$. We describe the behaviour of such solutions near T: first we show the convergence $u(t,x)\,(T-t)^{-1/(1-m)}$ to $S(x)$ in relative error, then we analyze the question of rates of convergence. For m close to 1 we obtain rates by means of entropy methods and weighted Poincaré inequalities.

Short Communications

Existence Results for p-superlinear Neumann Problems with a Nonhomogeneous Differential Operator

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The aim of this talk is to show some recent existence and multiplicity theorems for the following nonlinear Neumann problem driven by a nonhomogeneous differential operator:

$$-\mathrm{div}(a(z, Du(z)) = f(z, u(z)) \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \,. \qquad (0.3)$$

In our contest $a : \overline{\Omega} \times R^N \to R^N$ is a continuous map, strictly monotone in the $y \in R^N$ variable, and C^1 on $\overline{\Omega} \times (R^N \setminus \{0\})$. The reaction term $f(z, x)$ is a Carathéodory function that exhibits a $(p-1)$-superlinear growth near to $\pm\infty$ and need not satisfy the usual in such cases Ambrosetti-Rabinowitz condition.
Using variational methods based on the critical point theory coupled time by time with suitable truncation techniques, we show that problem (0.3) has four nontrivial smooth solutions of constant sign and a fifth nontrivial smooth solution, different from previous.
The results can be seen as the natural extension to the Neumann case of those obtained for Dirichlet problems driven by the p-Laplacian by several authors in the last years. For instance we can cite the papers:
T. Bartsch-Z. Liu-T. Weth: *Nodals solutions of a p-Laplacian equation*, Proc. London Math. Soc. **91** (2005), 129-152,
Filippakis-A. Kristaly-N. S. Papageorgiou: *Existence of five nonzero solutions with exact sign for a p-Laplacian equation*, Discrete Cont. Dyn. Systems **24** (2009), 405-440,
J. P. Garcia Azorero-J. Manfredi-I. Peral Alonso: *Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations*, Comm. Contemp. Math., **2** (2000), 385-404,
Z. Guo-Z. Zhang: *$W^{1,p}$ versus C^1 local minimizers and multiplicity results for quasilinear elliptic equations*, J. Math. Anal. Appl. **286** (2003), 32-50,
where the reaction term satisfies the AR-condition.

Qualitative behavior of global solutions to some nonlinear fourth order differential equations

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We consider global solutions to a fourth order semilinear ordinary differential equation which arises when dealing with radial solutions to a family of coercive Gelfand type biharmonic equations. We determine sufficient conditions (on the nonlinearity) that ensure global continuation of local solutions. Furthermore, we examine their qualitative behaviors such as oscillations and boundedness.

A Moser inequality for the 1-bilaplacian

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We present sharp inequalities within the context of second order limiting Sobolev embeddings, in the borderline case in which one assumes just the summability of the Laplacian. From one side this leads to establishing new sharp embeddings into Zygmund spaces which improve previous regularity results by Brezis-Merle for elliptic equations with L^1 data, from the other side these complement, in the second order case, Adams' extension of the Moser inequality and throws some new light on a long standing issue concerning the classical Pohozaev-Trudinger-Moser inequality.

Regularity of stable solutions of p-Laplace equations through geometric Sobolev type inequalities

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The purpose of this paper is twofold. On the one hand, we prove a Sobolev inequality and part of a Morrey inequality involving the mean curvature and the tangential gradient with respect to the level sets of the function that appears in both inequalities. Then, as an application, we prove *a priori* estimates for semi-stable solutions of $-\Delta_p u = g(u)$ in a smooth bounded domain $\Omega \subset \mathbb{R}^n$. These estimates leads to new regularity results for the extremal solution when the domain is convex. This is a joint work with Manel Sanchon (UB).

Higher integrability results for porous medium-type equations

Andrea Fugazzola

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In this talk I will consider non-negative, weak solutions to degenerate/singular parabolic equations modelled on

$$u_t - \Delta u^m = 0, \qquad m > 0.$$

Local higher integrability results for the spatial gradient follow from an intrinsic reverse Hölder inequality and a modification of Gehring's lemma. The notion of weak solution depends on the value of m, hence the degenerate and singular cases show different features.

A Discrete Bernoulli Problem

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We introduce a new type of free boundary problem for the p-Laplace operator, related to the so-called Bernoulli free boundary problem. In this new formulation, the classical boundary gradient condition is replaced by a condition on the distance between two different level surfaces of the solution. We show existence of solutions in convex setting, both for the interior and the exterior problem; for the latter we also prove uniqueness. Finally, convergence to the classical Bernoulli problem is shown. This is joint work with M. Gualdani and H. Shahgholian

Asymptotic behavior for the degenerate p-Laplacian equation in bounded domains

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We consider the Dirichlet problem for the p-Laplacian Equation $u_t = \Delta_p u$, where $p > 2$, posed in a bounded domain in $\mathbb{R}^N$ with homogenous boundary conditions and with nonnegative and integrable data. We study the large-time behavior by proving the uniform convergence to an unique asymptotic profile and give a rate for this convergence. As a consequence we also obtain the convergence in relative error.