Removability of singularities for a class of fully non-linear elliptic equations

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1 Introduction

Let B be the unit ball in $\mathbb{R}^n$, $n \geq 3$, carrying a conformally flat Riemannian metric $g = g_v = v^{-2}|dx|^2$ for some $v^{-1} > 0$; here $|dx|^2$ denotes the Euclidean metric. Construct the Schouten tensor as

$$\tilde{A}^g = \frac{1}{n-2}\left(Ric^g - \frac{1}{2(n-1)}R^g g\right)$$

where Ric and R are the Ricci tensor and the scalar curvature of the metric, respectively. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of the matrix $A^g = g^{-1}\tilde{A}^g$ at each point, and compute its k^{th} elementary symmetric function:

$$\sigma_k := \sigma_k(A^g) = \sum_{i_1<\dots<i_k} \lambda_{i_1}\dots\lambda_{i_k}$$

In the metric g_v the Schouten tensor becomes

$$A^{g_v} = v(D^2 v) - \frac{1}{2}|\nabla v|^2 I$$

and thus σ_k curvatures give rise to interesting elliptic fully non-linear equations of second order. As an example, for $k = 2$,

$$2\sigma_2(v) = \left[(\Delta v)^2 - |D^2 v|^2\right] v^2 - (n-1)\Delta v|\nabla v|^2 v + \frac{n(n-1)}{4}|\nabla v|^4$$

The problem is elliptic (but in general, not uniformly elliptic) in the positive cone

$$\Gamma_k^+ = \{v : \sigma_1, \dots, \sigma_k(v) > 0\}$$

These type of non-linear equations have an underlying divergence structure:

$$m\sigma_m = v\partial_j\left(v_i T_{ij}^{m-1}\right) - nT_{ij}^{m-1} v_i v_j + \tfrac{n-m+1}{2}\sigma_{m-1}|\nabla v|^2 \qquad (1)$$

where T_{ij} denotes the Newton tensor. The non-divergence terms are of lower order and indeed, they can be dealt through an inductive process. These facts were explored in the previous paper [12], and are described in section 2.

In this paper we are mostly interested in understanding the local behavior of singularities of the constant σ_k-curvature equation:

$$\begin{aligned} &\sigma_k(v) = 1 \quad \text{on} \quad B\backslash\Lambda \\ &v > 0, \quad v \in \Gamma_k^+, \quad n > 2k \end{aligned} \tag{2}$$

where $\Lambda \subset B$, the singular set, is a compact subset of the unit ball in $\mathbb{R}^n$.

The study of σ_k curvatures comes from conformal geometry: in fact, the full Riemannian curvature tensor of a manifold (M, g) can be computed in terms of the Schouten tensor through the formula (see [2]):

$$Riem = W + \tilde{A} \owedge g,$$

where $\owedge$ is the Kulkarni-Nomizu product, and W the Weyl tensor, a conformal invariant. In particular, the scalar curvature is simply

$$\sigma_1 = \lambda_1 + \ldots + \lambda_n = \frac{1}{2(n-1)} R$$

From the point of view of calculus of variations, fixed (M, g_0) a locally conformally flat manifold of dimension $n > 2k$, we have that $\sigma_k(v) =$ constant is precisely the Euler-Lagrange equation for the functional

$$\mathcal{F}_k(g) = (vol(g))^{-\frac{n-2k}{n}} \int_M \sigma_k(A^g) dvol_g, \tag{3}$$

where we take the infimum over all the metrics $g_v = v^{-2} g_0$, $v > 0$. This functional was first introduced by Viaclovsky [25], and it generalizes the Yamabe functional.

The sign of σ_k plays an important role understanding singularities. Indeed, in [12] we looked at the singular set $\tilde{\Lambda}$ of a complete metric g on $S^n\backslash\tilde{\Lambda}$, conformal to the standard metric g_c, for the case $n > 2k$. Under some positivity assumptions on $\sigma_1, \ldots, \sigma_k$ we were able to give an upper estimate for the Hausdorff dimension of this singular set: $\dim_{\mathcal{H}}(\tilde{\Lambda}) \leq \frac{n-2k}{2}$, together with some topological consequences. If we translate the problem from S^n to $\mathbb{R}^n$ through stereographic projection, actually need to study $\sigma_k(v)$ for some $g_v = v^{-2}|dx|^2$ defined on $\mathbb{R}^n\backslash\Lambda$. When the dimension is $n < 2k$ we saw that the singular set $\tilde{\Lambda}$ must be empty.

Here we are interested instead in the behavior of solutions of (2) near the singularity. We will give a sufficient condition so that the function v^{-1} is bounded. This condition is the smallness of volume of the metric g_v; in our notation, $vol(g_v) = \int_B v^{-n} dx$. The methods we use are integral estimates and thus, we do not require a priori smoothness of the function v^{-1}. Note that we will be dealing mostly with the case $n > 2k$.

First look at the isolated singularity of

$$\begin{aligned} &\sigma_k := \sigma_k(v) = 1 \quad \text{in} \quad B\backslash\{0\} \\ &v \in \Gamma_k^+, \quad v > 0, \quad n > 2k \end{aligned} \tag{4}$$

Theorem 1.1. *Let $v^{-1} \in \mathcal{C}^3(B\backslash\{0\})$ be a solution of (4) such that*

$$\int_{\rho<|x|<1} v^{-n} < a \quad \textit{independently of} \quad \rho \tag{5}$$

for some $a > 0$ small enough. Then

$$\int_{\rho<|x|<\frac{1}{2}} \sigma_{k-s}|\nabla v|^{2s} v^{-n} dx \leq C(a)$$

for all $s = 1, \ldots, k$, for some constant $C(a)$ not depending on $\rho > 0$. In particular, $v^{-\frac{n}{2k}} \in W^{1,2k}\left(B_{1/2}\right)$ and v^{-1} belongs to $L^{\tilde{q}}(B_{1/2})$ for some $\tilde{q} > n$.

Theorem 1.2. *Under the same hypothesis as the previous theorem, if $0 < R < 1/2$, then*

$$\|v^{-1}\|_{L^\infty(B_R)} \leq \frac{C}{R^{n/p}} \|v^{-1}\|_{L^p(B_{2R})} \tag{6}$$

for all

$$p > (n-2k)\frac{k}{k+1}$$

Remark. Note that because of the -somewhat arbitrary- notation $g = v^{-2}|dx|^2$, we are looking at regularity of the function v^{-1}.

Corollary 1.3. *In the same hypothesis as theorem 1.1, then also $v^{-1} \in W^{2,k}(B_{\frac{1}{4}})$.*

Corollary 1.4. *The hypothesis $v \in \mathcal{C}^3(B\backslash\{0\})$ can be relaxed to $v^{-\frac{n}{k}} \in W^{2,k}_{loc}(B\backslash\{0\})$ and $v^{-\frac{n}{2k}} \in W^{1,k}_{loc}(B\backslash\{0\})$ in theorems 1.1 and 1.2 (not in corollary 1.3).*

In the case $k = 1$ the complete picture is understood. Indeed, substituting $v^{-2} = u^{\frac{4}{n-2}}$, the equation $\sigma_1(v) = 1$ is equivalent to the constant scalar curvature equation for g_u

$$-\Delta u = u^{\frac{n+2}{n-2}} \tag{7}$$

Caffarelli-Gidas-Spruck [4] have given a complete local characterization of isolated singularities of this equation. Basically, if u is a positive solution of (7) on $B\backslash\{0\}$, then either the singularity is removable or the function has a determined asymptotic behavior at the origin

$$\frac{C_1}{|x|^{\frac{n-2}{2}}} \leq u(x) \leq \frac{C_2}{|x|^{\frac{n-2}{2}}} \quad \text{when} \quad |x| \to 0 \tag{8}$$

This is the type of result hoped for the σ_k problem; the present paper tries to give a step forward towards this classification of singularities.

From another point of view, the radial solutions of $-\Delta u = u^{\frac{n+2}{n-2}}$ in $\mathbb{R}^n\backslash\{0\}$ have been well understood ([23]). Viaclovsky [25], in some cases, and then a recent paper of Chang-Han-Yang [7], for the complete picture, have computed the radial singular solutions of $\sigma_k(v) = 1$ defined on an annulus. Although the computations are more delicate, it turns out radial solutions with an isolated singularity still behave asymptotically like (8) when $n > 2k$. This seems to be a more interesting case because when $n < 2k$, there are no singular radial solutions.

The global problem in $\mathbb{R}^n$ -that is equivalent to having an isolated singularity at infinity-

$$\sigma_k(v) = 1 \quad \text{in} \quad \mathbb{R}^n$$

has been well understood by Chang-Gursky-Yang [5], [6], Li-Li [18]. Indeed, a positive $\mathcal{C}^2(\mathbb{R}^n)$ solution must be of the form

$$v^{-1}(x) = c(n,k)\frac{a}{1 + a^2|x - \bar{x}|^2}$$

for some $a > 0$, $\bar{x} \in \mathbb{R}^n$, i.e., it comes from the standard metric on S^n (or its image under a conformal diffeomorphism).

Adding the finite volume condition to the theorem above, we are able to remove the a-priori smoothness assumption, and we can give the related result:

Theorem 1.5. *Let $p_1, \dots, p_N$ be a finite number of points, and v a solution of*

$$\begin{aligned} &\sigma_k(v) = 1 \quad in \quad \mathbb{R}^n \backslash \{p_1, \dots, p_N\} \\ &v > 0, \quad v \in \Gamma_k^+, \quad n > 2(k+1) \end{aligned} \tag{9}$$

satisfying $v^{-1} \in \mathcal{C}^3(\mathbb{R}^n \backslash \{p_1, \dots, p_N\})$ with finite volume $\int_{\mathbb{R}^n} v^{-n} < \infty$. Then the point singularities are removable and v comes from a conformal diffeomorphism of the sphere through stereographic projection, i.e., there exist $\bar{x} \in \mathbb{R}^n$, $a > 0$ such that

$$v^{-1}(x) = c(n,k)\frac{a}{1 + a^2|x - \bar{x}|^2}$$

Corollary 1.6. *The $v^{-1} \in \mathcal{C}^3(\mathbb{R}^n \backslash \{p_1, \dots, p_N\})$ assumption in the theorem above can be relaxed to $v^{-1} \in L^\infty_{loc} \cap W^{2,k+1}_{loc} \cap W^{1,2(k+1)}_{loc}(\mathbb{R}^n \backslash \{p_1, \dots, p_N\})$, and $D^3 v$ well defined in $\mathbb{R}^n \backslash \{p_1, \dots, p_N\}$.*

Remark. The assumption $n > 2(k+1)$ is a technical one and we expect the theorem to be true for $n > 2k$. In particular, for $k = 2$, the theorem was proved for $n \geq 5$ by Chang-Gursky-Yang [6]. Note that in their proof, the volume finiteness requirement is not needed for the case $n = 5$.

Note that Schoen [22] constructed a complete metric on $S^n \backslash \{p_1, \dots, p_N\}$ with constant scalar curvature. In particular, the previous theorem gives a partial inverse: there are no singular metrics on the sphere of finite volume with constant σ_k curvature.

As we mentioned before, the crucial idea in the proofs is the understanding the 'almost' divergence structure of the equation, (1). This summarized in section 2. With those we are able to prove a Sobolev type inequality of the form

$$\int \sigma_k dvol_{g_v} \gtrsim \sum_{s=1}^{k} \int \sigma_{k-s} |\nabla v|^{2s} dvol_{g_v}$$

A simple consequence of above formula is

$$\int \sigma_k dvol_{g_v} \geq C \ (vol(g_v))^{\frac{n-2k}{n}}$$

for functions in the positive cone Γ_k^+. This type of geometric inequalities has been developed in conformal geometry problems to understand the minimization of the functional (3). Note the related work of Guan-Wang [13]. For Hessian equations, $\sigma_k = \sigma_k(D^2 v)$, there exist similar Sobolev inequalities (see [3] for a reference), however, it seems that the structure of $\sigma_k(A^{g_v})$ is easier to understand that the one of $\sigma_k(D^2 v)$.

Now, this inequality allows to adapt the Moser iteration scheme for equations in divergence form, together with an inductive study of the errors. Here is where the small volume hypothesis is used in a crucial way. This proves the L^∞ estimate of theorem 1.2 (section 4).

The L^∞ estimates for $n = 4$, $k = 2$ have been obtained independently by Han [15], using also a divergence formula of the type (1). In fact, σ_k in dimension $n = 2k$ can be written in a purely divergence form without lower order terms.

Section 5 deals with the proof of theorem 1.5. The idea is to refine a standard Obata type argument, together with the L^∞ estimates obtained. The main improvement is that it does not require the gradient estimates by Guan-Wang [14] and thus, some smoothness hypothesis can be removed.

The same methods can be used to treat the more general problem (2), if we give some capacity conditions on the singular set Λ. The classical notion of capacity $c_{k,p}(\Lambda)$ was introduced to study linear and quasilinear PDE (a complete reference can be found in [19]). In particular, for the Laplacian problem (7) the Newtonian capacity $c_{1,2}$ is the suitable one; see for instance, Chen-Lin [8]. For fully nonlinear equations a new notion of capacity $\tilde{c}$ is required, this is done in section 6.

For Hessian equations of the type $\sigma_k(D^2 v)$ Trudinger-Wang [24], Labutin [16] considered a related notion of capacity in terms of potential theory, however, it is not known if this notion is equivalent to $\tilde{c}$.

Theorem 1.7. *Let $\Lambda \subset B_R \subset \mathbb{R}^n$ be a compact set, $R < 1$, with capacity*

$$\tilde{c}_{k,p}(\Lambda, B_R) = 0$$

for a given $2k < p \leq n$. Let v^{-1} in L^q for some

$$q \geq n \quad \text{and} \quad q > (n-2k)\frac{k}{k+1}\left(\frac{p}{p-2k}\right)$$

be a solution of (2) with

$$\|v^{-1}\|^n_{L^n} < a \tag{10}$$

for some $a > 0$ small enough. Then v^{-1} belongs to $L^{\tilde{q}}$ for some $\tilde{q} > n$ in a smaller ball. Also,

$$\|v^{-1}\|_{L^\infty(B_R)} \leq \frac{C}{R^{n/p}} \|v^{-1}\|_{L^p(B_{2R})} \tag{11}$$

for all

$$p > (n-2k)\frac{k}{k+1}$$

In particular, we have the analogous to theorem 1.5:

Corollary 1.8. *Let Λ be a finite disjoint union of compact, closed, smooth submanifolds Λ_i of dimensions $0 \leq k_i < \frac{n(n-2k)}{n+2k^2}$, and v a solution of*

$$\begin{aligned} &\sigma_k(v) = 1 \quad on \quad \mathbb{R}^n \backslash \Lambda \\ &v > 0, \quad v \in \Gamma_k^+, \quad n > 2(k+1) \end{aligned} \tag{12}$$

satisfying $v^{-\frac{n}{k}} \in W^{2,k}_{loc}(\mathbb{R}^n \backslash \Lambda)$ and $v^{-\frac{n}{2k}} \in W^{1,2k}_{loc}(\mathbb{R}^n \backslash \Lambda)$ with finite volume $\int_{\mathbb{R}^n} v^{-n} < \infty$, $D^3 v$ well defined on $\mathbb{R}^n \backslash \Lambda$. Then the singularities are removable and v comes from a conformal diffeomorphism of the sphere through stereographic projection, i.e., there exist $\bar{x} \in \mathbb{R}^n$, $a > 0$ such that

$$v^{-1}(x) = c(n,k) \frac{a}{1 + a^2 |x - \bar{x}|^2}$$

Remark. Note that Mazzeo-Pacard [20] have constructed a positive constant scalar curvature metric on S^n that is singular exactly along a finite disjoint union of smooth submanifolds Λ_i of dimensions $0 \leq k_i \leq \frac{n-2}{2}$.

The main open problem now is to get a complete classification of singularities for for (4) in the manner of Caffarelli-Gidas-Spruck [4]. Nevertheless, this classification is true for *subcritical* equations. Indeed, in the forthcoming paper [11] we prove:

Theorem 1.9. *Fix $\alpha \in (0, \alpha_0)$. Let v be a solution of*

$$\begin{aligned} &\sigma_k(v) = v^{\alpha} \quad in \quad B \backslash \{0\} \\ &v > 0, \quad v \in \Gamma_k^+ \end{aligned}$$

for α_0 small enough given in the proof, $n > 2(k+1)$, with $v^{-1} \in \mathcal{C}^3(B \backslash \{0\})$. If the function v^{-1} is not bounded near the origin, then there exists C_1, C_2 positive constants such that

$$\frac{C_1}{|x|^{\frac{2k}{2k-\alpha}}} \leq v^{-1}(x) \leq \frac{C_2}{|x|^{\frac{2k}{2k-\alpha}}} \quad when \quad |x| \to 0$$

The appendix contains some remarks about the relation between the PDE's we address in this paper and Hessian-type equations.

2 Algebraic properties of σ_k

For a general $n \times n$ matrix A, consider its eigenvalues $\lambda_1, \ldots, \lambda_n$, and construct the symmetric functions σ_k. The Newton tensor given by

$$T^k = \sigma_k - \sigma_{k-1} A + \ldots + (-1)^k A^k = \sigma_k I - T^{k-1} A \tag{13}$$

By definition, take $\sigma_0 := 1$ and $T^0_{ij} := \delta_{ij}$. Let us mention the work of Gårding [9], Reilly [21], Viaclovsky [25]. Some well known properties are summarized in the following lemma:

Lemma 2.1 ([9], [21]). *For A, T^k as above,*

a. $(n-k)\sigma_k = trace(T^k)$

b. $(k+1)\sigma_{k+1} = trace(AT^k)$

c. If $\sigma_1, \ldots, \sigma_k > 0$*, then* T^m *is positive definite for* $m = 1, \ldots k-1$.

d. If $\sigma_1, \ldots, \sigma_k > 0$*, then also*

$$\sigma_k \leq C_{n,k}(\sigma_1)^k$$

The key point is expression (15): it shows the 'almost' divergence structure for σ_m, that resembles the structure of some linear PDE, plus some terms of lower order $m-1$, that will be handled through the inductive formula (16). The proof of the following two lemmas can be found in the previous work [12]. Note that all the integrals are with respect to dx.

Lemma 2.2. *For a metric* $g_v = v^{-2}|dx|^2$*, if* $A = A^{g_v}$*, then*

$$A_{ij} = v_{ij}v - \tfrac{1}{2}|\nabla v|^2\delta_{ij} \tag{14}$$

$$m\sigma_m(A_g) = v\sum_{i,j}\partial_j\left(v_i T_{ij}^{m-1}\right) - n\sum_{i,j}T_{ij}^{m-1}v_iv_j + \tfrac{n-m+1}{2}\sigma_{m-1}|\nabla v|^2 \tag{15}$$

Lemma 2.3. *Let U be a domain in $\mathbb{R}^n$, $v^{-1} \in \mathcal{C}^\infty(U)$ and $\varphi \in \mathcal{C}_0^\infty(U)$ a smooth cutoff. Then for $1 \leq s \leq k \leq n$ integers and γ any real number,*

$$\begin{aligned}
&\int_U \sum_{i,j} T_{ij}^{k-s}v_iv_j|\nabla v|^{2(s-1)}\varphi^{2k}v^{-\gamma} \\
&= \left(1+\tfrac{k-s}{2s}\right)\int_U \sigma_{k-s}|\nabla v|^{2s}\varphi^{2k}v^{-\gamma} + \tfrac{s+n+1-\gamma}{2s}\int_U\sum_{i,j}T_{ij}^{k-s-1}v_iv_j|\nabla v|^{2s}\varphi^{2k}v^{-\gamma} \\
&-\tfrac{n-k+s+1}{4s}\int_U \sigma_{k-s-1}|\nabla v|^{2(s+1)}\varphi^{2k}v^{-\gamma} + \tfrac{k}{s}\int_U\sum_{i,j}T_{ij}^{k-s}v_j\varphi_i|\nabla v|^{2(s-1)}\varphi^{2k-1}v^{1-\gamma}
\end{aligned} \tag{16}$$

3 A Sobolev-type inequality for σ_k

Using the ingredients developed in the last section we can give an integral expression (17) relating σ_k to smaller σ_{k-s}. The proof of (17) follows the ideas in [12], but the difference (and main difficulty) is to get the coefficients in front of the integrals with the right sign.

Proposition 3.1. *Let U be a domain in $\mathbb{R}^n$, $n > 2k$. Assume that $v^{-1} \in \mathcal{C}_0^3(U)$, $v \in \Gamma_k^+$ and fix $\gamma \in \mathbb{R}$. Then, fixed $\{\alpha_{k-s}\}_{s=1}^{k-1}$ positive real numbers, there exists a decomposition*

$$k \int_U \sigma_k v^{-\gamma} = \sum_{s=1}^{k-1} d_{k-s}\mathcal{D}_{k-s} + \sum_{s=1}^{k-1} c_{k-s} \int_U \sigma_{k-s}|\nabla v|^{2s} v^{-\gamma} + c_{k-k} \int_U |\nabla v|^{2k} v^{-\gamma}, \quad (17)$$

where

$$\mathcal{D}_{k-s} = \left(-\tfrac{s+n-\gamma}{s} + \alpha_{k-s}\right) \int_U \sum_{i,j} T_{ij}^{k-s} v_i v_j |\nabla v|^{2(s-1)} v^{-\gamma}$$

for some constants d_{k-s}, $c_{k-s} \in \mathbb{R}$; c_{k-s} depending on the α's. Moreover,

1. $d_{k-s} > 0$
2. $\mathcal{D}_{k-s} \geq 0 \quad$ *if* $\quad \alpha_{k-s} \geq \frac{s+n-\gamma}{s}$
3. $c_{k-s} > 0 \quad$ *if* $\quad 0 < \alpha_{k-s} < \frac{n-k+s}{k+s}$

Remark. For $\gamma > n - \frac{n-2k}{k+1}$, we can pick some $\{\alpha_{k-s}\}_{s=1}^{k-1}$ such that both conditions *2.*, *3.* are satisfied at the same time: $c_{k-s} > 0$, $\mathcal{D}_{k-s} \geq 0$.

If v^{-1} does not have compact support, we can compute in the same fashion:

Corollary 3.2. *Let $v^{-1} \in \mathcal{C}^3(U)$, $v > 0$, $v \in \Gamma_k^+$, $n > 2k$. Then for all $\varphi \in \mathcal{C}_0^\infty(U)$,*

$$\int_U \sigma_k \varphi^{2k} v^{-\gamma} \geq \sum_{s=1}^{k} c_{k-s}(\gamma) \int_U \sigma_{k-s}|\nabla v|^{2s} \varphi^{2k} v^{-\gamma} + E(\varphi) \quad (18)$$

where

$$E(\varphi) \lesssim \sum_{s=1}^{k} \left| \int_U \sum_{i,j} T_{ij}^{k-s} v_j \varphi_i |\nabla v|^{2(s-1)} \varphi^{2k-1} v^{1-\gamma} \right|$$

and all the coefficients

$$c_{k-s}(\gamma) > 0 \quad \textit{for all} \quad \gamma > n - \frac{n-2k}{k+1}$$

The proof of above corollary will be postponed until the end of the section.

Corollary 3.3. *If $v^{-1} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $v > 0$, $v \in \Gamma_k^+$, $n > 2k$, then*

$$\int \sigma_k v^{-n} \geq \sum_{s=1}^{k} C_{k-s} \int \sigma_{k-s} |\nabla v|^{2s} v^{-n}$$

where $C_{k-s} > 0$ for all $s = 1, \ldots, k$. In particular, denoting $g_v = v^{-2}|dx|^2$,

$$\int \sigma_k(A^{g_v}) dvol_{g_v} \geq C \ (vol(g_v))^{\frac{n-2k}{n}} \quad (19)$$

where C depends on k, n but not on v, and $vol(g_v) = \int v^{-n} dx$.

Proof. Take $\gamma = n$ in proposition 3.1, and $\alpha_{k-s} = \frac{s+n-\gamma}{s}$ so that $\mathcal{D}_{k-s} = 0$. For the last part, simply use the Sobolev embedding $W^{1,2k} \hookrightarrow L^{p^*}$ with $\frac{1}{p^*} = \frac{1}{2k} - \frac{1}{n}$,

$$\int |\nabla v|^{2k} v^{-n} = C \int \left|\nabla \left(v^{-\frac{n-2k}{2k}}\right)\right|^{2k} \geq C \left(\int v^{-n}\right)^{\frac{n-2k}{n}}$$

□

Remark. This method does not give best constant in (19). However, in can be proved that the infimum of (3) is achieved at the sphere, through a gluing method in analogy to the Yamabe problem (see Lee-Parker [17] for a survey in the Yamabe problem). Note the work Guan-Wang [13].

Remark. As we have mentioned, this inequality resembles some isoperimetric inequalities for Hessian equations, $\sigma_k(D^2 v)$ (see [3] for a reference). However, it is precisely the special structure of the Schouten tensor $A_{ij}^{g_v}$ what allowed us to prove the inequality - the understanding of the lower orders played a crucial role.

Proof. of proposition 3.1: Fix $\gamma \in \mathbb{R}$, integrate expression (15) over U and use integration by parts:

$$\begin{aligned} k \int \sigma_k v^{-\gamma} &= \int \sum_{i,j} \partial_j \left(v_i \sum_{i,j} T_{ij}^{k-1} \right) v^{1-\gamma} - n \int \sum_{i,j} T_{ij}^{k-1} v_i v_j v^{-\gamma} \\ &+ \tfrac{n-k+1}{2} \int \sigma_{k-1} |\nabla v|^2 v^{-\gamma} \\ &= (-1+\gamma-n) \int \sum_{i,j} v_i v_j T_{ij}^{k-1} v^{-\gamma} + \tfrac{n-k+1}{2} \int \sigma_{k-1} |\nabla v|^2 v^{-\gamma} \end{aligned} \tag{20}$$

We could substitute directly expression (16) into (20) inductively. However, to get a precise control on the sign of the coefficients a more careful computation is needed and so we split in the following manner: for $s = 1, \ldots, k$, let

$$\begin{aligned} \mathcal{A}_{k-s} &= -\alpha_{k-s} \int \sum_{i,j} T_{ij}^{k-s} v_i v_j |\nabla v|^{2(s-1)} v^{-\gamma} + \tfrac{n-k+s}{2s} \int \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} \\ \mathcal{D}_{k-s} &= \left(-\tfrac{s+n-\gamma}{s} + \alpha_{k-s}\right) \int \sum_{i,j} T_{ij}^{k-s} v_i v_j |\nabla v|^{2(s-1)} v^{-\gamma} \end{aligned}$$

Substitute (16) into $\mathcal{A}_{k-s}$ to prove the induction step:

$$\begin{aligned} \mathcal{A}_{k-s} &= \left[\tfrac{n-k+s}{2s} - \alpha_{k-s}\left(1 + \tfrac{k-s}{2s}\right)\right] \int \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} \\ &+ \alpha_{k-s} \tfrac{s+1}{2s} \left(-\tfrac{s+n-\gamma+1}{s+1} \int \sum_{i,j} T_{ij}^{k-s-1} v_i v_j |\nabla v|^{2s} v^{-\gamma} \right) \\ &+ \alpha_{k-s} \tfrac{s+1}{2s} \left(\tfrac{n-k+s+1}{2(s+1)} \int \sigma_{k-s-1} |\nabla v|^{2(s+1)} v^{-\gamma} \right) \\ &= \tilde{c}_{k-s} \int \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} + \tilde{d}_{k-s} \left(\mathcal{A}_{k-s-1} + \mathcal{D}_{k-s-1}\right) \end{aligned} \tag{21}$$

for $\tilde{c}_{k-s} = \frac{n-k+s}{2s} - \alpha_{k-s}\left(1 + \frac{k-s}{2s}\right)$ and some constants $\tilde{d}_{k-s} > 0$. Now (17) follows by substituting (21) into (20) inductively.

Note that T^{k-s} is positive definite. We want $\tilde{c}_{k-s} > 0$, i.e.,

$$0 < \alpha_{k-s} < \tfrac{n-k+s}{k+s}$$

We also want $\mathcal{D}_{k-s} \geq 0$, i.e.,

$$\alpha_{k-s} \geq \tfrac{s+n-\gamma}{s}$$

□

Proof. of corollary 3.2: We just need to take into account the terms

$$\int_U \sum_{i,j} T_{ij}^{k-s} v_j \varphi_i |\nabla v|^{2(s-1)} \varphi^{2k-1} v^{1-\gamma}$$

that appear in the integration by parts (20) and (16) when there is a cutoff function φ involved. □

4 Moser iteration argument

Here we give the proofs of theorems 1.1 and 1.2. All the integrations will be Euclidean over the unit ball B unless it is written otherwise. Call $\chi = \frac{n}{n-2k} > 1$.

First note that v^{-1} is bounded from below by a positive constant, because of superharmonicity (lemma 6.5 for $\Lambda = \{0\}$).

Proof. of Theorem 1.1: The proof is basically a Moser-Trudinger iteration, using expression (18) with the right test function: fix $0 < \rho < 1$ small and let η be smooth cutoff such that

$$\eta = \begin{cases} 1 & \text{if} \quad 0 < |x| < \rho \\ 0 & \text{if} \quad 2\rho < |x| \end{cases} \tag{22}$$

and $\phi \in \mathcal{C}_0^\infty(B)$. Now $\varphi = (1-\eta)\phi$ has compact support on $B\backslash\{0\}$. Fix $\delta, \beta > n - \frac{n-2k}{k+1}$ real numbers, $m \in \mathbb{N}$ and take

$$V = V_m = \min\{v^{-\beta}, mv^{-\delta}\}$$

Call $B^\beta = \{x : V = v^{-\beta}\}$, $B^\delta = \{x : V = mv^{-\delta}\}$. We would like to use $V\varphi^{2k}$ as a test function. However, we need to rewrite the proof of (17) and (18) for this new function V instead of just $v^{-\gamma}$. Basically, the inductive process follows the same way just by taking into account the different subsets B^β, B^δ. So we have the analogous to (18)

$$\begin{aligned} &E(\varphi) + \int_B \sigma_k V \varphi^{2k} \\ &\quad \gtrsim \sum_{s=1}^k c_{k-s}(\beta) \int_{B^\beta} \sigma_{k-s} |\nabla v|^2 \varphi^{2k} v^{-\beta} + \sum_{s=1}^k d_{k-s}(\delta) m \int_{B^\delta} \sigma_{k-s} |\nabla v|^2 \varphi^{2k} v^{-\delta} \end{aligned} \tag{23}$$

with $c_s, d_s > 0$ because $\delta, \beta > n - \frac{n-2k}{k+1}$ and error

$$E(\varphi) \le \sum_{s=1}^{k} e_{k-s}(\beta) \left| \int_{B^\beta} \sum_{i,j} T_{ij}^{k-s} v_j \varphi_i |\nabla v|^{2(s-1)} \varphi^{2k-1} vV \right| + \sum_{s=1}^{k} e_{k-s}(\delta) \left| \int_{B^\delta} \sum_{i,j} T_{ij}^{k-s} v_j \varphi_i |\nabla v|^{2(s-1)} \varphi^{2k-1} vV \right| \tag{24}$$

Remark. For simplicity, we will just write

$$E(\varphi) \le \sum_{s=1}^{k} e_{k-s}(\beta, \delta) \left| \int_{B^\beta \cup B^\delta} \sum_{i,j} T_{ij}^{k-s} v_j \varphi_i |\nabla v|^{2(s-1)} \varphi^{2k-1} vV \right|$$

In general, we will use the following notation convention throughout the rest of the section:

$$c(\beta, \delta) \int_{B^\beta \cup B^\delta} \dots := c(\beta) \int_{B^\beta} \dots + c(\delta) \int_{B\delta} \dots$$

although the coefficients in front of the integration are different in each of the subsets B^β, B^δ.

To estimate this error term we need a little algebra lemma:

Lemma 4.1. *If $\sigma_1, \dots, \sigma_m > 0$, $m \le n$, then*

$$|T_{ij}^{m-1} x_i y_j| \le C_{m,n} \sigma_{m-1} |x||y|$$

for all vectors $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$.

Proof. Follows basically because T^{m-1} is positive definite. To estimate its norm we just need to look at the biggest eigenvalue. We are done because

$$trace(T^{m-1}) = (n - m)\sigma_{m-1}$$

□

Now we can say

$$E(\varphi) \lesssim \sum_{s=1}^{k} e_{k-s}(\beta, \delta) \int_{B^\beta \cup B^\delta} \sigma_{k-s} |\nabla \varphi| |\nabla v|^{2(s-1)+1} \varphi^{2k-1} vV$$

We want to use Hölder with some small $\epsilon > 0$; there exists C_ϵ such that

$$ab \le \epsilon a^p + C_\epsilon b^q, \quad 1 = \frac{1}{p} + \frac{1}{q}$$

Take $p = \frac{2s}{2(s-1)+1}$, $q = 2s$:

$$
\begin{aligned}
E(\varphi) \leq & \sum_{s=1}^{k} e^1_{k-s} \int_{B^\beta \cup B^\delta} \sigma_{k-s} |\nabla v|^{2s} \varphi^{2k} V \\
& + \sum_{s=1}^{k} e^2_{k-s} \int_{B^\beta \cup B^\delta} \sigma_{k-s} |\nabla \varphi|^{2s} \varphi^{2k-2s} v^{2s} V
\end{aligned} \tag{25}
$$

Fixed β, δ, we can choose $e^1_{k-s} = e^1_{k-s}(\beta, \delta)$ wisely small and absorb the first term of (25) into the right hand side of (23) and thus:

$$
\begin{aligned}
& \sum_{s=1}^{k} c_{k-s} \int_{B^\beta \cup B^\delta} \sigma_{k-s} |\nabla v|^{2s} \varphi^{2k} V \\
& \qquad \lesssim \int_{B^\beta \cup B^\delta} V \varphi^{2k} + \sum_{s=1}^{k} e^2_{k-s} \int_{B^\beta \cup B^\delta} \sigma_{k-s} |\nabla \varphi|^{2s} \varphi^{2k-2s} v^{2s} V
\end{aligned} \tag{26}
$$

We still need to control the 'bad' term in (26)

$$
\mathcal{B} := \int_B V \varphi^{2k} \tag{27}
$$

For that we will use the 'good' term when $s = k$ in the right hand side of (26):

$$
\mathcal{G} := c_{k-k} \int_{B^\beta \cup B^\delta} |\nabla v|^{2k} \varphi^{2k} V \tag{28}
$$

Lemma 4.2. *If $\int v^{-n} < a$,*

$$
\mathcal{B} \leq C_{\beta,\delta} \left(a^{\frac{2k}{n}} \mathcal{G} + E(\varphi) \right)
$$

with constant C depending on δ, β but not on v.

Proof. Call $F = V v^{2k} \varphi^{2k}$, then this good term satisfies

$$
\mathcal{G} \geq C_{\beta,\delta} \int_B |\nabla F^{\frac{1}{2k}}|^{\frac{1}{2k}} \tag{29}
$$

plus some terms in derivatives of φ that we will be able to absorb into $E(\varphi)$. F is a compactly supported function on the unit ball B. Sobolev embedding $W^{1,2k} \hookrightarrow L^{p^*}$ with $\frac{1}{p^*} = \frac{1}{2k} - \frac{1}{n}$ gives

$$
\int_B |\nabla F^{\frac{1}{2k}}|^{2k} \geq C \left(\int_B F^\chi \right)^{1/\chi} \tag{30}
$$

Now use Hölder with $p = \frac{n}{2k}$, $q = \chi$ to give the estimate for the bad term:

$$
\mathcal{B} = \int_B V \varphi^{2k} = \int_B F v^{-2k} \lesssim \left(\int_B F^\chi \right)^{1/\chi} \left(\int_B v^{-n} \right)^{\frac{2k}{n}} \tag{31}
$$

This proves the lemma. □

Let us continue with the proof of the theorem. By the assumption of smallness of volume and the lemma, for β, δ fixed we can absorb the bad term (27) into (28) and from (26) we get

$$\sum_{s=1}^{k} c_{k-s}(\beta,\delta) \int_{B^\beta \cup B^\delta} \sigma_{k-s} |\nabla v|^{2s} \varphi^{2k} V \le \sum_{s=1}^{k} d_{k-s}(\beta,\delta) \int_{B^\beta \cup B^\delta} \sigma_{k-s} |\nabla \varphi|^{2s} \varphi^{2k-2s} v^{2s} V \tag{32}$$

Remark. This is the place where the hypothesis of volume small is used in a crucial way.

We will refer to the terms

$$\sum_{s=1}^{k} \int \sigma_{k-s} |\nabla v|^{2s} \varphi^{2k} V$$

in the left hand side of (32) as 'good' terms, and the terms with derivatives in $\varphi = (1-\eta)\phi$ as 'error' terms. Because of our divergence formula

$$(k-s)\sigma_{k-s} = v\partial_j \left(v_i T_{ij}^{k-s-1} \right) - n T_{ij}^{k-s-1} v_i v_j + \tfrac{n-k+s+1}{2} \sigma_{k-s-1} |\nabla v|^2$$

we see that the error terms can be estimated by

$$\begin{aligned} \sum_{s=1}^{k} \int \sigma_{k-s} |\nabla \varphi|^{2s} \varphi^{2k-2s} v^{2s} V &\lesssim \sum_{s=1}^{k} \int \sigma_{k-s-1} |\nabla v|^2 |\nabla \varphi|^{2s} \varphi^{2k-2s} v^{2s} V \\ &+ \sum_{s=1}^{k} \int \sigma_{k-s-1} |\nabla v| \left| \nabla |\nabla \varphi|^{2s} \right| \varphi^{2k-2s} v^{2s+1} V \\ &+ \sum_{s=1}^{k} \int \sigma_{k-s-1} |\nabla v| |\nabla \varphi|^{2s+1} \varphi^{2k-2s-1} v^{2s+1} V \\ &=: \sum_{s=1}^{k} (I + II + III) \end{aligned} \tag{33}$$

Now let's study the errors I, II, III one by one: use Hölder with $p = s+1$, $q = \frac{s+1}{s}$ for some $\epsilon > 0$,

$$I \lesssim \epsilon \int \sigma_{k-s-1} |\nabla v|^{2(s+1)} \varphi^{2k} V + C_\epsilon \int \sigma_{k-s-1} |\nabla \varphi|^{2(s+1)} \varphi^{2k-2(s+1)} v^{2(s+1)} V$$

The first part can be absorbed in the good terms as soon as we take $\epsilon > 0$ small enough, and the second will be handled by an induction process, as we will see in a few lines. Hölder again with $p = 2(s+1)$, $q = \frac{2(s+1)}{2s+1}$ gives:

$$\begin{aligned} II &\lesssim \epsilon \int \sigma_{k-s-1} |\nabla v|^{2(s+1)} \varphi^{2k} V \\ &+ C_\epsilon \int \sigma_{k-s-1} \left| \nabla |\nabla \varphi|^{2s} \right|^{\frac{2(s+1)}{2(s+1)-1}} \varphi^{2k} \varphi^{-2s \frac{2(s+1)}{2s+1}} v^{2(s+1)} V \end{aligned}$$

Again, the first part gets absorbed and for the second, induction. And finally, take $p = q = 2$,

$$III \lesssim \epsilon \int \sigma_{k-s-1} |\nabla v|^2 |\nabla \varphi|^{2s} \varphi^{2k-2s} V + C_\epsilon \int \sigma_{k-s-1} |\nabla \varphi|^{2(s+1)} \varphi^{2k-2(s+1)} v^{2(s+1)} V$$

The first part is of type I and the second goes to the induction process. Basically, we are left with terms that contain $2(s+1)$ derivatives in φ, and those are the ones that will be handled by induction as follows: we will consider

$$U_s = U_s(\varphi) = \text{ group of derivatives of } \varphi \text{ of order exactly } 2s$$

They will be defined inductively. The starting point is $U_s = |\nabla \varphi|^{2s}$, $\alpha_U = 2s$; we have reduced the errors in (33) to errors containing U_{s+1}.

For the general induction step, assume we have an error term of the form

$$\int \sigma_{k-s} U_s \varphi^{2k} \varphi^{-\alpha_U} v^{2s} V \tag{34}$$

Analogous Hölder estimates allow us to reduce it to an error term in U_{s+1}, and continue the induction process until we arrive to U_k: as before, (34) can be estimated by the following three terms:

$$\begin{aligned}
I &= \int \sigma_{k-s-1} |\nabla v|^2 U_s \varphi^{2k-\alpha_U} v^{2s} V \\
&\lesssim \epsilon \int \sigma_{k-s-1} |\nabla v|^{2(s+1)} \varphi^{2k} V + C_\epsilon \int \sigma_{k-s-1} \left(U_s\right)^{\frac{s+1}{s}} \varphi^{2k-\alpha_U \frac{s+1}{s}} v^{2s+2} V \\
II &= \int \sigma_{k-s-1} |\nabla v| (\nabla U_s) \varphi^{2k-\alpha_U} v^{2s+1} V \\
&\lesssim \epsilon \int \sigma_{k-s-1} |\nabla v|^{2(s+1)} \varphi^{2k} V + C_\epsilon \int \sigma_{k-s-1} \left|\nabla U_s\right|^{\frac{2(s+1)}{2(s+1)-1}} \varphi^{2k-\alpha_U \frac{2(s+1)}{2(s+1)-1}} v^{2s+2} V \\
III &= \int \sigma_{k-s-1} |\nabla v| |\nabla \varphi| U_s \varphi^{2k-\alpha_U - 1} v^{2s+1} V \\
&\lesssim \epsilon \int \sigma_{k-s-1} |\nabla v|^2 U_s \varphi^{2k-\alpha_U} v^{2s} V + C_\epsilon \int \sigma_{k-s-1} \left[|\nabla \varphi|^2 U_s \right] \varphi^{2k-\alpha_U-2} v^{2(s+1)} V
\end{aligned}$$

Here we have terms that can be absorbed and terms in U_{s+1} only, so the induction step is proved.

At the last step $s = k$ we will have error terms that can be handled as follows

$$\int U_k(\varphi) \varphi^{2k-\alpha_U} v^{2k} V \leqslant \left(\int \left(U_k(\varphi) \varphi^{2k-\alpha_U} \right)^p \right)^{1/p} \left(\int \left(v^{2k} V \right)^q \right)^{1/q} \tag{35}$$

where $U_k = U_k(\varphi)$, $\varphi = (1-\eta)\phi$, $\frac{1}{p} + \frac{1}{q} = 1$. Note that we do not have to worry about the exponents $2k - \alpha_U$ being positive because we could have started with a higher power of φ. Now, the part of ϕ is controlled because it is a smooth fixed function on the ball, so ϕ and its derivatives are bounded by a constant. Thus we only need to worry about the cutoff η. Choose the η defined in (22) satisfying

$$U_k(\eta) \leq \frac{c}{\rho^{2k}}$$

We basically want (35) to tend to zero when $\rho \to 0$, i.e, we want both

$$\frac{1}{\rho^{2k}}\rho^{n/p} \to 0 \quad \text{when} \quad \rho \to 0$$

and

$$\int \left(v^{2k}V\right)^q \leq m \int v^{(2k-\delta)q} < \infty$$

at the same time. And it can be done: there exists $\delta > n - \frac{n-2k}{k+1}$ such that $q(2k-\delta) = -n$, and $\frac{n}{p} - 2k > 0$.

This basically shows that we can forget about the point singularity and that for all $\phi \in \mathcal{C}_0^\infty(B)$:

$$\sum_{s=1}^{k} c_{k-s}(\beta,\delta) \int_{B^\beta \cup B^\delta} \sigma_{k-s}|\nabla v|^{2s}\phi^{2k}V_m \lesssim \int_B U_k(\phi)\phi^{2k-\alpha_U}v^{2k}V_m \tag{36}$$

fixed δ as before, for all $\beta > n - \frac{n-2k}{k+1}$. We actually only need one good term in the sum (36) so

$$\int_B |\nabla v|^{2k}\phi^{2k}V_m \leq C(\beta,\delta)\int_{B^\beta\cup B^\delta} U_k(\phi)\phi^{2k-\alpha_U}v^{2k}V_m$$

We had $V_m = \inf\{v^{-\gamma}, mv^{-\delta}\}$. Note that v^{-1} is bounded from below and let $m \to \infty$. We arrive at

$$\int |\nabla v|^{2k}\phi^{2k}v^{-\beta} \leq c(\beta)\int U_k(\phi)\phi^{2k-\alpha_U}v^{2k-\beta} \tag{37}$$

ϕ is smooth, so $U_k(\phi)$ is bounded by constant. Call $-\tilde{\beta} = 2k - \beta$ and use Sobolev (don't worry about terms with derivatives in ϕ because they can be handled as the rest of the errors).

$$\int v^{-\tilde{\beta}} \gtrsim \int |\nabla v|^{2k}v^{-\beta} \gtrsim \int \left|\nabla\left(v^{-\frac{\tilde{\beta}}{2k}}\phi\right)\right|^{2k} \gtrsim \left(\int v^{-\chi\tilde{\beta}}\phi^{\chi}\right)^{1/\chi} \tag{38}$$

If we start with fixed $\tilde{\beta} = n$, the theorem is proved because $\chi = \frac{n}{n-2k} > 1$. □

Proof. of theorem 1.2: Once a $L^{\tilde{q}}$ estimate is reached for some $\tilde{q} > n$, the L^∞ estimate follows by a well known Moser iteration argument, by iteration of (37). Fix $R < \frac{1}{2}$ and choose a cutoff ϕ as

$$\phi = \begin{cases} 1 & \text{if} \quad 0 < |x| < R \\ 0 & \text{if} \quad 2R < |x| < 1 \end{cases}$$

and

$$U_k(\phi) \lesssim \frac{c}{R^{2k}}$$

Call $\tilde{\beta} = \beta - 2k$. From (37) and Sobolev embedding we get

$$\left(\int_{B_R} (v^{-1})^{\chi\tilde{\beta}}\right)^{1/\chi} \leq \frac{C(\beta)}{R^{2k}}\int_{B_{2R}} (v^{-1})^{\tilde{\beta}}$$

for $\tilde\beta > n - \frac{n-2k}{k+1} - 2k$. Take $\tilde\beta$−root on both sides

$$\|v^{-1}\|_{L^{\tilde\beta\chi}(B_R)} \le \left(\frac{C(\beta)}{R^{2k}}\right)^{\frac{1}{\tilde\beta}} \|v^{-1}\|_{L^{\tilde\beta}(B_{2R})} \tag{39}$$

for $\chi = \frac{n}{n-2k} > 1$.

We actually need to be careful with the dependence $C = C(\beta)$ because we will let $\beta \to \infty$. In particular, lemma 4.2 needs to be replaced by:

Lemma 4.3. *For any $\epsilon > 0$, if $v^{-1} \in L^{\tilde q}$ for some $\tilde q > n$, then*

$$\mathcal{B} \le C_1 \epsilon \mathcal{G} + C_2 E(\phi)$$

for some constant $C1 = C_1(\delta)$ but not depending on β, and $C_2 = C_2(\beta, \delta, \epsilon)$.

Proof. Denote $\frac{\tilde q}{2k} = q$, $1 = \frac{1}{p} + \frac{1}{q}$. By hypothesis, $1 < p < \chi = \frac{n}{n-2k}$. Call $F := Vv^{2k}\varphi^{2k}$. Using Hölder with those p, q, the bad term satisfies

$$\mathcal{B} = \int_B V\varphi^{2k} \le \left(\int v^{-\tilde q}\right)^{\frac{1}{q}} \left(\int F^p\right)^{\frac{1}{p}} \tag{40}$$

There exists $0 < \lambda < 1$ such that $p = \lambda + (1-\lambda)\chi$ so we can use interpolation to control the second term:

$$\left(\int F^p\right)^{\frac{1}{p}} \le \left(\int F\right)^{\frac{1-\lambda}{p}} \left(\int F^\chi\right)^{\frac{\lambda}{p}} = \left[\int F^\chi\right]^{\frac{1}{\chi}} \left[\left(\int F\right)\left(\int F^\chi\right)^{-\frac{1}{\chi}}\right]^{\frac{\lambda}{p}} \tag{41}$$

Fix $\epsilon > 0$ small. Since $\frac{\lambda}{p} < 1$, Young's inequality with $\tilde\epsilon$ reads

$$x^{\frac{\lambda}{p}} \le C_{\tilde\epsilon} x + \tilde\epsilon$$

If we substitute $x = \left(\int F\right)\left(\int F^\chi\right)^{-\frac{1}{\chi}}$, together with (41) we get

$$\left(\int F^p\right)^{\frac{1}{p}} \le \epsilon \left(\int F^\chi\right)^{\frac{1}{\chi}} + C_\epsilon \int F$$

Now, from (40), using the hypothesis on v,

$$\mathcal{B} \lesssim \tilde\epsilon \left(\int F^\chi\right)^{\frac{1}{\chi}} + C_{\tilde\epsilon} \int F \tag{42}$$

On the other hand, fixed δ, we had proved in (30) that

$$\mathcal{G} \ge C_\beta^{-1} \left(\int F^\chi\right)^{\frac{1}{\chi}} + C_\beta^{-1} E(\varphi)$$

For each β, choose $\tilde\epsilon = \tilde\epsilon(\beta, \epsilon)$ small enough and the lemma follows. (The part $\int F$ in (42) will be included in $E(\varphi)$). ∎

Taking into account that $\lim_{\beta\to\infty} C(\beta)^{1/\beta} = C_0$, expression (39) is the estimate we needed. From here we can iterate to get an L^∞ bound using the argument by Moser, that we sketch here (a good reference is [10], chapter 8): call $p_m = \chi^m p$, $R_m = R\left(1+\frac{1}{2}+\ldots+\frac{1}{2^m}\right)$. Applying (39) inductively,

$$\|v^{-1}\|_{L^{p_m}(B_R)} \leq C\left(\frac{1}{R}\right)^{\sum_{i=1}^{m-1}\frac{2k}{\chi^i p}}\left(\frac{C}{2}\right)^{\sum_{i=0}^{m-1}\frac{2k(i+1)}{\chi^i p}}\|v^{-1}\|_{L^p(B_{2R})}$$

Use that, for $\chi = \frac{n}{n-2k} > 1$,

$$\sum_{i=0}^{\infty}\frac{2k}{\chi^i p} = \frac{n}{p} \qquad \text{and} \qquad \sum_{i=0}^{\infty}\frac{i}{\chi^i} < \infty$$

to arrive at

$$\|v^{-1}\|_{L^\infty(B_R)} \leq \frac{C}{R^{n/p}}\|v^{-1}\|_{L^p(B_{2R})}$$

for all

$$p > (n-2k)\frac{k}{k+1}.$$

□

Corollary 4.4 (Harnack inequality). *Let v as in theorem 1.1. Then*

$$\sup_{B_{1/2}} v^{-1} \leq C \inf_{B_{1/2}} v^{-1}$$

for some constant C depending on a, k, n, but not on v.

Proof. Once we have the L^∞ estimate (6), the estimate of $\inf v^{-1}$ follows from standard elliptic theory since a function in the positive cone Γ_k^+ is automatically superharmonic.

□

Proof. of corollary 1.3: Once a L^∞ estimate is reached, it is well known that $\mathcal{C}^0$ estimates imply $\mathcal{C}^2$ estimates for the σ_k equation, by the gradient estimates of Guan-Wang [14], Li-Li [18] (theorem 1.6.). These estimates are true even if v^{-1} has an isolated singularity, using a simple covering argument by Li-Li that we sketch here:

Assume that we have proved $c_1 \leq v^{-1}(x) \leq c_2$ for all $x \in B$. Now, fixed $r < \frac{2}{3}$, define

$$v_r(y) = \frac{1}{r}v(ry) \quad \text{for} \quad y \in B_{\frac{3}{2}}, \quad x = ry$$

Cover the circle $|\bar{y}| = 1$ by balls $\left\{B_{\frac{1}{2}}(\bar{y}) \ : \ |\bar{y}| = 1\right\}$. This v_r is still a solution of $\sigma_k(v_R) = 1$ for all y in $B_{\frac{1}{2}}(\bar{y})$, smooth. By the gradient estimates we can conclude:

$$|\nabla \log v_r| \leq C, \quad |D^2 \log v_r| \leq C \quad \text{in} \quad B_{\frac{1}{4}}(\bar{y}),\ |\bar{y}| = 1$$

But this precisely tells us that, translating back to v,

$$|D^2 v|(x) \le \frac{C}{|x|^2} \quad \text{for all} \quad 0 < |x| < \frac{2}{3}$$

Now it is easy to see that $v \in W^{2,k}$ because

$$\int_{B_r} |D^2 v| dx \lesssim r^{n-2k} \to 0 \quad \text{as} \quad r \to 0.$$

□

Remark. The same proof gives $v \in W^{2,p}$ for $p < \frac{n}{2}$.

Remark. This argument does not allow to remove the regularity hypothesis $v^{-1} \in \mathcal{C}^3 \backslash \{0\}$.

5 Global problem

In this section we deal with the global problem (9). The main ingredient is basically a refinement of an Obata-type argument, developed in the forthcoming paper [11]. Obata-type proofs have proved to be very successful when dealing with this type of σ_k equations, see for instance, [25], [5], [6], [18].

Here we make use of the traceless Newton tensor $L^k := \frac{n-k}{n}\sigma_k I - T^k$. One of the properties we are interested in is that in the positive cone, $L^k_{ij} E_{ij} \ge 0$, where $E = L^1$ is constant times the traceless Ricci tensor; and it is zero just if $E \equiv 0$. In [11] we prove:

Proposition 5.1. *Let $n > 2k$, U domain in $\mathbb{R}^n$. Take $v^{-1} \in \mathcal{C}^3(U)$ and $\eta \in \mathcal{C}_0^\infty(U)$, $v \in \Gamma_k^+$. We have then*

$$\begin{aligned} \int_U \sum_{i,j} L^k_{ij} E_{ij} v^{-\delta} \eta + \sum_{s=1}^{k} (1+n-\delta) c_{k-s}(\delta) \int_U \sigma_{k-s} |\nabla v|^{2(s+1)} v^{-\delta} \eta \\ = E(\eta) + (1+n-\delta) \tfrac{k(n+2)}{2n} \int_U \sigma_k |\nabla v|^2 v^{-\delta} \eta \end{aligned} \tag{43}$$

for some constants c_{k-s} where

$$E(\eta) \lesssim \left| \int_U \sum_{i,j} L^k_{ij} v_i \eta_j v^{1-\delta} \right| + \sum_{s=1}^{k} \left| \int_U \sum_{i,j} T^{k-s}_{ij} v_j \eta_j |\nabla v|^{2s} v^{1-\delta} \right| \tag{44}$$

In addition, if $\delta < n+1$ and δ close enough $n+1$, all the coefficients $c_{k-s}(\delta)$ are positive.

Denote $A_{a,b}$ to be the annulus $\{a \le |x| \le b\}$. Fix $0 < \rho < R$. Construct a smooth cutoff function

$$\eta = \begin{cases} 1 & \text{if } x \in A_{\rho,R} \\ 0 & \text{if } 0 < |x| < \frac{\rho}{2}, \quad 2R < |x| \end{cases}$$

satisfying

$$|\nabla\eta| \lesssim \frac{1}{\rho}, \quad |D^2\eta| \lesssim \frac{1}{\rho^2} \quad \text{in} \quad A_{\frac{\rho}{2},\rho},$$
$$|\nabla\eta| \lesssim \frac{1}{R}, \quad |D^2\eta| \lesssim \frac{1}{R^2} \quad \text{in} \quad A_{R,2R}.$$

If we substitute this particular cutoff in (43), after an inductive study of the error terms in a similar manner as in the previous section, we can give an estimate for $E(\eta)$:

Lemma 5.2. *Fix $\epsilon > 0$. In the same hypothesis as the previous proposition, and the cutoff constructed above, then*

$$\begin{aligned} E(\eta) \leq & \epsilon \sum_{s=0}^{k} \int_{A_{\frac{\rho}{2},\rho} \cup A_{R,2R}} \sigma_{k-s} |\nabla v|^{2(s+1)} \eta v^{-\delta} \\ & + \frac{C_\epsilon}{\rho^{2(k+1)}} \int_{A_{\frac{\rho}{2},\rho}} v^{2(k+1)-\delta} + \frac{C_\epsilon}{R^{2(k+1)}} \int_{A_{R,2R}} v^{2(k+1)-\delta} \end{aligned} \tag{45}$$

Proof. of theorem 1.5: We can assume that there is only one singular point $p = 0$ but it is clear that a similar cutoff can be constructed in the general case.

Now, use expressions (43) and (45) for $\delta < n+1$, but sufficiently close to $n+1$, ϵ small enough, and note that $L^k_{ij} E_{ij} \geq 0$,

$$\begin{aligned} & \sum_{s=1}^{k} c_{k-s}(\delta) \int \sigma_{k-s} |\nabla v|^{2(s+1)} v^{-\delta} \eta \\ & \quad \lesssim \frac{1}{\rho^{2(k+1)}} \int_{A_{\rho,2\rho}} v^{2(k+1)-\delta} + \frac{1}{R^{2(k+1)}} \int_{A_{R,2R}} v^{2(k+1)-\delta} + \int_{A_{\rho,R}} \sigma_k |\nabla v|^2 v^{-\delta} \eta \end{aligned} \tag{46}$$

If we take ρ small enough, the volume on $B_{4\rho}$ can be made very small, so we can apply theorem 1.2 to get an L^∞ bound for v^{-1} in a small ball. Thus

$$\frac{1}{\rho^{2(k+1)}} \int_{A_{\rho,2\rho}} v^{2(k+1)-\delta} \leq C \rho^{n-2(k+1)}. \tag{47}$$

For the term in $A_{R,2R}$, near infinity, use Hölder with $p = \frac{n}{\delta - 2(k+1)}$, $q = \frac{p}{p-1}$

$$\frac{1}{R^{2(k+1)}} \int_{A_{R,2R}} v^{\delta-2(k+1)} \leq \left(\int v^{-n} \right)^{\frac{1}{p}} R^{\frac{n}{q}-2(k+1)} \to 0$$

when $R \to \infty$ because when $\delta > n$,

$$\frac{n}{q} < 2(k+1)$$

For the third term in (46), note that

$$\int_{\mathbb{R}^n \setminus \{0\}} |\nabla v|^2 v^{-\delta} \eta < \infty \tag{48}$$

This is so because near the origin, the hypothesis that the volume is finite allows to use theorems 1.1 and 1.2, and near infinity, the bound follows from lemma 5.3 below. We have proved, looking at (46), that

$$\sum_{s=1}^{k} \int_{\mathbb{R}^n \backslash \{0\}} \sigma_{k-s} |\nabla v|^{2(s+1)} v^{-\delta_0} \eta \leq C \tag{49}$$

for some $n < \delta_0 < n+1$. Now use again (43) with $\delta = n+1$ and a similar cutoff:

$$\begin{aligned} \int_{A_{\rho,R}} \sum_{i,j} L_{ij}^k E_{ij} v^{-n-1} \lesssim_\epsilon & \sum_{s=0}^{k} \int_{A_{\frac{\rho}{2},\rho} \cup A_{R,2R}} \sigma_{k-s} |\nabla v|^{2(s+1)} \eta v^{-n-1} \\ & + \frac{1}{\rho^{2(k+1)}} \int_{A_{\frac{\rho}{2},\rho}} v^{2(k+1)-n-1} + \frac{1}{R^{2(k+1)}} \int_{A_{R,2R}} v^{2(k+1)-n-1} \end{aligned} \tag{50}$$

The integral over $A_{\rho/2,\rho}$ goes to zero as $\rho \to 0$ by arguments as in (47), and the same can be said for the integral over $A_{R,2R}$. We will be done if

$$\sum_{s=0}^{k} \int_{A_{\frac{\rho}{2},\rho} \cup A_{R,2R}} \sigma_{k-s} |\nabla v|^{2(s+1)} \eta v^{-n-1} \to 0$$

when $\rho \to 0$, $R \to \infty$. First observe that in a small ball B_{ρ_0}, v^{-1} is bounded from above and below so (49) implies that

$$\sum_{s=0}^{k} \int_{A_{\frac{\rho}{2},\rho}} \sigma_{k-s} |\nabla v|^{2(s+1)} \eta v^{-n-1} \to 0 \quad \text{as} \quad \rho \to 0$$

From (49), and the fact that $\delta_0 < n+1$, we can also conclude that

$$\sum_{s=0}^{k} \int_{A_{R,2R}} \sigma_{k-s} |\nabla v|^{2(s+1)} \eta v^{-n-1} \to 0 \quad \text{as} \quad R \to \infty$$

because when $|x| > R_0$, for some R_0 big enough, $|v^{-1}(x)| < 1$, as a consequence of lemma 5.3.

In view of this discussion, (50) implies that

$$\int_{A_{\rho,R}} \sum_{i,j} L_{ij}^k E_{ij} v^{-n-1} \to 0 \quad \text{as} \quad \rho \to 0, R \to \infty$$

and as a consequence, $L_{ij}^k E_{ij} = 0$ on $\mathbb{R}^n \backslash \{0\}$. This implies that $E \equiv 0$. By the second Bianchi Identity, R_{g_v} is identically constant on $\mathbb{R}^n \backslash \{0\}$, let's say $R_{g_v} \equiv 1$, and we have reduced the problem to

$$-\Delta u = u^{\frac{n+2}{n-2}} \quad \text{in} \quad \mathbb{R}^n \backslash \{0\} \tag{51}$$

for $u^{\frac{4}{n-2}} = v^{-2}$. This problem cannot have isolated singularities because of the weak removable singularities theorem (it can be found in Lee-Parker [17], proposition 2.7.), so we must have that u is a weak solution of (51) in the whole $\mathbb{R}^n$.

Elliptic regularity for the Laplacian equation gives $u \in \mathcal{C}^\infty(\mathbb{R}^n)$, and the theorem is proved by the classification result of smooth solutions on $\mathbb{R}^n$. □

Remark. As we have mentioned previously, this theorem does not use the gradient estimates of Guan-Wang [13], and thus, it does not require a-priori smoothness of the function v^{-1}. Corollary 1.6 follows.

Lemma 5.3. *For R big enough, if v is as in theorem 1.5, $\delta > n$, then*

$$\int_{|x|>2R} |\nabla v|^2 v^{-\delta} dx < \infty$$

and $v^{-1} \leq \frac{C}{|x|^2}$ when $|x| \to \infty$.

Proof. First, make an inversion with respect to the unit circle

$$w(y) = |y|^2 v\left(\frac{y}{|y|^2}\right), \quad x = \frac{y}{|y|^2}$$

This w is still a solution of $\sigma_k(w) = 1$ for $y \in B_{\frac{1}{R}}$. Moreover, the volume of w in the ball $B_{\frac{1}{R}}$ is exactly

$$\int_{|x|>R} v^{-n} dx$$

and by hypothesis, this quantity is can be made as small as we want by taking R big enough. Thus we can apply theorem 1.1 and theorem 1.2 to w and obtain that w^{-1} is bounded from above and below near zero and

$$\int_{B_{\frac{1}{2R}}} |\nabla w|^{2k} dy < \infty$$

In particular,

$$\int_{B_{\frac{1}{2R}}} |\nabla w|^2 dy < \infty \tag{52}$$

Now, $|\nabla v|^2$ can be estimated in terms of $w^2|x|^2$ and $|\nabla w|^2$. But because of lemma 5.4, from (52) we quickly conclude

$$\int_{|x|>2R} |\nabla v|^2 v^{-\delta} dx \lesssim \int_{|y|<\frac{1}{2R}} |\nabla w|^2 |y|^{2(\delta-n)} w^{-\delta} dy < \infty$$

This proves the lemma. □

Lemma 5.4. *For the w constructed above,*

$$\int_B w^2 \frac{1}{|y|^2} dy \leq C \int_{2B} |\nabla w|^2 dy$$

Proof. If w has compact support, just use integration by parts and Hölder inequality:

$$\int_B w^2 \frac{1}{|y|^2} dy \lesssim \int_B |\nabla w| w \frac{1}{|y|} dy \leq \left(\int_B |\nabla w|^2 dy\right)^{\frac{1}{2}} \left(\int_B w^2 \frac{1}{|y|^2} dy\right)^{\frac{1}{2}}$$

If w does not have compact support, a cutoff η^2 needs to be introduced, but the proof is straightforward. □

Remark. For $n < 2k$, all the radial solutions are smooth.

Conjecture 5.5. *There are no singular solutions of $\sigma_k(A^{g_v}) = 1$ on $B\backslash\{0\}$ when $n < 2k$.*

6 A new notion of capacity

The classical notion of capacity was introduced to treat singularities of linear and quasilinear PDE (see Adams-Hedberg [1], Malý-Ziemer [19] for good introductions to the subject), and it is defined as:

Definition 6.1. *Let Λ be a compact subset of $\mathbb{R}^n$. We consider*

$$c_{k,p}(\Lambda) := \inf\left\{\|\eta\|^p_{W^{k,p}} \quad : \quad \eta \in \mathcal{C}_0^\infty,\ \eta \geq 1 \ \text{on}\ \Lambda\right\} \tag{53}$$

Note that the actual definition of $c_{k,p}$ is an equivalent notion to (53), but it does not make any significant difference in the following.

It is natural to introduce a new concept of capacity to treat specifically the type fully non-linear equations we are interested in this paper:

Definition 6.2. *Let Λ be a compact subset of $\mathbb{R}^n$. For $p \geq 2k$, define*

$$\tilde{c}_{k,p}(\Lambda) = \inf\left\{\|\eta\|^p_{L^p} + \sum_{U_k}\int (U_k(\eta))^{\frac{p}{2k}}\,dx \quad : \quad \eta \in \mathcal{C}_0^\infty,\ \eta \geq 1 \ \text{on}\ \Lambda\right\}$$

with the U_k constructed inductively in the proof of the theorem 1.1.

For $k = 1$ this corresponds to the classical definition of $(1,p)-$capacity,

$$c_{1,p}(\Lambda) = \inf\left\{\|\eta\|^p_{L^p} + \int \left(|\nabla\eta|^2\right)^{\frac{p}{2}}dx \quad : \quad \eta \in \mathcal{C}_0^\infty, \quad \eta \geq 1 \text{ on } \Lambda\right\}$$

The U_k in the definition of capacity can be computed inductively, and in particular for $k = 2$ we have

$$\tilde{c}_{2,p}(\Lambda) = \inf\left\{\|\eta\|^p_{L^p} + \int |\nabla\eta|^p dx + \int \left|\nabla|\nabla\eta|^2\right|^{\frac{p}{3}}dx \quad : \quad \eta \in \mathcal{C}_0^\infty, \quad \eta \geq 1 \text{ on } \Lambda\right\}$$

Note that a simple Hölder estimate with $p = \frac{3}{2}, q = 3$:

$$\int (\eta_{ij}\eta_j)^{\frac{p}{3}} \lesssim \int \eta_{ij}^{p/2} + \int \eta_k^p$$

gives

$$c_{2,p/2}(\Lambda) = c_{1,p}(\Lambda) = 0 \quad \Rightarrow \quad \tilde{c}_{2,p}(\Lambda) = 0$$

Lemma 6.3. *For general k:*

1. *If $c_{k,p/k}(\Lambda) = c_{k-1,p/(k-1)}(\Lambda) = \ldots = c_{1,p}(\Lambda) = 0$, then $\tilde{c}_{k,p}(\Lambda) = 0$*

2. If $dim_{\mathcal{H}}(\Lambda) < n-p$ for $n > p > 2k$, then $\tilde{c}_{k,p}(\Lambda) = 0$

Proof. Each of the $U_k(\eta)$ is bounded by

$$\int U_k(\eta)^{\frac{p}{2k}} \le \int \left(|D^k\eta|^{\beta_k} |D^{k-1}\eta|^{\beta_{k-1}} \dots |D\eta|^{\beta_1} \right)^{\frac{p}{2k}}$$

for $k\beta_k + (k-1)\beta_{k-1} + \dots + 1\beta_1 = 2k$, counting the orders. Using Hölder with

$$\left(\frac{2k}{k\beta_k}\right)^{-1} + \left(\frac{2k}{(k-1)\beta_{k-1}}\right)^{-1} + \dots + \left(\frac{2k}{\beta_1}\right)^{-1} = 1$$

gives

$$\int U_k(\eta)^{\frac{p}{2k}} \lesssim \int |D^k\eta|^{\frac{p}{k}} + \int |D^{k-1}\eta|^{\frac{p}{k-1}} + \dots + \int |D\eta|^{\frac{p}{1}}$$

and the first assertion follows.

For the second, just note that the restriction in the Hausdorff dimension implies that all $c_{k,p/k}(\Lambda) = c_{k-1,p/(k-1)}(\Lambda) = \dots = c_{1,p}(\Lambda) = 0$. □

Remark. It is not known if the converse to the first statement in lemma 6.3 is true, i.e., if this new notion of capacity is different from the standard one.

In general, to simplify the proofs, we will consider instead the related notion:

Definition 6.4. *($p \ge 2k$) Let Λ be a compact subset of the ball B_R for some $R > 0$. Define*

$$\tilde{c}_{k,p}(\Lambda, B_R) = \inf \left\{ \|\eta\|_{L^p}^p + \sum_{U_k} \int (U_k(\eta))^{\frac{p}{2k}} \, dx : \eta \in \mathcal{C}_0^\infty(B_R),\ \eta \ge 1 \text{ on } \Lambda \right\}$$

with the U_k constructed inductively in the proof of the theorem 1.1.

Lemma 6.5. *Let Λ be a compact subset of B with Newtonian capacity $c_{1,2}(\Lambda) = 0$, and let v with $v^{-1} \in \mathcal{C}^3(\bar{B} \backslash \{\Lambda\})$ be a solution of (2). Then v^{-1} is bounded from below by a positive constant on $\bar{B} \backslash \Lambda$.*

Proof. Rewriting $v^{-2} = u^{\frac{4}{n-2}}$, we see that u is a superharmonic function on $B \backslash \Lambda$:

$$-\Delta u = R_{g_u} u^{\frac{n+2}{n-2}} \ge 0 \quad \text{on} \quad B \backslash \Lambda$$

The estimate follows from standard elliptic theory, for instance, lemma 2.1. in [8]. □

Proof. of Theorem 1.7: The proof is the same as in the isolated singularity case except in the way we choose the cutoff $\varphi = (1-\eta)\phi$. By hypothesis, we know $\Lambda \subset\subset B_R$. Pick $R < R_1, R_2 < 1$ and choose ϕ a smooth function, $\phi \equiv 1$ in B_{R_1}, and zero outside B_{R_2}.

We need to estimate (35)

$$\int U_k(\varphi) v^{2k} V = \int_{B_R} U_k(\eta) v^{2k} V + \int_{R_1 < |x| < R_2} U_k(\phi) v^{2k} V$$

The second term is bounded by a constant depending on R_1, R_2. And for the first term,

$$\int_{B_R} U_k(\eta) v^{2k} V \leq \left(\int [U_k(\eta)]^{\frac{p}{2k}} \right)^{\frac{2k}{p}} \left(\int \left(v^{2k} V \right)^{\frac{p}{p-2k}} \right)^{\frac{p-2k}{p}} \to 0$$

by choosing a suitable sequence of cutoffs η such that $0 \leq \eta \leq 1$, $\{\eta = 1\} \supset \bar{\Lambda}$, and the fact that $\tilde{c}_{k,p}(\Lambda, B_R) = 0$. We will be done if we can choose $\delta > n - \frac{n-2k}{k+1}$ such that

$$\int v^{(2k-\delta)\frac{p}{p-2k}} < \infty$$

and this is precisely our hypothesis. □

Proof. of corollary 1.8: The only difference is the way the cutoff η is constructed near each singularity Λ_i. In particular, if ρ denotes the distance to Λ_i, (47) becomes

$$\frac{1}{\rho} \int_{A_{\rho,2\rho}} v^{2(k-1)-\delta} \leq C \rho^{n-k_i-2(k+1)}$$

Note that the dimension restriction implies that $k_i < n - 2(k+1)$, needed in the estimate of the errors. □

Corollary 6.6. *A sufficient condition on Λ to get the L^∞ estimates of theorem 1.7 is*

$$\dim_{\mathcal{H}}(\Lambda) < \frac{n(n-2k)}{n+2k^2}.$$

Proof. If we ask $(n-2k)\frac{k}{k+1}\left(\frac{p}{p-2k}\right) = n$ we the seeked value for p. Note that $2k < p < n$. And a sufficient condition to achieve $\tilde{c}_{k,p}(\Lambda) = 0$ is to ask

$$\dim_{\mathcal{H}}(\Lambda) < n - p = \frac{n(n-2k)}{n+2k^2}$$

□

Remark. We do not know if this condition is also necessary.

Hypothesis (10) is only used to control the left hand side of the equation as we did in lemma 4.2. If we had a very good LHS this condition could be removed, and for instance,

Corollary 6.7. *Let $\Lambda \subset B \subset \mathbb{R}^n$ be a compact set with capacity*

$$\tilde{c}_{k,p}(\Lambda) = 0, \quad 2k < p \leq n.$$

Let $\hat{\epsilon} > 0$ and v^{-1} in $L^{(n-2k)\frac{k}{k+1}\ \frac{p}{p-2k}\ +\hat{\epsilon}}(B)$ be a solution of

$$\sigma_k(v) = 0 \quad on \quad B \backslash \Lambda$$
$$v \in \overline{\Gamma_k^+}, \quad v > 0$$

Then v^{-1} is bounded near Λ.

A Appendix

In general, the most reasonable notation seems to be

$$v^{-2} = u^{\frac{2(k+1)}{n-2k}}$$

Call

$$(2k)^* = \frac{n(k+1)}{n-2k}$$

Then our equation $\sigma_k(A^{g_v}) = 1$ becomes

$$P_k(u) = \sigma_k(-D^2 u) + l.o.t. = u^{(2k)^*-1}, \quad u \in \Gamma_k^+$$

where D^2u is the Hessian matrix of u.

In particular, we have proved (see (19)):

Corollary A.1. *If $u \in \mathcal{C}_0^\infty$, $n > 2k$, $u \in \Gamma_k^+$,*

$$\int P_k(u)u \geq C \left(\int u^{(2k)^*} \right)^{\frac{n-2k}{n}}$$

Trudinger [24] has defined a notion of k-capacity for $\sigma_k(D^2u)$ from the notion of Hessian measures:

$$c_k'(\Lambda) := \sup \left\{ \int_\Lambda \sigma_k(D^2 u) : u \quad k\text{-convex}, \quad -1 < u < 0 \right\}$$

We know c_1' is equivalent to $\tilde{c}_{1,2}$ because of potential theory for the Laplacian, but the relation between the different capacities remains open for general k.

Conjecture A.2. $c_k' \sim \tilde{c}_{k,2k}$

As we have mentioned, we do not know if the capacity conditions in theorem 1.7 are also necessary, and we believe not. Look at the following construction: let Λ be an $(n-s)$-dim plane in $\mathbb{R}^n$, that has $c_{1,s}(\Lambda) = 0$, and consider the function

$$u(\rho) = \frac{1}{\rho^{\frac{p-2k}{k}}}$$

where ρ denotes the distance to Λ. This function would make Theorem 1.7 sharp but it is not a solution of $\sigma_k(A^{g_u}) = 1$ but of the Hessian problem

$$\sigma_k(D^2 u) = 0 \quad \text{in} \quad \mathbb{R}^n \backslash \Lambda$$

This gives some evidence that the new capacity defined here is the suitable one for Hessian equations.

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