Singular sets of a class of locally conformally flat manifolds

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Abstract

We look at complete, locally conformally flat metrics defined on a domain $\Omega \subset S^n$. There is a lot of information about the singular set $\partial\Omega$ contained in the positivity of σ_k and, in particular, we obtain a bound for the Hausdorff dimension of $\partial\Omega$, in relation to Schoen-Yau's work for the scalar curvature. On the other hand, since some locally conformally flat manifolds can be embedded into S^n, this dimension bound implies several topological corollaries.

1 Introduction

Let (M,g) be a Riemannian manifold of dimension $n \geq 3$, with a metric g. Denote by $Riem$, Ric, R, the Riemmanian curvature tensor, the Ricci tensor and the scalar curvature respectively. Construct the Schouten tensor as

$$\tilde{A}^g_{ij} = \frac{1}{n-2}\left(Ric^g_{ij} - \frac{1}{2(n-1)}R^g g_{ij}\right)$$

Now transform the (0,2)-tensor $\tilde{A}_{ij}$ to a $(1,1)$ tensor A_{ij} by $A^g = g^{-1}\tilde{A}^g$. Note that although the standard notation for a (1,1)-tensor is A_i^j, here we write both indexes as subindexes without risk of confusion.

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the matrix A^g at each point. The main object of study will be its k^{th}-elementary symmetric function:

$$\sigma_k := \sigma_k(A^g) = \sum_{i_1<\ldots<i_k} \lambda_{i_1}\ldots\lambda_{i_k}$$

These σ_k-curvatures have received a lot of attention recently. From the point of view of conformal geometry, we are interested in the study of the Schouten tensor because it contains all the conformal information about the curvatures of a manifold. This can be seen from the decomposition (see [1])

$$Riem = W + \tilde{A} \owedge g,$$

where $\owedge$ is the Kulkarni-Nomizu product, and W the Weyl tensor, a conformal invariant. For instance,

$$\sigma_1 = \lambda_1 + \ldots + \lambda_n = \frac{1}{2(n-1)}R$$

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so the study of σ_k gives a natural generalization of the Yamabe problem and scalar curvature related issues.

In particular, we will be interested in locally conformally flat manifolds. We say that a Riemmanian manifold (M, g) is locally conformally flat (l.c.f.) if the metric g can be written, locally, as $g = v^{-2}|dx|^2$ for some smooth $v^{-1} > 0$, where $|dx|^2$ is the usual Euclidean metric.

The sign of σ_k will play an important role understanding the geometry. Indeed, Guan-Viaclovsky-Wang [10] proved that l.c.f. metrics with $\sigma_1, \ldots, \sigma_k \geq 0$ for some $k > 1$ have

$$Ric^g \geq \frac{(2k-n)(n-1)}{(k-1)} \binom{n}{k}^{-1/k} {\sigma_k}^{1/k}(A^g)g \tag{1}$$

and thus $Ric^g > 0$ when $n < 2k$.

Form the analytical point of view, these symmetric functions on l.c.f. manifolds give rise to interesting elliptic fully non-linear equations of second order: in fact, for a metric $g_v = v^{-2}|dx|^2$, the Schouten tensor becomes

$$\tilde{A}^{g_v} = v^{-1}(D^2 v) - \frac{1}{2}|\nabla v|^2 v^{-2} I$$

and thus

$$\sigma_k(A^{g_v}) = v^k \sigma_k(D^2 v) + \text{ lower order terms}, \tag{2}$$

a Hessian type equation. For $k = 2$,

$$2\sigma_2(A^{g_v}) = \left[(\Delta v)^2 - |D^2 v|^2\right] v^2 - (n-1)\Delta v |\nabla v|^2 v + \frac{n(n-1)}{4}|\nabla v|^4$$

There has been a lot of recent work understanding this non-linear PDE. For instance, Chang-Gursky-Yang [4], [3], Gursky-Viaclovsky [12], Li-Li [14], Guan-Wang [11].

However, these σ_k have an underlying 'almost' divergence structure. This is the first result in the present paper, and allows to use certain integral estimates. In particular, we prove in section 4 that

$$m\sigma_m = v\partial_j\left(v_i T_{ij}^{m-1}\right) - nT_{ij}^{m-1}v_i v_j + \tfrac{n-m+1}{2}\sigma_{m-1}|\nabla v|^2 \tag{3}$$

This analysis is the key for the study of singularities of the σ_k equation in the papers [7] and [8].

We are interested in studying singular sets of locally conformally flat metrics with positive σ_k curvature, and the topological information they contain. Let's remark that the cases $n > 2k$, $n = 2k$ and $n < 2k$ have very different behaviors.

Let g be a complete metric on a domain $\Omega \subset S^n$, conformal to the standard metric on the sphere g_c. Assumptions on the positivity of the $\sigma_1, \ldots, \sigma_k$ curvatures and some technical assumptions on the scalar curvature will give an upper bound for the Hausdorff dimension of the singular set $\partial\Omega$:

Theorem 1.1. *Let g be a complete metric on a domain $\Omega \subset S^n$, conformal to g_c, satisfying*

$$\sigma_1(A^g) \geq C_0 > 0 \quad \textit{and} \quad \sigma_2(A^g), \ldots, \sigma_k(A^g) \geq 0 \tag{4}$$

for some integer $1 \leq k < n/2$. Then

$$\dim_{\mathcal{H}}(\partial\Omega) \leq \frac{n-2k}{2}$$

Theorem 1.2. *For the case $k > 1$ if, in addition, we have*

$$|R| + |\nabla_g R| \leq c_0, \tag{5}$$

then

$$\dim_{\mathcal{H}}(\partial\Omega) < \frac{n-2k}{2}$$

Remark. The case $k = 2$ was addressed by Chang-Hang-Yang in [5]. The ideas in their paper generalize for any k here once we understand the structure of (3). The main step in the proof is an integral estimate that follows from that structure; this is done in section 3.

In the paper [18] (see also chapter VI in [19] for a more detailed discussion), Schoen-Yau proved the case $k = 1$, that gives a dimension estimate $\dim_{\mathcal{H}}(\partial\Omega) < \frac{n-2}{2}$ for complete metrics of positive scalar curvature. Theorem 1.2 improves this estimate to $\frac{n-2k}{2}$ if we have the additional information on σ_k.

In the same crucial paper Schoen-Yau showed that any complete l.c.f. manifold of positive scalar curvature is conformally equivalent to a subdomain Ω of the sphere. Now, theorem 1.2 will give restrictions on the homotopy and cohomology groups of the original manifold. In particular, when $n = 2k+1$, $n = 2k+2$ it implies a classification result. These topological corollaries are resumed in section 2.

The last section of the paper will be a discussion about the sharpness of the bound $\frac{n-2k}{2}$. Unlike the case $k = 1$, where metrics with singular set of dimension close to $\frac{n-2}{2}$ can be constructed (see [15]), for general k the bound $\frac{n-2k}{2}$ seems not to be the best. However, the problem is still open. The appendix contains the necessary computations for the section.

We are left to study the singular set for the case $3 \leq n \leq 2k$. But an easy argument, summarized in section 7, gives that there is no singular set.

2 Topological corollaries

The significant work of Schoen-Yau ([19], chapter VI, theorem 4.1) tells us that, given any compact locally conformally flat Riemannian manifold M of positive scalar curvature, its universal covering can be viewed as a domain Ω in S^n through the developing map $\phi : \tilde{M} \to S^n$, conformal, one-to-one. As a consequence, the dimension estimate of

Theorem 1.2 applies for $\Omega := \Phi(\tilde{M}) \subset S^n$. In particular, we have proved a bound for the Schoen-Yau invariant (see section 3)

$$d(M) < \frac{n-2k}{2} \tag{6}$$

and thus several topological consequences follow.

A simple counting dimension argument in [19] (chapter VI, theorem 4.4) plus (6) gives:

Corollary 2.1. *Let (M^n, g), $n > 2k$, be a complete locally conformally flat manifold such that*

$$|R| + |\nabla_g R| \leq c_0$$

$$\sigma_1(A^g) > 0 \quad \textit{and} \quad \sigma_2(A^g), \dots, \sigma_k(A^g) \geq 0.$$

Then for any $2 \leq i \leq \left[\frac{n}{2}\right] + k - 1$, the homotopy group

$$\pi_i(M) = \{0\}$$

On the other hand, Schoen-Yau proved also that we can realize a compact manifold M as a Kleinian manifold Ω/Γ for $\Omega \subset S^n$ and Γ a Kleinian group. From the work of Nayatani [16] in Kleinian groups we have

Corollary 2.2. *Under the same assumptions as in corollary 2.1, if in addition M is compact, then the cohomology group*

$$H^i(M, \mathbb{R}) = \{0\} \tag{7}$$

for $\frac{n-2k}{2} + 1 \leq i \leq \frac{n+2k}{2} - 1$.

And from the work of Izeki [13],

Corollary 2.3. *Let (M^n, g) be a compact locally conformally flat manifold with $n = 2k+1$ or $n = 2k+2$ satisfying*

$$\sigma_1(A^g) > 0 \quad \textit{and} \quad \sigma_2(A^g), \dots, \sigma_k(A^g) \geq 0$$

Then there is a finite covering of M which is either diffeomorphic to S^n or a connected sum of m copies of $S^1 \times S^{n-1}$.

Remark. Guan-Lin-Wang have recently proved corollary 2.2 independently (see [9]). The case $k = 1$ already appeared in Bourguignon [2].

3 Proof of the theorems

Remark. The proof is based on the case $k = 2$ (covered by Chang-Hang-Yang in [5]), and the inductive formula developed in section 4.

The metric g in Ω is conformal to g_c so it can be written as $g = v_0^{-2} g_c$ for some $v_0 > 0$. Call $\Lambda = S^n \backslash \Omega$ the singular set. Assume, without loss of generality that the North Pole $N \in \Omega$, and for simplicity, that $\partial\Omega = S^n \backslash \Omega$. We can transform the problem from S^n to $\mathbb{R}^n$ through stereographic projection σ. Denote $\tilde{\Lambda} = \sigma(\Lambda) \subset \mathbb{R}^n$, that is a compact set. Take $r > 0$ big enough so that $\tilde{\Lambda} \subset\subset B_r$.

The original metric in Ω transforms to a metric in $\mathbb{R}^n$ that is conformal to the Euclidean one, call it $g_v = (\sigma^{-1})^* g = v^{-2}|dx|^2$ for

$$v^{-2}(x) = v_0^{-2}(\sigma^{-1}(x)) \left(\frac{2}{1 + |x|^2} \right)^2, \quad x \in \mathbb{R}^n$$

defined outside $\tilde{\Lambda}$. This v^{-1} is smooth as soon as we go far away from the singular set $\tilde{\Lambda}$. For instance, there exists $D > 0$, $0 > \theta > 1$ such that $|v^{-1}| + |Dv| + |D^2 v| \leq D$ if $|x| > r(1 - \theta)$.

Compute A^{g_v} and then $\sigma_k := \sigma_k(A^{g_v})$. Assumptions (4) and (5) translate to these new σ_k because of their conformal invariance property. We will need the following result from [5], proposition 8.1:

Lemma 3.1. *Let Ω be a domain in S^n, $n \geq 3$, be an open subset and $g = v_0^{-2} g_c$ a complete metric in Ω conformal to the standard one g_c. If $R_g \geq -C$ for some $C > 0$, then $v_0^{-1}(x) \to \infty$ as $x \to \partial\Omega$.*

In particular, we have for v,

$$\lim_{\text{dist}(x, \tilde{\Lambda}) \to 0} v^{-1}(x) = \infty \tag{8}$$

Now denote, for $\lambda > D$,

$$\Omega_\lambda = \left\{ x \in B_r \backslash \tilde{\Lambda} : v^{-1}(x) < \lambda \right\} \tag{9}$$

The aim is to get a bound

$$\int_{\Omega_\lambda} v^{-\gamma} dx \leq C(\gamma) < \infty \tag{10}$$

for some suitable real γ, C independent of λ. Then, taking $\lambda \to \infty$, and pulling back to the sphere we would obtain

$$\int_{\Omega} v_0^{-\gamma} dvol_{g_c} \leq C(\gamma) < \infty \tag{11}$$

The rest of the proof of theorems follows from in Schoen-Yau's work (cf. [19], chapter VI.2) for complete, l.c.f. manifolds of positive scalar curvature because an estimate (11) gives a bound for the invariant

$$d(\Omega) = \inf \left\{ n - \gamma \in \mathbb{R} : \int_{\Omega} v_0^{-\gamma} dvol_{g_c} < \infty \right\} \tag{12}$$

and the theorem will follow because

$$\dim_{\mathcal{H}}(\partial\Omega) \le d(\Omega)$$

Remark. The Hausdorff dimension estimate can be obtained without going through Schoen-Yau's work thanks to the following lemma and (21), for a suitable chosen γ. Nevertheless, it is interesting to explore the relation to $d(\Omega)$ as we have done.

Lemma 3.2. *Let F be a compact subset of $\mathbb{R}^n$ of empty interior. Assume that for some $r > 0$ and $\alpha \ge 1$ we have $F \subset B_r$ and*

$$\int_{B_r \setminus F} \left(\frac{1}{dist(x, F)} \right)^{\alpha} dx < \infty$$

then $\dim_{\mathcal{H}}(F) \le n - \alpha$.

Proof. It is well known and can be found, for instance, in [5]. □

Now, the bound for (10) will follow from the assumptions (4) on the positivity of σ_k, through an iteration method. The main lemma -which will be used repeatedly- is;

Lemma 3.3. *If $\tilde{\Lambda}$, v^{-1}, Ω_λ, D, as above, then for any real number γ there exists constants $c_{k-s}(\gamma)$ such that*

$$\int_{\Omega_\lambda} \sigma_k v^{-\gamma} dx + \sum_{s=1}^{k} c_{k-s}(\gamma) \int_{\Omega_\lambda} \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} dx \le C \tag{13}$$

for some C constant $C = C(\gamma, D)$ but C not depending on λ. Also, $c_{k-s}(\gamma) > 0$ for $s = 1, \ldots, k$ if

$$\gamma < n - \frac{n - 2k}{2} \tag{14}$$

The proof of the lemma will be postponed for next sections. Now we resume the proof of the theorem.

Claim 1. *If we have the extra hypothesis $\sigma_k \ge C_1 > 0$, then*

$$\int_{\Omega_\lambda} v^{-\gamma} dx < C \tag{15}$$

for any $\gamma < n - \frac{n-2k}{2}$ and some $C = C(D, \gamma, C_1)$ not depending on λ.

Proof. Follows easily from (13) because $\sigma_1, \ldots, \sigma_{k-1} > 0$. □

Claim 2. *The assertion of claim 1 is true without the additional hypothesis $\sigma_k \ge C_1 > 0$.*

Proof. Look at the term with $s = k$ in (13), since all the σ_{k-s} are positive by hypothesis, we get for any $\gamma < n - \frac{n-2k}{2}$

$$\int_{\Omega_\lambda} |\nabla v|^{2k} v^{-\gamma} dx \le C < \infty \tag{16}$$

where $C = C(\gamma, M)$. On the other hand, (13) for $k = 1$ and same γ gives

$$\int_{\Omega_\lambda} \sigma_1 v^{-\gamma} dx \le -c_{1-1}(\gamma) \int_{\Omega_\lambda} |\nabla v|^2 v^{-\gamma} dx + C \tag{17}$$

but we do not know the sign of the coefficient $(-c_{1-1})$. Still using Holder in (17) we arrive at

$$\int_{\Omega_\lambda} \sigma_1 v^{-\gamma} dx \le \tilde{C}_\epsilon \int_{\Omega_\lambda} |\nabla v|^{2k} v^{-\gamma} dx + \epsilon \int_{\Omega_\lambda} v^{-\gamma} dx + C \le C_\epsilon + \epsilon \int_{\Omega_\lambda} v^{-\gamma} dx \tag{18}$$

by (16). Taking $\epsilon = \epsilon(k, C_0, \gamma)$ small enough, the hypothesis $\sigma_1 \ge C_0 > 0$ gives

$$\int_{\Omega_\lambda} v^{-\gamma} dx \le C < \infty \quad \text{for all} \quad \gamma > n - \frac{n-2k}{2}$$

and C depends on D, γ, k, C_0 but not on λ. $\square$

Claim 3. *Theorem 1.1 follows.*

Proof. The integral estimate (15) gives that

$$\int_\Omega v_0^{-\gamma} dvol g_c \le C(\gamma) < \infty$$

for any $\gamma < n - \frac{n-2k}{2}$. This implies $\dim_{\mathcal{H}}(\partial\Omega) \le d(\Omega) \le \frac{n-2k}{2}$ as we wanted. Note that this part does not need the technical assumptions (5). $\square$

Claim 4. *If, in addition,* (5) *is satisfied, then there exists some* $\delta > 0$ *such that*

$$\int_{\Omega_\lambda} v^{-n+\frac{n-2k}{2}-\delta} dx < \infty \tag{19}$$

Proof. First of all, we have that $Ric^{g_v} \ge -C$ for some $C > 0$ as a consequence of the Ricci bound (1) for $k > 1$ and the hypothesis that the scalar curvature is bounded from below by a positive constant. This allows us to use Yau's gradient estimates (see for instance, [19], chapter VI, theorem 2.12), and so we have

$$|\nabla v| \le C \quad \text{in} \quad B_r \backslash \tilde{\Lambda} \subset \mathbb{R}^n \tag{20}$$

$$v^{-1}(x) \ge \frac{C}{\operatorname{dist}(x, \tilde{\Lambda})} \quad \text{for} \quad x \in B_r \backslash \tilde{\Lambda} \tag{21}$$

Now observe that (13) for $\gamma = n - \frac{n-2k}{2} + \delta$ will give:

$$\int_{\Omega_\lambda} \sigma_k v^{-\gamma} dx + \sum_{s=1}^{k-1} c_{k-s}(\gamma) \int_{\Omega_\lambda} \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} dx \leq C - c_{k-k} \int_{\Omega_\lambda} |\nabla v|^{2k} v^{-\gamma} dx \quad (22)$$

but now $-c_{k-k} = C(n,k)\delta > 0$, and $c_{k-s}(\gamma) > 0$ for δ small enough, $s = 1, \dots, k-1$. Using again gradient estimates (20)

$$\int_{\Omega_\lambda} |\nabla v|^{2k} v^{-\gamma} dx \lesssim \int_{\Omega_\lambda} |\nabla v|^{2(k-1)} v^{-\gamma} dx \quad (23)$$

Since $\sigma_1 \geq C_0 > 0$, together with (22) and (23) we arrive at

$$\begin{aligned} C_0 \int_{\Omega_\lambda} |\nabla v|^{2(k-1)} v^{-\gamma} dx \leq \int_{\Omega_\lambda} \sigma_1 |\nabla v|^{2(k-1)} v^{-\gamma} dx \leq C + C\delta \int_{\Omega_\lambda} |\nabla v|^{2k} v^{-\gamma} dx \\ \lesssim C + C\delta \int_{\Omega_\lambda} |\nabla v|^{2(k-1)} v^{-\gamma} dx. \end{aligned} \quad (24)$$

and thus, taking δ small enough we conclude from (24)

$$\int_{\Omega_\lambda} |\nabla v|^{2(k-1)} v^{-\gamma} dx < \infty \quad (25)$$

Putting together (23) and (25),

$$\int_{\Omega_\lambda} |\nabla v|^{2k} v^{-\gamma} dx < \infty$$

for $\gamma = n - \frac{n-2k}{2} + \delta$, and now proceeding as we did from (16) we achieve (19). □

Theorem 1.2 follows from this last claim.

Remark. The conclusion of theorem (1.2) is also true for $k = 1$ if we add the extra assumption $Ric \geq -C$.

4 An 'almost' divergence formula for σ_k

For a general $n \times n$ matrix A, consider its eigenvalues $\lambda_1, \dots, \lambda_n$ and construct the symmetric functions

$$\sigma_k(A) = \sum_{i_1 < \dots < i_k} \lambda_{i_1} \dots \lambda_{i_k}$$

Definition 4.1. *Denote $\sigma_k := \sigma_k(A)$; we can define the k^{th} Newton tensor as*

$$T^k = \sigma_k - \sigma_{k-1} A + \dots + (-1)^k A^k = \sigma_k I - T^{k-1} A \quad (26)$$

Take $\sigma_0 := 1$ and $T^0_{ij} := \delta_{ij}$. We can summarize a few well known properties in

Lemma 4.2 ([6],[17]). *We have*

a. $(n-k)\sigma_k(A) = trace(T^k)$

b. $(k+1)\sigma_{k+1}(A) = trace(AT^k)$

c. If $\sigma_1, \ldots, \sigma_k > 0$*, then* T^m *is positive definite for* $m = 1, \ldots k-1$.

d. If $\sigma_1, \ldots, \sigma_k > 0$*, then also*

$$\sigma_k \leq C_{n,k}{\sigma_1}^k$$

If we start with a manifold M with a metric $g_v = v^{-2}|dx|^2$, we can compute the Schouten tensor $\tilde{A}^{g_v}$ and the matrix $A = A^{g_v}$ at each point $x \in M$.

Remark. Here $\tilde{\partial}_j$ will denote the $j-th$ covariant derivative with respect to the g_v metric, and ∂_j denotes the usual Euclidean derivative. The strategy in the following is just to translate all the covariant derivatives into derivatives with respect to the Euclidean background metric, a task relatively straightforward due to the simple shape of the metric.

Lemma 4.3 ([20]). *Let* $A = A^{g_v}$*, then the Newton tensor* T^m *for* $m \leq n-1$ *defined as in* (26) *is divergence-free with respect to the metric* g_v*, i. e,*

$$\sum_j \tilde{\partial}_j T^m_{ij} = 0 \quad \textit{for all} \quad i$$

In this new metric $g = g_v = v^{-2}|dx|^2$, write $\sigma_k := \sigma_k(A^g)$ and compute:

- Christoffel symbols:

$$\Gamma^k_{ij} = \frac{1}{2}\sum_s g^{ks}\left(\partial_i g_{sj} + \partial_j g_{is} - \partial_s g_{ij}\right) = -v^{-1}\left[v_i\delta_{kj} + v_j\delta_{ik} - v_k\delta_{ij}\right] \tag{27}$$

- Covariant derivative of a (1,1)-tensor B_{ij}:

$$\tilde{\partial}_j B_{ij} = \partial_j B_{ij} - \sum_l \Gamma^l_{ij} B_{lj} + \sum_l \Gamma^j_{lj} B_{il} \tag{28}$$

- Schouten tensor:

$$\begin{aligned} \tilde{A}_{ij} &= v_{ij}v^{-1} - \tfrac{1}{2}|\nabla v|^2 v^{-2}\delta_{ij} \\ A_{ij} &= v^2\tilde{A}_{ij} = v_{ij}v - \tfrac{1}{2}|\nabla v|^2\delta_{ij} \end{aligned} \tag{29}$$

here v_i means the Euclidean partial derivative of v. We will prove

Lemma 4.4.

$$\sum_j \partial_j T^m_{ij} \quad = \quad -(n-m)\sigma_m v_i v^{-1} + n\sum_i T^m_{ij} v_i v^{-1} \quad \textit{for each} \quad i \tag{30}$$

$$m\sigma_m(A_g) \quad = \quad v\sum_{i,j}\partial_j\left(v_i T^{m-1}_{ij}\right) - n\sum_{i,j} T^{m-1}_{ij} v_i v_j + \tfrac{n-m+1}{2}\sigma_{m-1}|\nabla v|^2 \tag{31}$$

Remark. Expression (31) is the key point for the later proofs. It shows the 'almost" divergence structure for σ_m, that resembles the structure of some linear PDE, plus some terms of lower order $m-1$, that will be handled through an inductive formula developed in next section.

Proof. Using Lemma 4.3, and the definition of covariant derivative (28):

$$0 = \sum_j \tilde{\partial}_j T^m_{ij} = \sum_j \left(\partial_j T^m_{ij} - \sum_l \Gamma^l_{ij} T^m_{lj} + \sum_l \Gamma^j_{lj} T^m_{il} \right) \tag{32}$$

Now substitute (27),

$$0 = \sum_j \left(\partial_j T^m_{ij} + v^{-1} \left[v_i T^m_{jj} + v_j T^m_{ij} - v_j T^m_{ji} \right] - v^{-1} \left[n v_j T^m_{ij} + v_j T^m_{ij} - v_j T^m_{ij} \right] \right)$$

Lemma 4.2.a. gives the trace of T^m,

$$0 = \sum_j \partial_j T^m_{ij} + (n-m)\sigma_m v_i v^{-1} - n \sum_j T^m_{ij} v_j v^{-1},$$

this establishes (30). To get (31) we use b. and then a.

$$\begin{aligned} m\sigma_m &= \sum_{i,j} T^{m-1}_{ij} A_{ij} = \sum_{i,j} T^{m-1}_{ij} \left[v v_{ij} - \tfrac{1}{2} |\nabla v|^2 \delta_{ij} \right] \\ &= v \sum_{i,j} T^{m-1}_{ij} v_{ij} - \tfrac{n-m+1}{2} \sigma_{m-1} |\nabla v|^2 \end{aligned}$$

Substitute

$$\partial_j \left(T^{m-1}_{ij} v_i \right) = T^{m-1}_{ij} v_{ij} + v_i \partial_j T^{m-1}_{ij}$$

above and use (30) to get

$$\begin{aligned} m\sigma_m &= v \sum_{i,j} \partial_j \left(T^{m-1}_{ij} v_i \right) - v \sum_{i,j} v_i \left(\partial_j T^{m-1}_{ij} \right) - \tfrac{n-m+1}{2} \sigma_{m-1} |\nabla v|^2 \\ &= v \sum_{i,j} \partial_j \left(T^{m-1}_{ij} v_i \right) + (n-m+1)\sigma_{m-1} |\nabla v|^2 - n \sum_{i,j} v_i v_j T^{m-1}_{ij} \\ &\quad - \tfrac{n-m+1}{2} \sigma_{m-1} |\nabla v|^2 \\ &= v \sum_{i,j} \partial_j \left(v_i T^{m-1}_{ij} \right) - n \sum_{i,j} T^{m-1}_{ij} v_i v_j + \tfrac{n-m+1}{2} \sigma_{m-1} |\nabla v|^2 \end{aligned}$$

□

We will need a technical formula that constitutes the main induction step:

Lemma 4.5. *Let U be an open set in $\mathbb{R}^n$ with smooth boundary, and $v^{-1} \in \mathcal{C}^\infty(U)$. Denote $\nu = (\nu_1, \ldots, \nu_n)$ the outer normal to U and ds the $(n-1)$-area element. Then for $1 \leq s \leq k \leq n$ integers and any γ real number,*

$$\begin{aligned}
&\int_U \sum_{i,j} T_{ij}^{k-s} v_i v_j |\nabla v|^{2(s-1)} v^{-\gamma} dx \\
&= \left(1 + \tfrac{k-s}{2s}\right) \int_U \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} dx + \tfrac{s+n+1-\gamma}{2s} \int_U \sum_{i,j} T_{ij}^{k-s-1} v_i v_j |\nabla v|^{2s} v^{-\gamma} dx \\
&- \tfrac{n-k+s+1}{4s} \int_U \sigma_{k-s-1} |\nabla v|^{2(s+1)} v^{-\gamma} dx - \tfrac{1}{2s} \int_{\partial U} \sum_{i,j} T_{ij}^{k-s-1} v_i \nu_j |\nabla v|^{2s} v^{1-\gamma} ds
\end{aligned} \tag{33}$$

Proof. Using that

$$T_{ij}^m = \sigma_m \delta_{ij} - \sum_l A_{il} T_{lj}^{m-1},$$

and the expression for A_{ij} (29) we arrive at

$$\begin{aligned}
&\int_U \sum_{i,j} T_{ij}^{k-s} v_i v_j |\nabla v|^{2(s-1)} v^{-\gamma} dx \\
&\qquad = \int_U \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} dx - \int_U \sum_{i,j,l} A_{il} T_{lj}^{k-s-1} v_i v_j |\nabla v|^{2(s-1)} v^{-\gamma} dx \\
&\qquad = \int_U \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} dx - \int_U \sum_{i,j,l} v_{il} T_{lj}^{k-s-1} v_i v_j |\nabla v|^{2(s-1)} v^{1-\gamma} dx \\
&\qquad + \tfrac{1}{2} \int_U \sum_{ij} T_{ij}^{k-s-1} v_i v_j |\nabla v|^{2s} v^{-\gamma} dx
\end{aligned} \tag{34}$$

Now look at the middle term above: through integration by parts we get

$$\begin{aligned}
&- \int_U \sum_{i,j,l} v_{il} T_{lj}^{k-s-1} v_i v_j |\nabla v|^{2(s-1)} v^{1-\gamma} dx \\
&= -\tfrac{1}{2s} \int_U \sum_{j,l} \partial_l \left(|\nabla v|^{2s}\right) v_j T_{lj}^{k-s-1} v^{1-\gamma} dx \\
&= \tfrac{1-\gamma}{2s} \int_U \sum_{j,l} T_{lj}^{k-s-1} v_l v_j |\nabla v|^{2s} v^{-\gamma} dx + \tfrac{1}{2s} \int_U \sum_{j,l} \partial_l \left(v_j T_{lj}^{k-s-1}\right) |\nabla v|^{2s} v^{1-\gamma} dx \\
&- \tfrac{1}{2s} \int_{\partial U} \sum_{j,l} T_{lj}^{k-s-1} v_j \nu_l |\nabla v|^{2s} v^{1-\gamma} ds
\end{aligned} \tag{35}$$

Substitute (31) into (35)

$$
\begin{aligned}
&\int_U \sum_{i,j,l} v_{il} T_{lj}^{k-s-1} v_i v_j |\nabla v|^{2(s-1)} v^{1-\gamma} dx \\
&\qquad = \tfrac{1+n-\gamma}{2s} \int_U \sum_{i,j} T_{ij}^{k-s-1} v_i v_j |\nabla v|^{2s} v^{-\gamma} dx + \tfrac{k-s}{2s} \int_U \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} dx \\
&\qquad - \tfrac{n-k+s+1}{4s} \int_U \sigma_{k-s-1} |\nabla v|^{2(s+1)} v^{-\gamma} dx - \tfrac{1}{2s} \int_{\partial U} \sum_{j,l} T_{lj}^{k-s-1} v_j \nu_l |\nabla v|^{2s} v^{1-\gamma} ds
\end{aligned}
\tag{36}
$$

We get (33) by substituting (36) into (34). □

5 Proof of lemma 3.3

Integrate (31) over Ω_λ and use integration by parts:

$$
\begin{aligned}
k \int_{\Omega_\lambda} \sigma_k v^{-\gamma} = & \int_{\Omega_\lambda} \sum_{i,j} \partial_j \left(v_i T_{ij}^{k-1} \right) v^{1-\gamma} - n \int_{\Omega_\lambda} \sum_{i,j} T_{ij}^{k-1} v_i v_j v^{-\gamma} \\
& + \tfrac{n-k+1}{2} \int_{\Omega_\lambda} \sigma_{k-1} |\nabla v|^2 v^{-\gamma} \\
= & (\gamma - n - 1) \int_{\Omega_\lambda} \sum_{i,j} T_{ij}^{k-1} v_i v_j v^{-\gamma} + \tfrac{n-k+1}{2} \int_{\Omega_\lambda} \sigma_{k-1} |\nabla v|^2 v^{-\gamma} \\
& + \int_{\partial\Omega_\lambda} \sum_{i,j} T_{ij}^{k-1} v_i \nu_j v^{1-\gamma} ds,
\end{aligned}
\tag{37}
$$

where $\nu = (\nu_1, \ldots, \nu_n) = -\frac{\nabla v}{|\nabla v|}$ is the outer normal to $\partial\Omega_\lambda$. Now look at the boundary of Ω_λ

$$
\partial\Omega_\lambda = \partial B_r \cup \left\{ x \in B_r \backslash \tilde{\Lambda} : v^{-1}(x) = \lambda \right\}.
$$

but actually for $\lambda \geq D$ the two sets are disjoint because of $\tilde{\Lambda} \subset\subset B_r$ and the behavior of v, so we get two boundary terms in (37):

$$
\int_{\partial\Omega_\lambda} \sum_{i,j} T_{ij}^{k-1} v_i \nu_j v^{1-\gamma} ds = \int_{\partial B_r} \ldots + \int_{\{v^{-1}=\lambda\}} \ldots \tag{38}
$$

The first part can be controlled by a constant $C(D)$, not depending on λ. The second boundary term is of the form

$$
-\int_{\{v^{-1}=\lambda\}} \sum_{i,j} T_{ij}^{k-1} \frac{v_i v_j}{|\nabla v|} v^{1-\gamma} ds
$$

and it has a sign because T^{k-1} is positive definite. Consequently, from (37) we get

$$
k \int_{\Omega_\lambda} \sigma_k v^{-\gamma} \leq (\gamma - n - 1) \int_{\Omega_\lambda} \sum_{i,j} T_{ij}^{k-1} v_i v_j v^{-\gamma} + \tfrac{n-k+1}{2} \int_{\Omega_\lambda} \sigma_{k-1} |\nabla v|^2 v^{-\gamma} + C \tag{39}
$$

Fix k and $1 \le s \le k$ integer. Define

$$I_{k-s} := \left(\tfrac{\gamma-n}{s} - 1\right) \int_{\Omega_\lambda} \sum_{i,j} T_{ij}^{k-s} v_i v_j |\nabla v|^{2(s-1)} v^{-\gamma} + \tfrac{n-k+s}{2s} \int_{\Omega_\lambda} \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} \tag{40}$$

then we see that expression (39) is nothing but the first step:

$$k \int_{\Omega_\lambda} \sigma_k v^{-\gamma} \le I_{k-1} + C \tag{41}$$

For the general step s, substitute (33) into (40)

$$\begin{aligned} I_{k-s} &= \left(\tfrac{n-2k}{2s} + \tfrac{(\gamma-n)(k+s)}{2s^2}\right) \int_{\Omega_\lambda} \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} + \left(1 - \tfrac{\gamma-n}{s}\right) \tfrac{s+1}{2s} I_{k-s-1} \\ &+ \left(\tfrac{n-\gamma}{s} + 1\right) \tfrac{1}{2s} \int_{\partial\Omega_\lambda} \sum_{i,j} T_{ij}^{k-s-1} v_i \nu_j |\nabla v|^{2s} v^{1-\gamma} ds \end{aligned} \tag{42}$$

The boundary term (actually, the two boundary terms) can be estimated in an analogous manner as we did in (38), and as soon as $\frac{n-\gamma}{s} + 1 > 0$ we conclude from (42)

$$I_{k-s} \le \left(\tfrac{n-2k}{2s} + \tfrac{(\gamma-n)(k+s)}{2s^2}\right) \int_{\Omega_\lambda} \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} + \left(1 - \tfrac{\gamma-n}{s}\right) \tfrac{s+1}{2s} I_{k-s-1} + C \tag{43}$$

Call $a_{k-s} = -\left(\frac{n-2k}{2s} + \frac{(\gamma-n)(k+s)}{2s^2}\right)$, $s = 1, \ldots, k$. Substitute (43) into (41); an inductive process using (43) several times gives

$$\int_{\Omega_\lambda} \sigma_k v^{-\gamma} + \sum_{s=1}^{k} (a_{k-s} b_{k-s}) \int_{\Omega_\lambda} \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} \le C$$

for some constants b_{k-s} that are positive because $1 - \frac{\gamma-n}{s} > 0$ if $\gamma < n - \frac{n-2k}{2}$. Also, all the coefficients a_{k-s} are positive in this range of γ, so the lemma is proved.

6 Discussion about dimension $\frac{n-2k}{2}$

For $k = 1$ we have the dimension estimate $\dim_{\mathcal{H}}(\partial\Omega) \le \frac{n-2}{2}$. This is sharp, in the sense that we can construct examples with singular set of dimensions as close as we want to $\frac{n-2}{2}$. For instance, Mazzeo-Pacard [15] construct metrics of constant positive scalar curvature that are singular at any given disjoint union of smooth submanifolds of S^n of dimensions $0 < k_i \le \frac{n-2}{2}$.

The model problem is when the singular set is a meridian $\lambda = S^l \subset S^n$, here the construction is simple because:

Lemma 6.1. *$S^n \backslash S^l$ is conformal to $S^{n-l-1} \times \mathbb{H}^{l+1}$ with its standard metric. In this metric the Schouten tensor is diagonal and modulo a constant, its eigenvalues are* 1 *and* -1*, with multiplicities* $n-l-1$ *and* $l+1$*, respectively:*

$$A = \frac{1}{n-2}\left(Ric - \frac{R}{2(n-1)} g\right) = c_n \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \\ & & & & & -1 \end{pmatrix}$$

Proof. $S^n \backslash S^l$ transforms to $\mathbb{R}^n \backslash \mathbb{R}^l$ through stereographic projection. Taking coordinates $y \in \mathbb{R}^l$, $(r, \theta) \in \mathbb{R}^{n-l}$ polar, the flat metric in $\mathbb{R}^n$ is written as $g_0 = dr^2 + r^2 d\theta^2 + dy^2$. Consider now the conformal metric

$$g = \frac{1}{r^2} g_0 = d\theta^2 + \frac{dr^2 + dy^2}{r^2}$$

and this is the product metric on $S^{n-l-1} \times \mathbb{H}^{l+1}$. $\square$

In this section we will assume, without loss of generality, that $c_n = 1$.

This example has scalar curvature $R = n - 2l - 2$, constant, positive when $l < \frac{n-2}{2}$. So basically, it shows that the dimension estimate is sharp. However, for bigger k this example does not reach $\frac{n-2k}{2}$. Indeed, the results of this section will tell us that the best we can do in this particular example for fixed $k > 1$ and $n >> k$ is $\dim_{\mathcal{H}}(\Lambda) \sim \frac{n}{2} - O(\sqrt{n})$, $n \to \infty$. The answer to the question: *is* $\frac{n-2k}{2}$ *is sharp?* is not known so far, and maybe other type of constructions is needed.

Let's go back to $\mathbb{H}^{l+1} \times S^{n-l-1}$:

$$\sigma_r(\mathbb{H}^{l+1} \times S^{n-l-1}) = \sum_{i=0}^{r} \binom{n-l-1}{i} \binom{l+1}{r-i} (-1)^{r-i} \tag{44}$$

Fix $2 \leq k < \frac{n}{2}$ and define $P_r(l)$ to be the polynomial of degree m given by expression (44) in the variable l, for $r = 1, \ldots, k$. We seek

$$l_k = \sup \{ l \geq 0 : P_1(l), \ldots, P_k(l) > 0 \} \tag{45}$$

Lemma 6.2. *In this setting,*

$$l_k \leq l_2 \leq \frac{n-2}{2} - \frac{1}{2}\sqrt{n}.$$

Proof. Note that $P_1(l) > 0$ if an only if $l < \frac{n-2}{2}$. Now just look at P_2, a second order polynomial with roots $l = \frac{n-2+\sqrt{n}}{2}$ and $l = \frac{n-2-\sqrt{n}}{2}$. $\square$

Let's now look at the roots of P_k for $k \geq 2$ fixed and study the behavior of l_k when $n \to \infty$.

Proposition 6.3. *Fixed $k > 1$, let $n \to \infty$. Then*

$$l_k \sim \frac{n-2}{2} - 0(\sqrt{n})$$

more precisely

$$\frac{n}{2} - C(k)\sqrt{n} \leq l_k \leq \frac{n}{2} - \frac{2+\sqrt{n}}{2}$$

for some constant $C(k)$.

Proof. P_k it is an even or odd polynomial around $l = 1 - \frac{n}{2}$, according to the parity of k. Now change variables to $l = -1 + \frac{n}{2} + \tilde{l}$, so $\tilde{P}_k(\tilde{l})$ is symmetric (even or odd) around zero and

$$\tilde{P}_k(\tilde{l}) = \sum_{s=0}^{k} \binom{\frac{n}{2} - \tilde{l}}{s} \binom{\frac{n}{2} + \tilde{l}}{k-s} (-1)^{k-s} \tag{46}$$

Lemma 6.4 does not give an exact answer but it tells us that

$$l_k \geq \frac{n}{2} - O(\sqrt{n})$$

The other side inequality follows from lemma 6.2. □

Lemma 6.4. *For $k = 2m$ even, if the roots of $\tilde{P}_k$ are*

$$-r_1 \leq -r_2 \leq \ldots \leq -r_m \leq r_m \leq \ldots \leq r_1,$$

or for $k = 2m + 1$ odd, if $\tilde{P}_k$ has roots

$$-r_1 \leq -r_2 \leq \ldots \leq -r_m \leq 0 \leq r_m \leq \ldots \leq r_1,$$

then $r_1 = O(\sqrt{n})$, when $n \to \infty$, fixed k. This means,

$$r_1 \leq C(k)\sqrt{n} \quad \textit{for} \quad n \to \infty$$

Proof. Let's do k even (the odd case follows similarly). By definition,

$$\tilde{P}_k(\tilde{l}) = \prod_{i=1}^{m} \left(\tilde{l}^2 - r_i^2\right) = \tilde{l}^k + a_{k-2}\tilde{l}^{k-2} + \ldots + a_0$$

where

$$a_{k-2} = -\sum_{i=1}^{m} r_i^2 \tag{47}$$

On the other hand, we see from direct computation in (46) that

$$a_{k-2} = \frac{1}{4k!} A_k n^2 + O(n) \tag{48}$$

with

$$A_k = \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} \left[\binom{s}{2} - \binom{s}{1}\binom{k-s}{1} + \binom{k-s}{2} \right] \tag{49}$$

It will be proved in the appendix that $A_k = 0$, thus the lemma follows from (47) and (48). □

Remark. As an illustrative example, for $k = 3$,

$$l_3 = \frac{n - 2 - \sqrt{-2 + 3n}}{2}$$

and this is still far from $\frac{n-6}{2}$.

7 Case $n < 2k$

Proposition 7.1. *Let k be an integer such that $3 \leq n < 2k$, and let g be a complete metric on a domain $\Omega \subset S^n$, conformal to g_c with*

$$\sigma_1(A^g), \ldots, \sigma_k(A^g) \geq C_0 > 0$$

Then $\Omega = S^n$, i.e., there is no singular set.

Proof. The positivity of the k-curvatures implies that the Ricci tensor is positive definite because of the estimate (1). Now, by the Bonnet-Myers theorem Ω must be compact, and since Ω was by definition open, it must be the whole S^n. $\square$

Remark. When the dimension of the manifold is exactly $n = 2k$, estimate (1) only gives $Ric \geq 0$. Nevertheless, Guan-Viaclovsky-Wang proved, in the same paper, that actually the Ricci tensor is positive definite at each point. Thus is we start with a compact manifold M we can allow $n = 2k$:

Corollary 7.2. *Let (M^n, g) be a compact l.c.f. manifold with*

$$\sigma_1(A^g), \ldots, \sigma_k(A^g) > 0, \quad 3 \leq n \leq 2k$$

Then $\tilde{M}$ is compact and moreover the image of the developing map is $\phi(\tilde{M}) = S^n$ so the singular set $\Lambda = S^n \backslash \phi(\tilde{M})$ is empty.

A Appendix

Here we use a simple combinatorics argument to prove that

Lemma A.1. $A_k = 0$ *for all $k \in \mathbb{N}$, where A_k is defined as in* (49).

Remark. By convention, $\binom{a}{b} = 0$ if $a < b$.

Proof. Define

$$E_s^k = \binom{s}{2} - \binom{s}{1}\binom{k-s}{1} + \binom{k-s}{2}$$

so

$$A_k = \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} E_s^k$$

To understand the meaning of expression above, consider the following situation: fix $0 \leq s \leq k$. From a set K of k elements, pick s and assign them the value -1. Assign the value 1 to the rest $(k-s)$ elements. Now pick any two elements in K and multiply their values. Sum over all the possible combinations, that gives precisely the value of E_s^k.

The proof goes by induction on k. $k = 1$ and $k = 2$ are easily checked. Assume that $A_k = 0$, let's try to prove that

$$A_{k+2} := \sum_{r=0}^{k+2} \binom{k+2}{r} (-1)^{k+2-r} E_r^{k+2} = 0.$$

Following our interpretation, fix $0 \leq r \leq k+2$; we first need to pick r elements from the $k+2$. For that, we select first k of the $k+2$ and then we pick; three possibilities can happen:

1. All the r elements are among the selected k.
2. We pick $r-1$ elements from those k, and then one of the two that were left. Call $r-1=s$.
3. Pick $r-2$ elements from those k, and the two that were left apart. Denote $r-2=t$.

Now we need to pick the pairs of ones and minus ones, and sum over all of them. With this observations in sight we conclude:

$$\begin{aligned}\binom{k+2}{k}^{-1} A_{k+2} = &\sum_{r=0}^{k}(-1)^{k+2-r}\left[\binom{r}{2}-\binom{r}{1}\binom{k+2-r}{1}+\binom{k+2-r}{2}\right]\\ &+\sum_{s=0}^{k}2\binom{k}{s}(-1)^{k-s+1}\left[\binom{s}{2}+\binom{k-s}{2}-\binom{s}{1}\binom{k-s}{1}-1\right]\\ &+\sum_{t=0}^{k}\binom{k}{t}(-1)^{k-t}\left[\binom{t}{2}+\binom{k-t}{2}-\binom{t}{1}\binom{k-t}{1}+4t-2k+1\right]\end{aligned}$$

To finish, we just need to relate expression above to A_k:

$$\begin{aligned}\binom{k+2}{k}^{-1} A_{k+2} = &\sum_{r=0}^{k}\binom{k}{r}(-1)^{k-r}\left[E_r^k+2k-4r+1\right]\\ &-\sum_{s=0}^{k}2\binom{k}{s}(-1)^{k-s}\left[E_s^k-1\right]\\ &+\sum_{t=0}^{k}\binom{k}{t}(-1)^{k-t}\left[E_t^k-2k+4t+1\right]\end{aligned}$$

Thus, $A_{k+2}=0$ because of our induction hypothesis. □

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