Classification of singularities for a subcritical fully non-linear problem

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Abstract

We address the study of isolated singularities for a fully non-linear elliptic PDE, of subcritical type, arising in conformal geometry. This equation appears when dealing with the k-curvature of a locally conformally flat manifold, that generalizes the scalar curvature. We give a classification result: either the function is bounded near the singularity or it has a specific asymptotic behavior.

1 Introduction

The study of singularities for the the subcritical problem

$$-\Delta u = u^{\beta} \quad \text{in} \quad B\backslash\{0\}, \quad \beta \in \left(\frac{n}{n-2}, \frac{n+2}{n-2}\right) \tag{1}$$

has received a lot of attention. In particular, Gidas-Spruck [5] gave a classification result: a positive solution of (1) with a non-removable singularity at zero must behave like

$$v^{-1}(x) = (1+o(1))\frac{c_0}{|x|^{\frac{2}{\beta-1}}} \quad \text{near} \quad x=0$$

for some $c_0 = c_0(\beta, n)$. In this paper we deal with a more general subcritical equation, of the form

$$\sigma_k(A^g) = v^{\alpha} \quad \text{in} \quad B\backslash\{0\}, \quad \alpha > 0, \tag{2}$$

where $g = v^{-2}|dx|^2$, $v > 0$, is a locally conformally flat metric on the unit ball $B \subset \mathbb{R}^n$ with an isolated singularity at the origin. The matrix A^g is given by $A^g = g^{-1}\tilde{A}^g$, $\tilde{A}^g$ is the Schouten tensor

$$\tilde{A}^g_{ij} = \frac{1}{n-2}\left(Ric_{ij} - \frac{R}{2(n-1)}g_{ij}\right)$$

and Ric, R denote the Ricci tensor and the scalar curvature of g, respectively. In this metric, the Schouten tensor becomes

$$A^{g_v} = v(D^2 v) - \frac{1}{2}|\nabla v|^2 I$$

These σ_k curvatures are defined as the symmetric functions of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the $(1,1)$-tensor A^g,

$$\sigma_k := \sigma_k(A^g) = \sum_{i_1 < \ldots < i_k} \lambda_{i_1} \ldots \lambda_{i_k}$$

The scalar curvature is simply

$$\sigma_1 = \lambda_1 + \ldots + \lambda_n = \frac{R}{2(n-1)}$$

Here we see that problem (2) for $k = 1$ becomes the well known (1). In fact, if we write $u^{\frac{4}{n-2}} = v^{-2}$ and $1 + \frac{n}{2} - \frac{n-2}{2}\beta = \alpha$, then both problems are equivalent. Note that the critical exponent is $\beta = \frac{n+2}{n-2}$ or $\alpha = 0$.

For general k, we are dealing with a fully non-linear equation of second order. The problem is elliptic (but in general, not uniformly elliptic) in the positive cone

$$\Gamma_k^+ = \{v : \sigma_1(A^{g_v}), \ldots, \sigma_k(A^{g_v}) > 0\}$$

However, it still carries a divergence structure

$$m\sigma_m = v\partial_j \left(v_i T_{ij}^{m-1} \right) - nT_{ij}^{m-1} v_i v_j + \tfrac{n-m+1}{2} \sigma_{m-1} |\nabla v|^2$$

that was explored in the previous paper [7].

The main result of this paper is a classification of isolated singularities of (2).

Theorem 1.1. *Let $\alpha \in (0, k)$, $n > 2(k+1)$, and take v a solution of*

$$\begin{aligned} &\sigma_k(v) = v^\alpha \quad \text{in} \quad B\backslash\{0\} \\ &v > 0, \quad v \in \Gamma_k^+ \end{aligned} \tag{3}$$

with $v^{-1} \in \mathcal{C}^3(B\backslash\{0\})$. Then

$$v^{-1}(x) \le \frac{C}{|x|^{\frac{2k}{2k-\alpha}}} \quad \text{near} \quad x = 0$$

Theorem 1.2. *Let v be a solution of (3) for $\alpha \in (0, \frac{2k}{k+1})$, $n > 2(k+1)$ with $v^{-1} \in \mathcal{C}^3(B\backslash\{0\})$. Then, if the function v^{-1} is not bounded near the origin, there exists $c_1, c_2 > 0$ such that*

$$\frac{c_1}{|x|^{\frac{2k}{2k-\alpha}}} \le v^{-1}(x) \le \frac{c_2}{|x|^{\frac{2k}{2k-\alpha}}} \quad \text{near} \quad x = 0$$

On the other hand, the local behavior of singularities for the critical problem $\sigma_k(v) = 1$ has been addressed in the previous paper [6]. There we gave a sufficient condition for the function to be bounded near the singularity: the finiteness of volume of the metric g_v, for the case $n > 2k$. The same result was given by Hang [8] for $n = 2k$. For the Laplacian ($k = 1$) problem, a complete classification of solutions was obtained by Caffarelli-Gidas-Spruck in the significant paper [1]. This the result hoped for the σ_k equation, however, so far are just able to deal with the subcritical version of the problem.

One of the motivations for the study of (1) is because it appears in the resolution of the Yamabe problem (see [9] for a very good survey). We can establish an analogous

k-Yamabe problem: find the infimum over all the metrics $g_v = v^{-2}g_0$, $v > 0$, of the functional

$$\mathcal{F}_k(g) = (vol(g))^{-\frac{n-2k}{n}} \int_M \sigma_k(A^g) dvol_g, \tag{4}$$

This functional was first introduced by Viaclovsky [14], and it generalizes the Yamabe functional. Its Euler equation is precisely $\sigma_k(v) = 1$.

The global subcritical problem has been understood by Li-Li in [10]. Indeed, if v is a positive solution of

$$\sigma_k(v) = v^\alpha \quad \text{in} \quad \mathbb{R}^n$$

for $\alpha \geq 0$ that satisfies $v^{-1} \in \mathcal{C}^2(\mathbb{R}^n)$, then either $v \equiv$ *constant* or $\alpha = 0$ and

$$v^{-1}(x) = \left(\frac{a}{1 + b^2|x - \bar{x}|^2} \right)$$

for some $\bar{x} \in \mathbb{R}^n$ and some positive constants a, b.

The methods of Gidas-Spruck [5] for the problem for $k = 1$ can be generalized to our case. The key ingredient in the present paper is to understand the structure of σ_k and, in particular, to replace the traceless Ricci tensor by the traceless k-Newton tensor (6).

The paper is structured as follows: in section 2 we give some properties of σ_k that will be crucial in the proofs. We use the divergence structure of σ_k (9), an inductive process (11), and the properties of the traceless Newton tensor (6).

In section 3 we prove the expression that will allow us to obtain the necessary L^p estimates, through a generalization of an argument due originally to Obata and that has been very successfully used by Chang-Gursky-Yang [2], and then by Li-Li [11]. In particular, we give a more refined formula (13). This is precisely the ingredient missing in the critical problem. The L^p estimates are found in section 4, and in the last two sections we give the proof of the theorems.

2 Algebraic properties of σ_k

For a general $n \times n$ matrix A, consider its eigenvalues $\lambda_1, \ldots, \lambda_n$, construct the symmetric functions σ_k, and the two tensors

- k^{th} Newton tensor

$$T^k := \sigma_k - \sigma_{k-1}A + \ldots + (-1)^k A^k = \sigma_k I - T^{k-1}A \tag{5}$$

- Traceless Newton tensor

$$L^k := \frac{n-k}{n}\sigma_k I - T^k \tag{6}$$

Remark. Note that although the standard notation for a (1,1)-tensor is A_i^j, here we write both indexes as subindexes without risk of confusion. Take $\sigma_0 := 1$ and $T^0_{ij} := \delta_{ij}$.

Lemma 2.1 ([4], [13]). *We have*

a. $(n-k)\sigma_k = trace(T^k)$

b. $(k+1)\sigma_{k+1} = trace(AT^k)$

c. $trace(L^k) = 0$

d. If $\sigma_1, \ldots, \sigma_k > 0$*, then* T^m *is positive definite for* $m = 1, \ldots k-1$.

e. If $\sigma_1, \ldots, \sigma_k > 0$*, then also*

$$\sigma_k \leq C_{n,k}(\sigma_1)^k$$

In particular, if $A = A^{g_v}$ for $g_v = v^{-2}|dx|^2$, the Schouten tensor becomes

$$A_{ij} = v_{ij}v - \tfrac{1}{2}|\nabla v|^2 \delta_{ij} \tag{7}$$

and the traceless Ricci tensor (actually, a constant multiple of the actual traceless Ricci tensor):

$$E_{ij} := L^1_{ij} = v v_{ij} - \tfrac{1}{n} v \Delta v \delta_{ij} \tag{8}$$

Lemma 2.2 (Viaclovsky [14]). *Let* $g = v^{-2}|dx|^2$*. Then the Newton tensor* T^m *for* $m \leq n-1$ *is divergence-free with respect to this metric* g_v*, i.e,*

$$\sum_j \tilde{\partial}_j T^m_{ij} = 0 \quad \textit{for all} \quad i$$

As a consequence,

$$\sum_j \tilde{\partial}_j L^m_{ij} = \tfrac{n-m}{n} \partial_i \sigma_m(A^{g_v})$$

where $\tilde{\partial}_j$ *is the* $j-th$ *covariant derivative with respect to the* g_v *metric, and* ∂_j *denotes the usual Euclidean derivative.*

The following two lemmas were proved in the previous paper [7]. Expression (10) shows the 'almost' divergence structure of σ_m, and (11) is an inductive formula that allows to handle the non-divergence terms (of order $m-1$) that appear in (10).

Lemma 2.3. *In this setting,*

$$\begin{aligned} \sum_j \partial_j T^m_{ij} &= -(n-m)\sigma_m v_i v^{-1} + n \sum_i T^m_{ij} v_i v^{-1} \quad \textit{for each} \quad i && (9) \\ m\sigma_m(A_g) &= v \sum_{i,j} \partial_j \left(v_i T^{m-1}_{ij} \right) - n \sum_{i,j} T^{m-1}_{ij} v_i v_j + \tfrac{n-m+1}{2} \sigma_{m-1} |\nabla v|^2 && (10) \end{aligned}$$

Lemma 2.4. *Let U be a domain in $\mathbb{R}^n$, $v^{-1} \in \mathcal{C}^\infty(U)$ and $\varphi \in \mathcal{C}_0^\infty(U)$ a smooth cutoff. Then for $1 \le s \le k \le n$ integers and any γ real number,*

$$\begin{aligned} &\int_U \sum_{i,j} T_{ij}^{k-s} v_i v_j |\nabla v|^{2(s-1)} \varphi^{2k} v^{-\gamma} dx \\ &= \left(1 + \tfrac{k-s}{2s}\right) \int_U \sigma_{k-s} |\nabla v|^{2s} \varphi^{2k} v^{-\gamma} dx + \tfrac{s+n+1-\gamma}{2s} \int_U \sum_{i,j} T_{ij}^{k-s-1} v_i v_j |\nabla v|^{2s} \varphi^{2k} v^{-\gamma} dx \\ &- \tfrac{n-k+s+1}{4s} \int_U \sigma_{k-s-1} |\nabla v|^{2(s+1)} \varphi^{2k} v^{-\gamma} dx + \tfrac{k}{s} \int_U \sum_{i,j} T_{ij}^{k-s} v_j \varphi_i |\nabla v|^{2(s-1)} \varphi^{2k-1} v^{1-\gamma} dx \end{aligned} \tag{11}$$

We will need a similar formula for the traceless Newton tensor in section 3.

Corollary 2.5. *Fixed i,*

$$\sum_j \partial_j \left(L_{ij}^m\right) = \frac{n-m}{n} \partial_i \sigma_k + n \sum_j L_{ij}^m v_i v^{-1} \tag{12}$$

Proof. Follows easily from (9) and (6). □

Now we estimate the norm of the Newton tensor:

Lemma 2.6. *If $\sigma_1, \ldots, \sigma_m > 0$, $m \le n$, then*

$$\|T_{ij}^{m-1}\| \le C_{m,n} \sigma_{m-1}$$

Proof. Because of lemma 2.1, T^{m-1} is positive definite and thus to estimate its norm we just need to look at the biggest eigenvalue. We are done because

$$trace(T^{m-1}) = (n-m)\sigma_{m-1}$$

□

One of the main properties of the Traceless Newton tensor is the following well known lemma:

Lemma 2.7. *For any $1 \le k \le n-1$, if we have a metric $g = v^{-2}|dx|^2$ in the positive cone Γ_k^+,*

$$\sum_{i,j} L_{ij}^k E_{ij} \ge 0,$$

with equality if and only if $E = 0$.

Proof. Because E_{ij} is traceless,

$$\sum_{i,j} L_{ij}^k E_{ij} = -\sum_{i,j} T_{ij}^k E_{ij}$$

But using that $E_{ij} = \frac{-1}{n}\sigma_1\delta_{ij} + A_{ij}$, and

$$(k+1)\sigma_{k+1} = T^k_{ij}A_{ij}, \quad T^k_{ij}\delta_{ij} = (n-k)\sigma_k,$$

we see that

$$\sum_{i,j} T^k_{ij}E_{ij} = -\frac{n-k}{n}\sigma_k\sigma_1 + (k+1)\sigma_{k+1}$$

The result follows by the general inequality for matrices in the positive cone Γ^+_k

$$\sigma_{k+1} \leq \frac{n-k}{n(k+1)}\sigma_1\sigma_k$$

with equality if and only if $E \equiv 0$. $\square$

3 Obata type formula

Obata's original result (see [12]) states that if we have a metric g on the unit sphere S^n, conformal to the standard one g_c, of constant scalar curvature, then $E \equiv 0$, i.e., g is the standard metric g_c or it is obtained from g_c by a conformal diffeomorphism of the sphere . His method uses crucially the traceless Ricci tensor $E_{ij} = vv_{ij} - \frac{1}{n}v\Delta v\delta_{ij}$, and the Bianchi identity $\nabla^i E_{ij} = \nabla^j R$. Indeed, the main step is to prove that

$$\sum_{i,j}\int_{S^n} E_{ij}E_{ij}v^{-1}dvol_{g_c} = 0.$$

and thus, g is an Einstein metric on S^n.

This same argument was generalized for $\sigma_k = cst$ instead of R by Viaclovsky [14], here the role of E is played by L^k, and the Bianchi identity is replaced by (12). If the metric is defined on $\mathbb{R}^n$ instead of S^n, an analogous argument works but a cutoff η is introduced and in order to get the same conclusion a careful estimate of the error terms is needed. Let's mention the work of Chang-Gursky-Yang [2], [3], and then by Li-Li [11].

However, we are now interested in the subcritical problem approach by Gidas-Spruck [5]. They refined the computation of

$$0 \leq \int_B \sum_{i,j} E_{ij}E_{ij}v^{-\delta}\eta dx = \dots$$

for any $\delta \in \mathbb{R}$. The main result of this section is the corresponding refinement for σ_k.

Proposition 3.1. *Let $\alpha > 0$, $n > 2k$. Take $v^{-1} \in \mathcal{C}^3(U)$ solution of $\sigma_k(v) = v^\alpha$ in U, $v \in \Gamma^+_k$, $v > 0$, and $\eta \in \mathcal{C}^\infty_0(U)$, U domain in $\mathbb{R}^n$, θ big positive integer. We have then*

$$\begin{aligned}\int_U \sum_{i,j} L^k_{ij}E_{ij}v^{-\delta}\eta^\theta + \left(\tfrac{n-k}{n}\alpha - (1+n-\delta)\tfrac{k(n+2)}{2n}\right)\int_U v^\alpha|\nabla v|^2 v^{-\delta}\eta^\theta \\ +(1+n-\delta)\sum_{s=1}^k d_{k-s}\int_U \sigma_{k-s}|\nabla v|^{2(s+1)}v^{-\delta}\eta^\theta \;=\; E_1(\eta)\end{aligned} \tag{13}$$

for some constants d_{k-s} where

$$E_1(\eta) \lesssim \left| \int_U \sum_{i,j} L^k_{ij} v_i \eta_j v^{1-\delta} \eta^{\theta-1} \right| + \sum_{s=1}^{k} \left| \int_U \sum_{i,j} T^{k-s}_{ij} v_j \eta_j |\nabla v|^{2s} v^{1-\delta} \eta^{\theta-1} \right| \tag{14}$$

In addition, if $\delta < n+1$ and δ close enough $n+1$, all the coefficients in front of the integrals in the left hand side of (13) side are positive.

Proof. It uses the inductive method developed in the previous papers [7] and [6], and the properties of L^k. In view of (8), integrate over U,

$$\int \sum_{i,j} L^k_{ij} E_{ij} v^{-\delta} \eta^\theta = \int \sum_{i,j} L^k_{ij} v_{ij} v^{1-\delta} \eta^\theta - \tfrac{1}{n} \int \sum_{i,j} L^k_{ij} (\Delta v) v^{1-\delta} \delta_{ij} \eta^\theta \tag{15}$$

The second term in (15) vanishes since L^k is trace-free, and thus, integrating by parts and (12),

$$\begin{aligned}
\int \sum_{i,j} L^k_{ij} E_{ij} v^{-\delta} \eta^\theta &= -\int \sum_{i,j} \left(\partial_i L^k_{ij}\right) v_j v^{1-\delta} \eta^\theta - (1-\delta) \int \sum_{i,j} L^k_{ij} v_i v_j v^{-\delta} \eta^\theta \\
&- \int \sum_{i,j} L^k_{ij} v_i \eta_j v^{1-\delta} \eta^{\theta-1} \\
&= -\tfrac{n-k}{n} \int (\partial_i \sigma_k) v_i v^{1-\delta} \eta^\theta - (1+n-\delta) \int \sum_{i,j} L^k_{ij} v_i v_j v^{-\delta} \eta^\theta \\
&- \int \sum_{i,j} L^k_{ij} v_i \eta_j v^{1-\delta} \eta^{\theta-1}
\end{aligned}$$

Group in $E_1(\eta)$ all the terms with derivatives in η. Now compute, using (5), (6) and (7)

$$\begin{aligned}
\int \sum_{i,j} L^k_{ij} v_i v_j v^{-\delta} \eta^\theta &= \tfrac{n-k}{n} \int \sigma_k |\nabla v|^2 v^{-\delta} \eta^\theta - \int \sum_{i,j} T^k_{ij} v_i v_j v^{-\delta} \eta^\theta \\
&= -\tfrac{k}{n} \int \sigma_k |\nabla v|^2 v^{-\delta} \eta^\theta + \int \sum_{i,j,l} T^{k-1}_{il} A_{lj} v_i v_j v^{-\delta} \eta^\theta \\
&= -\tfrac{k}{n} \int \sigma_k |\nabla v|^2 v^{-\delta} \eta^\theta + \int \sum_{i,j,l} T^{k-1}_{il} v_{lj} v_i v_j v^{1-\delta} \eta^\theta \\
&- \tfrac{1}{2} \int \sum_{i,j} T^{k-1}_{ij} v_i v_j v^{-\delta} \eta^\theta
\end{aligned} \tag{16}$$

The middle term above can be handled in a similar manner as in [7], section 4:

$$\begin{aligned}
\int \sum_{i,j,l} T^{k-1}_{il} v_{lj} v_i v_j v^{1-\delta} \eta^\theta &= \tfrac{1}{2} \int \sum_{i,l} \partial_l \left(|\nabla v|^2\right) T^{k-1}_{il} v_i v^{1-\delta} \eta^\theta \\
&= -\tfrac{\delta-1}{2} \int \sum_{i,l} T^{k-1}_{il} v_i v_l |\nabla v|^2 v^{-\delta} \eta^\theta - \tfrac{1}{2} \int \sum_{i,l} \partial_l \left(T^{k-1}_{il} v_i\right) |\nabla v|^2 v^{1-\delta} \eta^\theta \\
&- \tfrac{1}{2} \int \sum_{i,l} T^{k-1}_{il} v_i \eta_l |\nabla v|^2 v^{1-\delta} \eta^{\theta-1}
\end{aligned} \tag{17}$$

To eliminate the term $\partial_l\left(T^{k-1}_{il} v_i\right)$ in (17) just use the equality (9) and then substitute (17) into (16). We obtain

$$\begin{aligned}\int L^k_{ij} v_i v_j v^{-\delta}\eta^\theta &= -k\tfrac{n+2}{2n}\int \sigma_k|\nabla v|^2 v^{-\delta}\eta^\theta - \tfrac{2+n-\delta}{2}\int\sum_{i,j} T^{k-1}_{ij} v_i v_j |\nabla v|^2 v^{-\delta}\eta^\theta\\ &\quad+\tfrac{n-k+1}{4}\int \sigma_{k-1}|\nabla v|^4 v^{-\delta}\eta^\theta + E_1(\eta)\\ &= -k\tfrac{n+2}{2n}\int \sigma_k|\nabla v|^2 v^{-\delta}\eta^\theta + \mathcal{B}_{k-1} + E_1(\eta)\end{aligned} \tag{18}$$

where we have defined for fixed k, $s = 1,\ldots,k-1$,

$$\mathcal{B}_{k-s} = -\tfrac{s+1+n-\delta}{s+1}\int\sum_{i,j} T^{k-s}_{ij} v_i v_j|\nabla v|^{2s} v^{-\delta}\eta^\theta + \tfrac{n-k+s}{2(s+1)}\int \sigma_{k-s}|\nabla v|^{2(s+1)} v^{-\delta}\eta^\theta$$

The computations in (17) can be redone for T^{k-s} and thus

$$\mathcal{B}_{k-s} = \tilde{d}_{k-s}\int \sigma_{k-s}|\nabla v|^{2(s+1)} v^{-\delta}\eta^\theta + \tilde{c}_{k-s-1}\mathcal{B}_{k-s-1} + E_1(\eta) \tag{19}$$

with

$$\tilde{d}_{k-s} = -\tfrac{s+n+1-\delta}{s+1}\left(1+\tfrac{k-s}{2(s+1)}\right) + \tfrac{n-k+s}{2(s+1)}$$

and

$$\tilde{c}_{k-s} = \tfrac{(s+n+1-\delta)(s+2)}{2(s+1)^2}$$

The last step is

$$\mathcal{B}_1 = \tilde{d}_1\int \sigma_1|\nabla v|^{2k} v^{-\delta}\eta + \tilde{c}_1\tilde{d}_0\int|\nabla v|^{2(k+1)} v^{-\delta}\eta$$

Substitute (19) into (18), inductively. This proves (13) for some constants c_{k-s}, d_{k-s} obtained from $\tilde{c}_{k-s}$, $\tilde{d}_{k-s}$. Note that that $c_{k-s} > 0$ if $\delta < n+1$. We also want $d_{k-s} > 0$ for $s = 1,\ldots,k$, and this is achieved when δ is close enough to $n+1$ because $n > 2k$. □

Lemma 3.2. *With the same hypothesis as in the previous lemma,*

$$\int_U v^{\alpha/k-\gamma}\eta^\theta \lesssim \left(-1+\gamma-\tfrac{n}{2}\right)\int_U \sigma_k|\nabla v|^2 v^{-\gamma}\eta^\theta + E_2(\eta) \tag{20}$$

where

$$E_2(\eta) \lesssim \left|\int_U\sum_i v_i\eta_i v^{1-\gamma}\eta^{\theta-1}\right| \tag{21}$$

Proof. Since $\sigma_k(v) = v^\alpha$ and $\sigma_k \le C(n,k)\sigma_1^k$ (lemma 2.1) we get

$$\sigma_1(v) \gtrsim v^{\alpha/k}.$$

It is easy to see that

$$\int \sigma_1 v^{-\gamma}\eta^\theta = \left(-1+\gamma-\tfrac{n}{2}\right)\int|\nabla v|^2 v^{-\gamma}\eta^\theta + E_2(\eta)$$

and the lemma is proved. □

4 Main estimates

Here we obtain the L^p estimate needed as a consequence of (13). The terms on the left hand side of (13) will be 'good' terms, and we will give an estimate of the error terms.

Proposition 4.1. *Let $n > 2k$, $\alpha \in (0,k)$, v solution of (3). We have*

$$\int_{\rho<|x|<M\rho} v^{\alpha\frac{k+1}{k}-\delta} \lesssim \frac{1}{\rho^{2(k+1)}} \int_{A_\rho \cup A_{M\rho}} v^{2(k+1)-\delta} + \frac{1}{\rho^2} \int_{A_\rho \cup A_{M\rho}} v^{2+\alpha-\delta} \tag{22}$$

for $\delta < n+1$ close enough to $n+1$, where $A_\rho = \{\frac{\rho}{2} < |x| < \rho\}$, $A_{M\rho} = \{M\rho < |x| < 2M\rho\}$, and the constants depend on M but not in ρ.

Proof. If we take $\alpha - \delta = -\gamma$, then $-1 - \frac{n}{2} + \gamma > 0$ and the lemma above allows us to replace

$$\int |\nabla v|^2 v^{\alpha-\delta}\eta^\theta \quad \text{by} \quad \int v^{\frac{k+1}{k}\alpha-\delta}\eta^\theta + E_2(\eta)$$

in expression (13). Let η be a smooth cutoff such that

$$\eta = \begin{cases} 1 & \text{if } \rho < |x| < M\rho \\ 0 & \text{if } 0 < |x| < \frac{\rho}{2}, \quad 2M\rho < |x| \end{cases}$$

and

$$|\nabla \eta| \lesssim \frac{1}{\rho}, \quad |D^2\eta| \lesssim \frac{1}{\rho^2}.$$

The errors $E_1(\eta)$ in (14) are of one of these two types:

$$E_{11}(\eta) \lesssim \left| \int_{A_\rho \cup A_{M\rho}} \sum_{i,j} L^k_{ij} v_i \eta_j v^{1-\delta}\eta^{\theta-1} \right|$$

$$E_{12}(\eta) \lesssim \sum_{s=1}^{k} \left| \int_{A_\rho \cup A_{M\rho}} \sum_{i,j} T^{k-s}_{ij} v_j \eta_j |\nabla v|^{2s} v^{1-\delta}\eta^{\theta-1} \right|$$

They will get handled as in the proof of theorem 1.1. in [6]. However, here we give a clearer proof for this particular cutoff.

To understand the part E_{11} substitute $L^k = \frac{n-k}{n}\sigma_k I - T^k$ so

$$E_{11}(\eta) \lesssim \int_{A_\rho \cup A_{M\rho}} \sigma_k v_i \eta_i v^{1-\delta}\eta^{\theta-1} + \int_{A_\rho \cup A_{M\rho}} T^k_{ij} v_i \eta_j v^{1-\delta}\eta^{\theta-1} \tag{23}$$

However, here we cannot use the standard trick to estimate the norm $\|T^k\| \lesssim \sigma_k$ as in lemma 2.6 because we cannot conclude that T^k is positive definite from the information on $\sigma_1, \ldots, \sigma_k$ and we need to write everything in terms of smaller T^{k-s}. An inductive process is needed.

Substitute $T^k_{ij} = \sigma_k \delta_{ij} - A_{il} T^{k-1}_{lj}$ and $A_{il} = v v_{il} - \frac{1}{2}|\nabla v|^2 \delta_{il}$ in (23), together with lemma 2.6,

$$\begin{aligned} E_{11}(\eta) \lesssim &\int \sigma_k |\nabla v||\nabla \eta| v^{1-\delta}\eta^{\theta-1} + \int \sigma_{k-1} |\nabla v|^3 |\nabla \eta| v^{1-\delta}\eta^{\theta-1} \\ &+ \left| \int T^{k-1}_{lj} v_{il} v_i \eta_j v^{2-\delta}\eta^{\theta-1} \right| \end{aligned} \tag{24}$$

Now, for the last term, proceed as in (17),

$$\begin{aligned}\int T_{lj}^{k-1} v_{il} v_i \eta_j v^{2-\delta} \eta^{\theta-1} &= \tfrac{1}{2} \int \partial_l \left(|\nabla v|^2\right) T_{lj}^{k-1} \eta_j v^{2-\delta} \eta^{\theta-1} \\ &= -\tfrac{1}{2} \int \left(\partial_l T_{lj}^{k-1}\right) |\nabla v|^2 \eta_j v^{2-\delta} \eta^{\theta-1} \\ &\quad - \tfrac{1}{2} \int T_{lj}^{k-1} |\nabla v|^2 \eta_{lj} v^{2-\delta} \eta^{\theta-2} - \tfrac{2-\delta}{2} \int T_{lj}^{k-1} \eta_l v_j |\nabla v|^2 v^{1-\delta} \eta^{\theta-1}\end{aligned} \tag{25}$$

Note that (9) helps to compute $\partial_l T_{lj}^{k-1}$ and thus from (25) and lemma 2.6 we get

$$\left| \int T_{lj}^{k-1} v_{il} v_i \eta_j v^{2-\delta} \eta^{\theta-1} \right| \lesssim \int \sigma_{k-1} |D^2 \eta| |\nabla v|^2 v^{2-\delta} \eta^{\theta-2} + \int \sigma_{k-1} |\nabla v|^3 |\nabla \eta| v^{1-\delta} \eta^{\theta-1} \tag{26}$$

Young's inequality for a small ϵ, (24) and (26) give

$$\begin{aligned} E_{11}(\eta) \lesssim \epsilon \int \sigma_k |\nabla v|^2 \eta^\theta v^{-\delta} + \frac{C_\epsilon}{\rho^2} \int_{A_\rho \cup A_{M\rho}} \sigma_k v^{2-\delta} \eta^{\theta-2} \\ + \epsilon \int \sigma_{k-1} |\nabla v|^4 \eta^\theta v^{-\delta} + \frac{C_\epsilon}{\rho^4} \int_{A_\rho \cup A_{M\rho}} \sigma_{k-1} v^{4-\delta} \eta^{\theta-4} \end{aligned} \tag{27}$$

To finish the estimate we just need (29) from the lemma below, applied iteratively. Thus

$$E_{11}(\eta) \lesssim \epsilon \sum_{s=0}^{k} \int \sigma_{k-s} |\nabla v|^{2(s+1)} \eta^\theta v^{-\delta} + \frac{C_\epsilon}{\rho^{2(k+1)}} \int_{A_\rho \cup A_{M\rho}} v^{2(k+1)-\delta} \tag{28}$$

The estimate for $E_{12}(\eta)$ follows in a similar manner. For the errors in $E_2(\eta)$, defined in (21), use Young's inequality with $p = q = 2$:

$$E_2(\eta) \lesssim \int |\nabla v| |\nabla \eta| v^{1-\gamma} \eta^{\theta-1} \lesssim \epsilon \int |\nabla v|^2 v^{\alpha-\delta} \eta^\theta + \frac{C_\epsilon}{\rho^2} \int_{A_\rho \cup A_{M\rho}} v^{2+\alpha-\delta}$$

Putting all together in (13), and taking into account that $\sum_{i,j} L_{ij}^k E_{ij} \geq 0$

$$\int_{\rho<|x|<M\rho} v^{\alpha \frac{k+1}{k} - \delta} \leq \int v^{\alpha \frac{k+1}{k} - \delta} \eta^\theta \lesssim \frac{1}{\rho^{2(k+1)}} \int_{A_\rho \cup A_{M\rho}} v^{2(k+1)-\delta} + \frac{1}{\rho^2} \int_{A_\rho \cup A_{M\rho}} v^{2+\alpha-\delta}$$

□

Lemma 4.2. *For all $\epsilon > 0$, $s = 0, \ldots, k-1$, θ big positive integer,*

$$\begin{aligned} \frac{1}{\rho^{2(s+1)}} \int \sigma_{k-s} v^{2(s+1)-\delta} \eta^{\theta-2(s+1)} &\leq \epsilon \int \sigma_{k-s-1} |\nabla v|^{2(s+2)} \eta^\theta v^{-\delta} \\ &\quad + \frac{C_\epsilon}{\rho^{2(s+2)}} \int_{\{|\nabla \eta| \neq 0\}} \sigma_{k-s-1} \eta^{\theta-2(s+2)} v^{2(s+2)-\delta} \end{aligned} \tag{29}$$

Proof. First use the 'divergence' formula (10) for σ_{k-s} and integration by parts:

$$
\begin{aligned}
(k-s)\int \sigma_{k-s} v^{2(s+1)-\delta}\eta^{\theta-2(s+1)} &= \tfrac{n-k+s+1}{2}\int \sigma_{k-s-1}|\nabla v|^2\eta^{\theta-2(s+1)}v^{2(s+1)-\delta} \\
&\quad -(n+2(s+1)-\delta+1)\int T_{ij}^{k-s-1}v_i v_j \eta^{\theta-2(s+1)}v^{2(s+1)-\delta} \\
&\quad -\int T_{ij}^{k-s-1}v_i\eta_j\eta^{\theta-2(s+1)-1}v^{2(s+1)-\delta+1}
\end{aligned}
\tag{30}
$$

Now use lemma 2.6 again to bound the norm of the Newton tensor in (30),

$$
\begin{aligned}
\int \sigma_{k-s} v^{2(s+1)-\delta}\eta^{\theta-2(s+1)} &\lesssim \int \sigma_{k-s-1}|\nabla v|^2\eta^{\theta-2(s+1)}v^{2(s+1)-\delta} \\
&\quad +\frac{1}{\rho}\int \sigma_{k-s-1}|\nabla v|\eta^{\theta-2(s+1)-1}v^{2(s+1)-\delta+1}
\end{aligned}
\tag{31}
$$

Young's inequality with ϵ, $p=s+2$, $q=\frac{s+2}{s+1}$ reads

$$
\begin{aligned}
\int \sigma_{k-s-1}|\nabla v|^2\eta^{\theta-2(s+1)}v^{2(s+1)-\delta} &\lesssim \epsilon\rho^{2(s+1)}\int \sigma_{k-s-1}|\nabla v|^{2(s+1)}\eta^{\theta}v^{-\delta} \\
&\quad +\frac{C_\epsilon}{\rho^2}\int \sigma_{k-s-1}\eta^{\theta-2(s+2)}v^{2(s+2)-\delta}
\end{aligned}
\tag{32}
$$

And for the second part in (31), take $p=2(s+2)$, $q=\frac{2(s+2)}{2(s+2)-1}$,

$$
\begin{aligned}
\frac{1}{\rho}\int \sigma_{k-s-1}|\nabla v|\eta^{\theta-2(s+2)-1}v^{2(s+1)-\delta+1} &\lesssim \epsilon\rho^{2(s+1)}\int \sigma_{k-s-1}|\nabla v|^{2(s+2)}\eta^{\theta}v^{-\delta} \\
&\quad +\frac{C_\epsilon}{\rho^2}\int \sigma_{k-s-1}\eta^{\theta-2(s+2)}v^{2(s+2)-\delta}
\end{aligned}
\tag{33}
$$

The lemma is proved by substituting (32) and (33) into (31). $\square$

Proposition 4.3. *For $n \geq 2(k+1)$, $\alpha \in (0,k)$, v solution of (3) we have*

$$
\int_{\rho<|x|<M\rho} v^{\alpha\frac{k+1}{k}-\delta} \leq C\rho^{n-\frac{\delta-\alpha\frac{k+1}{k}}{1-\frac{\alpha}{2k}}},
\tag{34}
$$

where C depends on M and δ but not in ρ.

Proof. Use Hölder with

$$
p=\frac{\delta-\alpha\frac{k+1}{k}}{\delta-2(k+1)}, \quad q=\frac{p}{p-1}
$$

to get

$$
\frac{1}{\rho^{2(k+1)}}\int_{A_\rho\cup A_{M\rho}} v^{2(k+1)-\delta} \leq \epsilon\int_{A_\rho\cup A_{M\rho}} v^{\alpha\frac{k+1}{k}-\delta}+C_\epsilon\rho^{n-2(k+1)q}
\tag{35}
$$

for some ϵ small chosen later. Also, a Hölder estimate with

$$\tilde{p} = \frac{\delta - \alpha\frac{k+1}{k}}{\delta - 2 - \alpha}, \quad \tilde{q} = \frac{\tilde{p}}{\tilde{p} - 1}$$

gives

$$\frac{1}{\rho^2}\int_{A_\rho \cup A_{M\rho}} v^{2+\alpha-\delta} \leq \epsilon \int_{A_\rho \cup A_{M\rho}} v^{\alpha\frac{k+1}{k}-\delta} + C_\epsilon \rho^{n-2\tilde{q}} \tag{36}$$

Note that when $\alpha \in (0, k)$ and δ close enough to $n+1$, then both $p, \tilde{p} > 1$. Now, look at the powers of ρ in (35) and (36):

$$n - 2(k+1)q = n - 2\tilde{q} = n - \frac{\delta - \alpha\frac{k+1}{k}}{1 - \frac{\alpha}{2k}}$$

Choosing ϵ small enough we conclude from (22)

$$\int_{\rho<|x|<M\rho} v^{\alpha\frac{k+1}{k}-\delta} \leq C\rho^{n - \frac{\delta - \alpha\frac{k+1}{k}}{1-\frac{\alpha}{2k}}}$$

□

5 Proof of theorem 1.1

The following proposition is the analogous to the study of the critical problem in [6]. In particular, a "volume finiteness" condition gives regularity near the singularity.

Proposition 5.1. *Let $\alpha \in (0, k)$, $n > 2k$, $v > 0$, $v \in \Gamma_k^+$ be a solution of (3), $B_\rho(x_0) \subset B$.*

1. *If the integral*

$$\int_{B_\rho(x_0)} v^{(\alpha-2k)\frac{n}{2k}} \leq a \tag{37}$$

for some a small enough (not depending on ρ) then

$$\sup_{B_{\rho/2}(x_0)} |v^{-1}| \leq \frac{C}{\rho^{n/p}}\|v^{-1}\|_{L^p(B_\rho(x_0))} \tag{38}$$

for all

$$p > (n-2k)\frac{k}{k+1}$$

2. *In particular, if*

$$\int_{\epsilon<|x|<1} v^{(\alpha-2k)\frac{n}{2k}} < C < \infty \tag{39}$$

for some constant C independent of ϵ, then the function is bounded near the origin.

Proof. Similar to Theorem 1.2. in [6] for the critical problem. Condition (39) is the analogous to the "smallness volume" condition there. $\square$

Proof. of Theorem 1.1: Fix x_0 small enough and take $2R = |x_0|$. First note that Hölder estimates with

$$r = \frac{\delta - \frac{k+1}{k}}{\frac{n}{2k}(2k-\alpha)} > 1, \quad 1 = \frac{1}{r} + \frac{1}{s}$$

give

$$\int_{B_R(x_0)} v^{(\alpha-2k)\frac{n}{2k}} \leq \left(\int_{R\leq|x|\leq 3R} v^{\frac{k+1}{k}-\delta} \right)^{\frac{1}{r}} \epsilon^{\frac{n}{s}} \lesssim R^{\left(n - \frac{\delta - \alpha\frac{k+1}{k}}{1-\frac{\alpha}{2k}}\right)\frac{1}{r}} R^{\frac{n}{s}} \lesssim R^0 < \infty \tag{40}$$

independently of x_0. We cannot apply proposition 5.1 directly to v, but however, we could have started with the function $\tilde{v}(y) = A^{\frac{2k}{2k-\alpha}} v(\frac{y}{A})$ for some A big enough of the form

$$A = (\text{constant}) \int_{R\leq|x|\leq 3R} v^{(\alpha-2k)\frac{n}{2k}},$$

that still satisfies the same equation $\sigma_k(\tilde{v}) = \tilde{v}^\alpha$.

Since we are interested just in the local behavior near zero, we can assume that (38) gives an estimate for v

$$\sup_{B_{R/2}(x_0)} |v^{-1}| \leq \frac{C}{R^{n/p}} \|v^{-1}\|_{L^p(B_R(x_0))}$$

for all $p > (n-2k)\frac{k}{k+1}$, and C depending on

$$\int_{R\leq|x|\leq 3R} v^{(\alpha-2k)\frac{n}{2k}}$$

that is anyway uniformly bounded independently of R by a constant because of (40). In any case, it is also true

$$\sup_{B_{R/2}(x_0)} |v^{-1}| \leq \frac{C}{|x_0|^{n/p}} \|v^{-1}\|_{L^p(\{R\leq|x|\leq 3R\})} \tag{41}$$

for all $p > (n-2k)\frac{k}{k+1}$. Let $p = \delta - \alpha\frac{k+1}{k}$; this choice is valid when $\alpha \in (0,k)$, $n > 2k$; and now use (34) again

$$\int_{\{R\leq|x|\leq 3R\}} v^{-p} \leq C|x_0|^{n - \frac{p}{1-\frac{\alpha}{2k}}}$$

and thus from (41) we arrive at

$$v^{-1}(x_0) \leq \frac{C}{|x_0|^{\frac{2k}{2k-\alpha}}}$$

as desired. $\square$

Corollary 5.2 (Harnack). *In these hypothesis, there exists $M_0 > 0$ such that for all $\rho > 0$, $M \leq M_0$,*

$$\sup_{\rho \leq |x| \leq \rho M} v^{-1} \leq C \inf_{\rho \leq |x| \leq \rho M} \tag{42}$$

where C is independently of v, ρ and M.

Proof. Once we get a sup estimate (41) for a ball, the inf estimate follows from standard elliptic theory. In particular, writing $v^{-2} = u^{\frac{2}{n-2}}$, then u is a superharmonic function. To finish, we need to use a covering argument for the annulus $\{\rho \leq |x| \leq \rho M\}$. □

Corollary 5.3. *If v is a solution of (3), either v^{-1} is bounded near the origin or*

$$v^{-1}(x) \to \infty \quad as \quad x \to 0$$

Proof. Follows the steps of corollary 3.3. in [5], using the second part of proposition 5.1. □

6 Proof of theorem 1.2

We have proved the estimate

$$v^{-1}(x) \leq \frac{C}{|x|^{\frac{2k}{2k-\alpha}}} \tag{43}$$

Now we would like to get the opposite inequality. Suppose that

$$\lim_{x\to 0} \inf |x|^{\frac{2k}{2k-\alpha}} v^{-1}(x) = 0$$

By the Harnack estimate (42), also

$$\lim_{x\to 0} |x|^{\frac{2k}{2k-\alpha}} v^{-1}(x) = 0 \tag{44}$$

We want to see that in this case the function v^{-1} is bounded near the origin and thus the theorem follows. So it suffices to establish (39).

Let's review two results from [6]:

Proposition 6.1. *Let $v^{-1} \in \mathcal{C}^3(U)$, $v > 0$, $v \in \Gamma_k^+$, $n > 2k$. Then for all $\varphi \in \mathcal{C}_0^\infty(U)$, θ a big positive integer,*

$$\int_U \sigma_k \varphi^\theta v^{-\gamma} \geq \sum_{s=1}^{k} c_{k-s}(\gamma) \int_U \sigma_{k-s} |\nabla v|^{2s} \varphi^\theta v^{-\gamma} + E(\varphi) \tag{45}$$

where

$$E(\varphi) \lesssim \sum_{s=1}^{k} \left| \int_U \sum_{i,j} T_{ij}^{k-s} v_j \varphi_i |\nabla v|^{2(s-1)} \varphi^{\theta-1} v^{1-\gamma} \right| \tag{46}$$

and all the coefficients $c_{k-s}(\gamma) > 0$ for all for all

$$\gamma > n - \frac{n-2k}{k+1} \tag{47}$$

Proposition 6.2. *For all $\epsilon > 0$, the error terms (46) can be estimated by*

$$E(\varphi) \le \epsilon \sum_{s=1}^{k} \int \sigma_{k-s} |\nabla v|^{2s} \varphi^{\theta} v^{-\gamma} + C_\epsilon \sum_{U_k} \int U_k(\varphi) \varphi^{\theta - \alpha_k} v^{2k-\gamma}$$

where the $U_k(\varphi)$ are groups of derivatives of φ of order $2k$, and $\alpha_k \in \mathbb{R}$ depending on each of the U_k; these concepts are defined inductively in the following manner: the starting point is, fixed $s = 1, \ldots, k$

$$U_s(\varphi)\varphi^{\alpha_s} = |\nabla \varphi|^{2s} \varphi^{-2s}$$

For each integer $l = 0, 1, \ldots$, given $U_{s+l}\varphi^{\alpha_{s+l}}$, the following step $s + l + 1$ is of these three shapes: (call $s + l = m$)

$$U_{m+1}\varphi^{-\alpha_{m+1}} = \begin{cases} U_m^{\frac{m+1}{m}} \varphi^{-\alpha_m \frac{m+1}{m}} \\ |\nabla U_m|^{\frac{2(m+1)}{2(m+1)-1}} \varphi^{-\alpha_m \ \frac{2(m+1)}{2(m+1)-1}} \\ \left[|\nabla \varphi|^2 U_m\right] \varphi^{-\alpha_m - 2} \end{cases} \tag{48}$$

the ending point is $s + l = k$.

We will use (45) for a suitable cutoff function. Take $\varphi = \eta r$, where $\eta \in \mathcal{C}_0^\infty(B\backslash\{0\})$, such that

$$\eta = \begin{cases} 1 & \text{if} \quad \epsilon < |x| < R \\ 0 & \text{if} \quad |x| < \epsilon/2, |x| > 2R \end{cases}$$

and such that the derivatives the derivatives have a good bound $\epsilon/2 < |x| < \epsilon$, $R < |x| < 2R$. The value of γ will be chosen later. Now rewrite (45) as

$$\int \sigma_k v^{-\gamma} \varphi^\theta \gtrsim \sum_{s=1}^{k} \int \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} \varphi^\theta - \int T_{ij}^{k-1} v_i \varphi_j \varphi^{\theta-1} v^{1-\gamma} + \tilde{E}(\varphi) \tag{49}$$

with

$$\tilde{E}(\varphi) \lesssim \sum_{s=2}^{k} \left| \int T_{ij}^{k-s} v_i \varphi_j |\nabla v|^{2(s-1)} \varphi^{\theta-1} v^{1-\gamma} \right|$$

since we will look more carefully at the term in T^{k-1}. Integration by parts gives:

$$\begin{aligned} &-\int \sum_{i,j} T_{ij}^{k-1} v_i \varphi_j \varphi^{\theta-1} v^{1-\gamma} \\ &\qquad = -\frac{1}{2-\gamma} \int \sum_{i,j} T_{ij}^{k-1} \partial_i \left(v^{2-\gamma}\right) \varphi_j \varphi^{\theta-1} = \frac{1}{2-\gamma} \int \sum_{i,j} T_{ij}^{k-1} \varphi_{ij} \varphi^{\theta-1} v^{2-\gamma} \\ &\qquad + \frac{1}{2-\gamma} \int \sum_{i,j} \partial_i \left(T_{ij}^{k-1}\right) \varphi_j \varphi^{\theta-1} v^{2-\gamma} + \frac{\theta-1}{2-\gamma} \int \sum_{i,j} T_{ij}^{k-1} \varphi_i \varphi_j \varphi^{\theta-2} v^{2-\gamma} \end{aligned} \tag{50}$$

Substituting (9) into (50) we get:

$$-(n+2-\gamma)\int\sum_{i,j}T_{ij}^{k-1}v_i\varphi_j\varphi^{\theta-1}v^{1-\gamma}=\int\sum_{i,j}T_{ij}^{k-1}\varphi_{ij}\varphi^{\theta-1}v^{2-\gamma}$$
$$-(n-k+1)\int\sum_{i}\sigma_{k-1}v_i\varphi_i\varphi^{\theta-1}v^{1-\gamma}+(\theta-1)\int\sum_{i,j}T_{ij}^{k-1}\varphi_i\varphi_j\varphi^{\theta-2}v^{2-\gamma} \tag{51}$$

Now substitute (51) into (49)

$$\begin{aligned}\int\sigma_k\varphi^\theta v^{-\gamma}\geq&\sum_{s=1}^{k}\int\sigma_{k-s}|\nabla v|^{2s}\varphi^\theta v^{-\gamma}\\&+\frac{1}{n+2-\gamma}\left(\int\sum_{i,j}T_{ij}^{k-1}\varphi_{ij}\varphi^{\theta-2}v^{2-\gamma}+(\theta-1)\int\sum_{i,j}T_{ij}^{k-1}\varphi_i\varphi_j\varphi^{\theta-2}v^{2-\gamma}\right)\\&-\frac{n-k+1}{n+2-\gamma}\int\sum_i\sigma_{k-1}v_i\varphi_i\varphi^{\theta-1}v^{1-\gamma}+\tilde{E}(\varphi)\end{aligned} \tag{52}$$

Group all the error terms in the line above in

$$E(\varphi)\lesssim\sum_{s=1}^{k}\int\sigma_{k-s}|\nabla\varphi||\nabla v|\varphi^{\theta-1}v^{1-\gamma}$$

Compute

$$\varphi_i=\frac{x_i}{r}\eta+E_1(\varphi)$$

$$\varphi_{ij}=r^{-1}\left[-\frac{x_ix_j}{r^2}+\delta_{ij}\right]\eta+E_1(\varphi)$$

$$\sum_{i,j}T_{ij}^{k-1}\varphi_{ij}=r^{-1}\left[-\sum_{i,j}T_{ij}^{k-1}\frac{x_ix_j}{r^2}+(n-k+1)\sigma_{k-1}\right]\eta+E_1(\varphi)$$

Since T^{k-1} is positive definite and $\text{trace}(T^{k-1})=(n-k+1)\sigma_{k-1}$, as long as we keep $1<\theta$ we have

$$\sum_{i,j}T_{ij}^{k-1}\left[\varphi_{ij}+(\theta-1)\varphi_i\varphi_jr^{-\theta}\right]\geq C(\theta)\sigma_{k-1}r^{-1}\eta^2+E_1(\varphi)$$

for some $C(\theta)>0$. If we keep $\gamma<n+2$ we can conclude from (52):

$$E(\varphi)+E_1(\varphi)+\int\sigma_k\varphi^\theta v^{-\gamma}\gtrsim\sum_{s=1}^{k}\int\sigma_{k-s}|\nabla v|^{2s}\varphi^\theta v^{-\gamma}+\int\sigma_{k-1}r^{-2}\varphi^\theta v^{2-\gamma} \tag{53}$$

We have not been very precise with the errors $E_1(\varphi)$, however, they are of a similar type to $E(\varphi)$ and they are treated in the same manner. Note that, in the positive cone,

$$\sigma_{k-1}\gtrsim\sigma_k^{\frac{k-1}{k}}=v^{\alpha\frac{k-1}{k}}$$

so we actually have proved from (53)

$$E(\varphi) \gtrsim \int \left[v^{\frac{k-1}{k}\alpha+2-\gamma} r^{-2} - v^{\alpha-\gamma}\right] \varphi^{\theta} + \sum_{s=1}^{k} \int \sigma_{k-s} |\nabla v|^{2s} \varphi^{\theta} v^{-\gamma} \tag{54}$$

To handle $E(\varphi)$ we need to control the error terms that appear in proposition 6.2. Using the lemma below,

$$\begin{aligned} \int U_k(\varphi) \varphi^{\theta-\alpha_k} v^{2k-\gamma} \lesssim & \int r^{-2k} \varphi^{\theta} v^{2k-\gamma} \\ & + \frac{1}{\epsilon^{2k}} \int_{\epsilon/2<|x|<\epsilon} r^{\theta} v^{2k-\gamma} + \frac{1}{R^{2k}} \int_{R<|x|<2R} r^{\theta} v^{2k-\gamma} \end{aligned} \tag{55}$$

Looking at the terms above one by one:

$$\frac{1}{\epsilon^{2k}} \int_{\epsilon/2<|x|<\epsilon} r^{\theta} v^{2k-\gamma} \to 0 \quad \text{as} \quad \epsilon \to 0$$

using the previous estimate (43) and the definition of η, as soon as we choose some

$$\gamma > n - \alpha \left(\frac{n-2k}{2k}\right) \tag{56}$$

An analogous argument gives

$$\frac{1}{R^{2k}} \int_{R<|x|<2R} r^{\theta} v^{2k-\gamma} \leq C$$

The other integral in (55) is bounded by

$$\int r^{-2k} \varphi^{\theta} v^{2k-\gamma} \lesssim \int \left(v^{\frac{k-1}{k}\alpha+2-\gamma} r^{-2}\right) \left(v^{-\frac{k-1}{k}\alpha-2+2k} r^{2-2k}\right) \varphi^{\theta}$$

But because our assumption (44),

$$v^{-\frac{k-1}{k}\alpha-2+2k} r^{2-2k} = o(1)$$

and thus from (54) we obtain:

$$C \gtrsim \int \left[v^{\frac{k-1}{k}\alpha+2-\gamma} r^{-2} - v^{\alpha-\gamma}\right] \varphi^{\theta}$$

Note that again, because of (44),

$$r^2 v^{\frac{\alpha-2k}{2}} = o\left(1\right)$$

Now, theorem 1.1 gives

$$v^{\alpha-\gamma} \lesssim \left(v^{\alpha\frac{k-1}{k}+2-\gamma} r^{-2}\right) \left(r^2 v^{\frac{\alpha-2k}{2}}\right) \tag{57}$$

Comparing the orders of (57), we quickly obtain

$$\int v^{\alpha\frac{k-1}{k}} r^{-2} v^{2-\gamma} \varphi^{\theta} < \infty \tag{58}$$

But this is precisely the term (39) that we need to estimate because

$$\begin{aligned}\int_{\epsilon\leq|x|\leq R} v^{(\alpha-2k)\frac{n}{2k}} &= \int v^{\frac{k-1}{k}\alpha+2-\gamma} v^{-\frac{k-1}{k}\alpha-n+\alpha\frac{n}{2k}-2+\gamma}\eta^{\theta} \\ &\lesssim \int v^{\frac{k-1}{k}\alpha+2-\gamma} r^{\left(-\frac{k-1}{k}\alpha-n+\alpha\frac{n}{2k}-2+\gamma\right)\left(\frac{2k}{2k-\alpha}\right)}\eta^{\theta} \\ &= \int v^{\frac{k-1}{k}\alpha+2-\gamma} r^{-2}\varphi^{\theta}\end{aligned} \tag{59}$$

using Theorem 1.1, and where the choice of θ, γ is

$$\left(-\frac{k-1}{k}\alpha - n + \alpha\frac{n}{2k} - 2 + \gamma\right)\left(\frac{2k}{2k-\alpha}\right) = -2+\theta \tag{60}$$

i.e.

$$\gamma = n - \alpha\left(\frac{n-2k}{2k}\right) + \theta\left(1 - \frac{\alpha}{2k}\right)$$

This is an admissible value for γ because when $\alpha < \frac{2k}{k+1}$, it can be chosen to satisfy (47), (56), $\gamma < n+2$ and $\theta > 1$.

Lemma 6.3. *For the cutoff $\varphi = r\eta$ constructed in the previous proof,*

$$U_k(\varphi)\varphi^{\theta-\alpha_k} \lesssim r^{-2k}\varphi^{\theta} + \epsilon^{-2k}r^{\theta}\chi_{\{\epsilon/2\leq|x|<\epsilon\}} + R^{-2k}r^{\theta}\chi_{\{R\leq|x|<2R\}}$$

Proof. The definition of the U_k is given in proposition 6.2. We are just interested in the orders of r and ϵ. Fixed $s = 1, \ldots, k$, the initial step is

$$U_s(\varphi)\varphi^{\theta-2s} = |\nabla\varphi|^{2s}\varphi^{\theta-2s} \lesssim |\nabla r|^{2s}\varphi^{\theta-2s}\eta^{2s} + |\nabla\eta|^{2s}r^{2s}\varphi^{\theta-2s} \lesssim r^{-2s}\varphi^{\theta} + \epsilon^{-2s}r^{\theta}\eta^{2s}$$

Next, assume that the result is true for $s+l$, (call $s+l=m$)

$$U_m(\varphi)\varphi^{\theta-\alpha_m} \lesssim r^{-2m}\varphi^{\theta} + \epsilon^{-2m}r^{\theta}\eta^{2m}$$

the proof for $m+1$ follows easily from (48). $\square$

Remark. We hope that theorem 1.2 is true also for $n = 2k+1$, but as in the case of the Laplacian, it needs different estimates in (34).

References

[1] L. A. Caffarelli, B. Gidas, and J. Spruck. Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. *Comm. Pure Appl. Math.*, 42(3):271–297, 1989.

[2] S.-Y. A. Chang, M. J. Gursky, and P. Yang. An a priori estimate for a fully nonlinear equation on four-manifolds. *J. Anal. Math.*, 87:151–186, 2002. Dedicated to the memory of Thomas H. Wolff.

[3] S.-Y. A. Chang, M. J. Gursky, and P. C. Yang. Entire solutions of a fully nonlinear equation. Preprint.

[4] L. Gårding. An inequality for hyperbolic polynomials. *J. Math. Mech.*, 8:957–965, 1959.

[5] B. Gidas and J. Spruck. Global and local behavior of positive solutions of nonlinear elliptic equations. *Comm. Pure Appl. Math.*, 34(4):525–598, 1981.

[6] M. d. M. González. Removability of singularities for a class of fully non-linear elliptic equations. Preprint.

[7] M. d. M. González. Singular sets of a class of locally conformally flat manifolds. Preprint, May 2004.

[8] Z.-C. Han. Local pointwise estimates for solutions of the σ_2 curvature equation on 4 manifolds. Preprint, 2004.

[9] J. M. Lee and T. H. Parker. The Yamabe problem. *Bull. Amer. Math. Soc. (N.S.)*, 17(1):37–91, 1987.

[10] A. Li and Y. Li. Further results on Liouville type theorems for some conformally invariant fully nonlinear equations. arXiv:math.AP/0301254 v1 22 Jan 2003, 9 pages.

[11] A. Li and Y. Li. On some conformally invariant fully nonlinear equations. *C. R. Math. Acad. Sci. Paris*, 334(4):305–310, 2002.

[12] M. Obata. Certain conditions for a riemannian manifold to be isometric with a sphere. *Jour. Math. Soc. Japan*, (14):333–340, 1962.

[13] R. C. Reilly. On the Hessian of a function and the curvatures of its graph. *Michigan Math. J.*, 20:373–383, 1973.

[14] J. A. Viaclovsky. Conformal geometry, contact geometry, and the calculus of variations. *Duke Math. J.*, 101(2):283–316, 2000.