Some non-standard Sobolev spaces, interpolation and its application to PDE

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Abstract

We consider some nonstandard Sobolev spaces in one dimension, in which functions have different regularity in different subsets. These spaces are useful in the study of some nonlinear parabolic equations where the nonlinearity is highly degenerate and depends on the smoothness of the solution at a certain subset (that may vary with time). An example of application is a diffusion equation with a smooth free boundary, and a moving source/sink where the solution has singularity. The main new idea here is to characterize the functional space setting that is needed for semigroup theory to apply.

1 Introduction

This note is motivated by the following problem: consider the non-linear parabolic equation for $f(x,t)$ given by

$$\begin{cases} f_t - \Delta f = \mu(f|_{I_t}, f_x|_{I_t}), \quad x \in \Omega \subset \mathbb{R}, \ t > 0 \\ f(\cdot, 0) = f_I, \\ f(\cdot, t) = 0 \ \text{ on } \partial\Omega, \ \ t > 0, \end{cases} \tag{1.1}$$

where I_t is a subset of Ω and μ is a measure, nonlinear in f, f_x with support compactly contained in $\Omega \setminus I_t$.

If the nonlinear operator μ satisfies some boundedness conditions, it is well known that $\Delta + \mu$ shares the same spectral properties as the Laplacian operator Δ (see for example [8] and [6]). Thus semigroup theory can be applied to prove existence of solutions of (1.1), convergence to equilibrium and stability.

The aim of this paper is to define suitable function spaces that *(i)* allow to treat μ as a bounded perturbation of the Laplacian operator and, *(ii)* at the same time, give suitable regularity for the solutions of (1.1).

The problem of characterization of function spaces for semilinear parabolic equations is a very classical one. The novelty in the current work relies on the non-standard form of the nonlinearity μ. We allow very general nonlinearities and only require that μ is defined in distributional sense.

More precisely, we need to find three function spaces

$$Z \subset Y \subset X,$$

where $Y := X^\alpha$, $0 < \alpha < 1$, is the interpolation space between X and Z such that

$$\Delta : X \to X, \quad \text{unbounded linear operator, closed,} \tag{1.2}$$

$$-\Delta \quad \text{is a sectorial operator in } X, \tag{1.3}$$

$$Z = D(\Delta), \quad \text{densely defined in } X, \tag{1.4}$$

$$\mu : Y \to X, \quad \text{bounded operator.} \tag{1.5}$$

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Once we achieve this functional space setting, local existence, uniqueness and stability near equilibrium for (1.1) follow from standard arguments in semigroup theory (see Theorem 3.3.3. and 5.1.1. in [8]):

Theorem 1.1. *Assume that $\mu : U \subset Y \to X$ is locally Lipschitz in U, i.e, given any $f \in U$, there exists a smaller neighborhood $f \in V \subset U$ such that for all $f_1, f_2 \in V$ we have that*

$$\|\mu(f_1) - \mu(f_2)\|_X \leq L \|f_1 - f_2\|_Y , \tag{1.6}$$

for some constant $L > 0$, and $\mu(0) = 0$. Then, for any $f_I \in U \subset Y$, there exists $T = T(f_I) > 0$ such that (1.1) has a unique solution $f(x,t)$ for all times $t \in [0,T)$ with initial value $f(\cdot, 0) = f_I$.

If, in addition, the origin is an equilibrium point, $\|\mu(f)\|_X = o(\|f\|_Y)$ when $\|f\|_Y \to 0$, and the spectrum of $(-\Delta)$ is contained in the region $\{Re(z) > \beta\}$ for some $\beta > 0$, then the origin is uniformly asymptotically stable.

Note that hypothesis (1.6) together with $\mu(0) = 0$ implies that $\mu : Y \to X$ is a bounded operator (1.5).

The choice of functional spaces, in order to satisfy conditions (1.2)-(1.5), is delicate and based on the shape of μ; this is the main purpose of the present paper. Our initial assumptions on μ, beside (1.6) are the following:

(*H1*) Assume that $\mu(f, f_x)$ is a (maybe) rough distribution with coefficients depending on the value of the function f and f_x restricted to a smaller subset $I_t \subset \Omega$, in such a way that $\mu(f|_{I_t}, f_x|_{I_t})$ belongs to $H^r(\Omega)$ for some $r < 0$ at each fixed $t > 0$. In addition, assume that the distribution μ has support compactly contained in the complementary set of I_t. We fix an interval I such that $I_0 \subset I$, and $\mu|_{t=0}$ is supported outside $\Omega \backslash I$.

(*H2*) Suppose that f is more regular when restricted to set I, more precisely it should be H^s_ϕ, where ϕ is a smooth cutoff function supported on $I_1 \supset I$ and $\phi_{|_I} = 1$.

We choose $X = H^r$, $r \in \mathbb{R}$, from hypothesis (*H1*), so that $\Delta : H^r \to H^r$, $Z = D(\Delta) = H^{r+2}$, $Y = H^{r+2\alpha}$ for any $0 < \alpha < 1$, then the Laplacian operator Δ satisfies the first three properties (1.2)-(1.4). However, in order to handle (1.5) and the Lipschitz property of μ from (1.6), more regularity is needed at the set I. Therefore, we will ask X to be

$$X = H^r \cap H^s_\phi, \tag{1.7}$$

for $r < s$ and $s > 0$, that is defined through the norm

$$\|f\|_X := \|f\|_{H^r} + \|\phi f\|_{H^s} , \tag{1.8}$$

Given this X, the domain of the Laplacian operator becomes, in a first approximation,

$$Z = H^{r+2} \cap H^{s+2}_\phi.$$

Proposition 2.1 makes this last statement rigorous. The main remaining issue now is to find the intermediate space Y which allows (1.5). This space is important because, given $f_I \in Y$, any solution f of (1.1) will belong to Y for $t > 0$, as stated in Theorem 1.1. Moreover the *time distance* between any solution and an equilibrium point of (1.1) will be measured in the Y-norm. So it is clear that we should define Y to be the *real interpolation* between Z and X:

$$Y := (X, Z)_{\alpha,2}.$$

Given the non-standard definition of the spaces X and Z, it is not always immediate to write Y explicitly as intersection of Sobolev spaces. Hence, in Section 3 we provide a bound for Y in terms of two Sobolev spaces; more precisely we will define $\hat{Y}$ and $\tilde{Y}$ in (3.6) and (3.4) such that

$$Z \subset \hat{Y} \subset Y \subset \tilde{Y} \subset X.$$

Thus, a sufficient condition for (1.6) to hold is that the nonlinearity μ satisfies the following bound:

$$\|\mu(f_1) - \mu(f_2)\|_X \leq L \|f_1 - f_2\|_{\tilde{Y}} .$$

Such a bound for μ is immediate to check since $\tilde{Y}$ is explicitly defined in (3.4).

The first PDE application of our result can be found in [7]. There the authors study the linear stability of a free boundary problem of mean-field type (see [11], [4] and references therein): the nonlinear part of the problem is of the form

$$\mu = f_x(p(t), t) \left[\delta_{x=p(t)-a} - \delta_{x=p(t)+a}\right] , \tag{1.9}$$

where $p(t)$ is the set of points such that $f(p(t), t) = 0$. It is clear that (1.9) belongs to the class of equations (1.1), for $I_t = \{p(t)\}$. Given suitable functional spaces X, Z and Y such that (1.2)-(1.6) are satisfied, existence and asymptotics for initial data close to equilibrium follow easily from Theorem 1.1. Another possible target of study from this point of view is a PDE model for neuronal network dynamics. This model has been introduced in [1] and recently studied in [3]. In its simplest form it consists of a Fokker-Planck equation with a non-linear drift term together with delta functions, that has been considered in the works [2], [5]. In general, we expect this functional space setting to be useful in other evolution problems that involve dynamically evolving points of singularity, such as non-linear Stefan problems and more general ones.

Summarizing, the aim of the present paper is twofold: *(i)* provide a rigorous proof of properties (1.2)-(1.5), when the function space X is of the form (1.7) and the nonlinear operator μ satisfies minimal regularity assumptions, *(ii)* give a rigorous characterization of the real interpolation space Y.
We have not found any reference in the literature that deals with these non-standard intersection spaces. In particular, we concentrate on the issue of denseness of Z in X (Section 2) and on interpolation between X and Z (Section 3).

The paper is organized as follows: in Section 2 we rigorously define the spaces X and Z and prove that Z is a dense subset of X. In Section 3 a characterization and related properties of the *real interpolation space* $Y := (X, Z)_{\alpha,2}$ is given.

2 Function spaces

Periodic functions on $\mathbb{R}$ with period $[-1, 1]$ can be considered as defined on the one-dimensional torus $\mathbb{T}^1$. Such functions have Fourier expansion as

$$f \sim \sum \hat{f}_n e_n,$$

where $\{e_n\}$ is an orthonormal basis in $\mathbb{T}^1$ and $\hat{f}_n$ are the Fourier coefficients of f. The Hilbert space $H^p(\mathbb{T}^1)$ for any $p \in \mathbb{R}$ is given by the norm

$$\|f\|_{H^p} := \sum (1 + n^2)^p |\hat{f}_n|^2.$$

Let ϕ be a smooth cutoff function with support on $[-1/2, 1/2]$, identically one on $[-1/4, 1/4]$. We also consider the space $H^p_\phi(\mathbb{T}^1)$ given by the norm

$$\|f\|_{H^p_\phi(\mathbb{T}^1)} := \|\phi f\|_{H^p(\mathbb{T}^1)} .$$

Given $r \in \mathbb{R}$, $s > 0$, $r < s$, we will work with the intersection space $X(\mathbb{T}^1) := H^r(\mathbb{T}^1) \cap H^s_\phi(\mathbb{T}^1)$, with norm defined as in (1.8). From now on, we will drop the $\mathbb{T}^1$ in the notation of the spaces and norms.

Consider now the Laplacian operator acting on X,

$$\Delta f = \sum \lambda_n \hat{f}_n e_n, \quad \lambda_n \sim -n^2.$$

The coefficients λ_n denotes the eigenvalues of the Laplacian, that are strictly negative except the value zero, that can be deleted if we exclude the constant functions.

Define also the space

$$Z = H^{r+2} \cap H^{s+2}_{\hat{\phi}},$$

where $\hat{\phi}$ is another cutoff function such that $\hat{\phi} \equiv 1$ on the support of ϕ. The following proposition justifies the choice of Z as the domain of $\Delta + \mu$.

Proposition 2.1. *Assume that μ satisfies (H1), (H2), and that $\mu(0) = 0$. Consider the function spaces*

$$X = H^r \cap H^s_{\phi}, \quad Z = H^{r+2} \cap H^{s+2}_{\hat{\phi}}, \quad Y = (X, Z)_{\alpha,2}.$$

If μ satisfies (1.6) then it holds:

1. *$\Delta + \mu : Z \to X$ is a bounded operator.*
2. *Z is a dense subset of X.*

Proof. For the first assertion, just note that the choice of the cutoff functions ϕ and $\hat{\phi}$ implies:

$$\phi\Delta(\hat{\phi} f) = \phi\Delta(f).$$

Consequently, $\Delta : Z \to X$ is a bounded operator. The bound of $\Delta + \mu$ easily follows from (1.6) and the fact that $\mu(0) = 0$:

$$\|(\Delta + \mu)f\|_X \leq \|\Delta f\|_X + \|\mu f\|_X \leq c_1\|f\|_Z + c_2\|f\|_Y \leq C\|f\|_Z.$$

For the second assertion, we will prove that periodic $\mathcal{C}^\infty$-functions are dense in X: given $f_0 \in X$ and $\epsilon > 0$, we will find a sequence of smooth functions f_ϵ such that $\|f_\epsilon - f_0\|_X < \epsilon$.

The idea is the following: given a collection $\mathcal{U}$ of overlapping intervals covering $[-1, 1]$, we construct a good approximation for f_0 in each subintervals of $\mathcal{U}$. Consider for example

$$\mathcal{U} = \{[-1, -1/2], [-5/8, -1/4], [-3/8, 3/8], [1/4, 5/8], [-1/2, 1]\}.$$

The convergence in the whole interval $[-1, 1]$ follows from the convergence in the single subintervals by a partition of unity argument, as explained in the following.

The convergence in H^r is guaranteed by the regularity of the heat equation. Indeed, let $f(x, t)$ be the solution of the heat equation $f_t = \Delta f$ with initial condition f_0. Then

$$f(x, t) = \sum e^{\lambda_n t} \hat{f}_n e_n$$

satisfies $f \to f_0$ in H^r when $t \to 0$.

Moreover, also $f \to f_0$ in H^s_{ϕ} on $[-3/8, 3/8]$, away from the vanishing points of the cutoff ϕ (we remind the reader that ϕ is a smooth function with support on $[-1/2, 1/2]$, identically one on $[-1/4, 1/4]$).

The delicate part is the convergence near $x = \pm 1/2$. For that, on the subinterval $[-5/8, -1/4]$ we regularize the function f_0 as $f_\epsilon = \theta_\epsilon * f_0$, $\theta_\epsilon(x) := \frac{1}{\epsilon}\theta(x/\epsilon)$ with θ is a mollifier supported in $(0, 1)$. Since $f_0 \in H^r$, it is known that $f_\epsilon \to f_0$ in H^r, while it is not clear that ϕf_ϵ converges to ϕf_0 because of the inability of switching the order of convolution and cutoff.

This problem can be handled as follows: by assumption, $\phi f_0 \in H^s$. Consider a small shifting, $\phi(x-t)f_0(x) \in H^s$ for $t \in (0,\epsilon)$, so that $\phi(x)f_0(x+t) \in H^s$. Thus when we average with respect to the weight θ_ϵ, the function $\phi(x)f_\epsilon(x)$, defined as

$$\phi(x)f_\epsilon(x) = \phi(x)\int_0^\epsilon f_0(x+t)\theta_\epsilon(t)dt, \tag{2.1}$$

still belongs to H^s with norm uniformly bounded with respect to ϵ, depending on $\|\phi f_0\|_{H^s}$.

We claim that ϕf_ϵ is very close to ϕf_0 on $[-5/8,-1/4]$. First note that for all $\delta > 0$, there exists a function $\psi \in \mathcal{C}^\infty$ with compact support in the interval $[-5/8,-1/4]$ such that $\|\phi(f_0-\psi)\|_{H^s} \leq \delta/3$. Consider now $\psi_\epsilon := \theta_\epsilon * \psi$. It holds

$$\|\phi f_\epsilon - \phi f_0\|_{H^s} \leq \|\phi(f_\epsilon - \psi_\epsilon)\|_{H^s} + \|\phi(\psi_\epsilon - \psi)\|_{H^s} + \|\phi(f_0-\psi)\|_{H^s}.$$

The first term in the sum can be estimated as in (2.1), and in particular, it can be uniformly bounded by $\|\phi(f_0-\psi)\|_{H^s} < \delta/3$. The second term can be made small thanks to a standard property of mollifiers, and the third is bounded by hypothesis of the initial data.

Thus we have that $\|\phi f_\epsilon - \phi f_0\|_{H^s} \leq \delta$ for ϵ small enough, as desired. □

3 Interpolation

Consider the unbounded linear operator $\Delta : X \to X$, with dense domain $Z = D(X)$. We seek for a characterization of the intermediate space $Y := D((-\Delta)^\alpha) = X^\alpha$, for $0 < \alpha < 1$, that is the domain of a fractional power of the Laplacian operator.

In the case that the function space is $X := H^p$ for $p \in \mathbb{R}$, everything can be written explicitly using Fourier series. The operator $(-\Delta)^\alpha$ for $0 < \alpha < 1$ is the fractional Laplacian defined by semigroup theory as

$$(-\Delta)^\alpha = \int_0^\infty t^{\alpha-1}(-\Delta)(tI + (-\Delta))^{-1}dt, \tag{3.1}$$

and it can be written using Fourier series as

$$(-\Delta)^\alpha f = \sum_n \hat{f}_n \lambda_n^\alpha e_n(x),$$

where the domain of $(-\Delta)^\alpha$ is

$$X^\alpha = H^{2\alpha+p}.$$

For a general space X, such as for the one defined in (1.7), we can use real interpolation:

Theorem 3.1 (Theorem 1.18.10. in [17])**.** *Let L be a positive self-adjoint operator. If γ_1 and γ_2 are two complex numbers, $0 \leq \Re\gamma_1 < \Re\gamma_2 < \infty$ and $0 < \alpha < 1$, then*

$$(D(L^{\gamma_1}), D(L^{\gamma_2}))_{\alpha,2} = D(L^{\gamma_1(1-\alpha)+\gamma_2\alpha}).$$

In particular,

$$D(L^\alpha) = (X, D(L))_{\alpha,2}. \tag{3.2}$$

Thus we are interested on this interpolation space between $X = H^r \cap H^s_\phi$ and $Z := H^{r+2} \cap H^{s+2}_\phi$. Note that, for simplicity and without loss of generality for the purposes of this work, we are taking the same cutoff function for both X and Z, which differs from the previously defined Z defined in Proposition 2.1. For the original case, see the remark at the end.

For any $0 < \alpha < 1$, we are interested in the (real) interpolation space

$$Y := (X, Z)_{\alpha,2}$$

(see [15], [16], [17] for the precise definitions). Although Y cannot be characterized exactly due to the non-standard form of the space X, we will prove that:

Theorem 3.2. *There exist two function spaces $\hat{Y}$ and $\tilde{Y}$ such that*

$$\hat{Y} \subset Y \subset \tilde{Y},$$

where $\hat{Y}$ is defined in (3.6)*, and $\tilde{Y}$ in* (3.4)*.*

Finding $\hat{Y}$ involves understanding pseudo-differential operators, while for the space $\tilde{Y}$ a careful look at interpolation theory is required.

3.1 Characterization of $\tilde{Y}$

The definition of *(real) interpolation* of spaces uses interpolation functors (see [16], section 1.6.2). Let A_0 and A_1 be two Banach spaces. We say that $\{A_0, A_1\}$ is an *interpolation couple* if there exists a linear Hausdorff space A such that both A_0 and A_1 are linearly and continuously embedded in A. Then the spaces $A_0 \cap A_1$ and $A_0 + A_1$ with

$$A_0 + A_1 = \{a \in A \;:\; a = a_0 + a_1 \text{ for some } a_0 \in A_0, a_1 \in A_1\}$$

are well defined. For each $t > 0$, we consider the Peetre's K-functional as

$$K(t, a) = \inf_{(a_0, a_1)} \left\{ \|a_0\|_{A_0} + t\, \|a_1\|_{A_1} \right\},$$

for each $a \in A$, where the infimum is taken among all the representations $a = a_0 + a_1$ with $a_0 \in A_0$, $a_1 \in A_1$.

Given $0 < \alpha < 1$, the (real) interpolation space $(A_0, A_1)_{\alpha,2}$ is defined to be the set of functions $a \in A_0 + A_1$ such that the norm

$$\|a\|_{(A_0,A_1)_{\alpha,2}} := \left(\int_0^\infty |t^{-\alpha} K(t, a)|^2 \frac{dt}{t} \right)^{1/2} \tag{3.3}$$

is finite.

The definition above gives the standard interpolation between Sobolev spaces (see Chapter 2 in [15]). In particular we have that

Theorem 3.3 (Theorem 2.4.2. in [15])**.** *Let Ω be a bounded $\mathcal{C}^\infty$ domain in $\mathbb{R}^n$. Fix real numbers $s_0, s_1 \in \mathbb{R}$ and $0 < \alpha < 1$, satisfying*

$$s = (1 - \alpha)s_0 + \alpha s_1.$$

Then we have the following (real) interpolation equality

$$(H^{s_0}(\Omega), H^{s_1}(\Omega))_{\alpha,2} = H^s(\Omega).$$

Proposition 3.4. *Let*

$$\tilde{Y} := \left\{ f \in H^{r+2\alpha} \;:\; \phi f \in H^{s+2\alpha} \right\}, \tag{3.4}$$

with the norm $\|f\|_{\tilde{Y}} := \|f\|_{H^{r+2\alpha}} + \|\phi f\|_{H^{s+2\alpha}}$. It holds that

$$\|f\|_{\tilde{Y}} \le \|f\|_Y \quad \textit{for all } f \in Y,$$

and consequently, $Z \subset Y \subset \tilde{Y}$.

Proof. Consider the Peetre's K-functional

$$K(t,f) = \inf\left\{\|f_0\|_X + t\,\|f_1\|_Z\right\},$$

where $f = f_0 + f_1$, $f_0 \in X$, $f_1 \in Z$ and

$$\begin{aligned}\|f_0\|_X &= \|f_0\|_{H^r} + \|\phi f_0\|_{H^s}\,,\\ \|f_1\|_Z &= \|f_1\|_{H^{r+2}} + \|\phi f_1\|_{H^{s+2}}\,.\end{aligned}$$

Since $\inf\{x_i + y_j\} \geq \inf(x_i) + \inf(x_j)$, we obtain that

$$K(t,f) \geq K_1(t,f) + K_2(t,f),$$

where K_1 is the Peetre's K-functional of the interpolation

$$\left(H^r, H^{r+2}\right)_{\alpha,2} = H^{r+2\alpha},$$

and K_2 the one corresponding to the interpolation

$$\left(H^s_\phi, H^{s+2}_\phi\right)_{\alpha,2},$$

where we have defined the space

$$H^p_\phi := \{f : \phi f \in H^p\}.$$

We claim that

$$\left(H^s_\phi, H^{s+2}_\phi\right)_{\alpha,2} \subseteq H^{s+2\alpha}_\phi. \tag{3.5}$$

To show it, let $f \in \left(H^s_\phi, H^{s+2}_\phi\right)_{\alpha,2}$. Its norm can be computed with the formula (3.3) where the K-functional is given by

$$K_2(t,f) = \inf\left\{\|\phi a_0\|_{H^s} + t\,\|\phi a_1\|_{H^{s+2}}\right\},$$

where the inf is taken among all the pairs such that $a_0 + a_1 = f$. On the other hand, when we compute the H^s norm of ϕf we need to consider the functional

$$K_3(t,\phi f) = \inf\left\{\|\bar{a}_0\|_{H^s} + t\,\|\bar{a}_1\|_{H^{s+2}}\right\}$$

among all the pairs $\bar{a}_0, \bar{a}_1$ such that $\bar{a}_0 + \bar{a}_1 = \phi f$. It is clear then that $K_2 \geq K_3$ because the infimum for K_3 is computed on a bigger set that may contain more pairs. ☐

3.2 Construction of $\hat{Y}$

This is the most delicate step in the characterization of the space Y. Our reference space X is locally defined (it depends on the cutoff ϕ), however, the fractional Laplacian is a non-local operator. The compromise between the local and non-local setting requires a different cutoff $\hat{\phi}$ such that *supp* $\hat{\phi} \supset$ *supp* ϕ, as it was done in Section 2.

Proposition 3.5. *Let $0 < \alpha < 1$ and $Y = (X, Z)_{\alpha,2}$. Define the new function space $\hat{Y}$ as*

$$\hat{Y} := \left\{f \in H^{r+2\alpha} \;:\; \hat{\phi} f \in H^{s+2\alpha}\right\}, \tag{3.6}$$

with $\hat{\phi}$ a smooth cut-off function such that $\hat{\phi} = 1$ on the supp ϕ, and norm

$$\|f\|_{\hat{Y}} = \|f\|_{H^{r+2\alpha}} + \left\|\hat{\phi} f\right\|_{H^{s+2\alpha}}.$$

It holds

$$\hat{Y} \subset Y.$$

Proof. We need to show that, given any $f \in \hat{Y}$, then $(-\Delta)^\alpha f \in X$. It is clear that, since $f \in H^{r+2\alpha}$, then $(-\Delta)^\alpha f \in H^r$.

The main problem is to understand the localization through the cutoff ϕ. For functions of compact support on an interval, we can use pseudo-differential calculus on $\mathbb{R}$. More precisely, we will show that given $(-\Delta)^\alpha(\hat{\phi} f) \in H^s(\mathbb{R})$, this implies $\phi(-\Delta)^\alpha(f) \in H^s(\mathbb{R})$. It holds

$$\phi(-\Delta)^\alpha(f) = \phi(-\Delta)^\alpha(\hat{\phi} f) + G(f),$$

where

$$Gf := \phi[(-\Delta)^\alpha(f) - (-\Delta)^\alpha(\hat{\phi} f)].$$

Take the H^s norm, then

$$\|\phi(-\Delta)^\alpha(f)\|_{H^s} \leq \left\|(-\Delta)^\alpha(\hat{\phi} f)\right\|_{H^s} + \|G(f)\|_{H^s}.$$

We will show that G is a smoothing operator, i.e.,

$$\|G(f)\|_{H^s} \leq C\,\|f\|_{H^{r+2\alpha}} \tag{3.7}$$

for $s > r + 2\alpha$, and this last norm is bounded by hypothesis.

We use some pseudo-differential operator tools to give information about G. Some standard references are [14], [9], [10]. The advantage of working with pseudo-differential calculus in $\mathbb{R}$ is that the principal symbol of the fractional Laplacian is just $\sigma((-\Delta)^\alpha) = |\xi|^{2\alpha}$. On the other hand, the principal symbol of the multiplication operator $M(f) := \hat{\phi} f$ is $\sigma(M) = \hat{\phi}(x)$.

Next, we compute the principal symbol of the composition operator $(-\Delta)^\alpha(\hat{\phi}\,\cdot)$. Its asymptotic expansion is given by

$$\sigma((-\Delta)^\alpha(\hat{\phi}\,\cdot)) = \sum_{\theta \geq 0} \frac{1}{\theta!}(iD_\xi)^\theta |\xi|^{2\alpha} D_x^\theta \hat{\phi}.$$

The previous formula can be found, for instance, in [14], Chapter II, Section 4. In addition, the principal symbol for $\phi(-\Delta)^\alpha(\cdot)$ is just

$$\sigma\left(\phi(-\Delta)^\alpha(\cdot)\right) = \phi(x)|\xi|^{2\alpha}.$$

Then the principal symbol of the operator G reduces to

$$\sigma(G) = \begin{cases} 0 & \text{on supp } \phi, \text{ since } \hat{\phi} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that the operator G is zero modulus an operator with true order $-\infty$. We recall that the order of an operator A is any number q such that

$$\|Au\|_{H^p} \leq \|u\|_{H^{p+q}},$$

for each real p. The infimum of all such orders q is called the *true order* of G.

Since the operator G has true order $-\infty$, estimate (3.7) holds. In particular we have shown that

$$\phi(-\Delta)^\alpha(f) \in H^s,$$

as we desired. $\square$

Remark. Back to our problem, consider the function spaces X and Z defined as in Proposition 2.1

$$X = H^r \cap H^s_\phi, \quad Z = H^{r+2} \cap H^{s+2}_{\hat{\phi}}.$$

The interpolation space $Y := (X, Z)_{\alpha,2}$ is a subset of

$$\tilde{Y} = \{f \in H^{r+2\alpha} : \phi f \in H^{s+2\alpha}\},$$

and contains the space

$$\bar{Y} = \{f \in H^{r+2\alpha} : \bar{\phi} f \in H^{s+2\alpha}\},$$

where $\bar{\phi}$ is a cut-off function such that $\bar{\phi} = 1$ on supp $\hat{\phi}$.

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