OPTIMAL CONFIGURATION AND SYMMETRY BREAKING PHENOMENA IN THE COMPOSITE MEMBRANE PROBLEM WITH FRACTIONAL LAPLACIAN

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Abstract. We consider the following eigenvalue optimization in the composite membrane problem with fractional Laplacian: given a bounded domain $\Omega \subset \mathbb{R}^n$, $\alpha > 0$ and $0 < A < |\Omega|$, find a subset $D \subset \Omega$ of area A such that the first Dirichlet eigenvalue of the operator $(-\Delta)^s + \alpha\chi_D$ is as small as possible. The solution D is called as an optimal configuration for the data (Ω, α, A). Looking at the well-known extension definition for the fractional Laplacian, in the case $s = 1/2$ this is essentially the composite membrane problem for which the mass is concentrated at the boundary as one is trying to minimize the Steklov eigenvalue.

We prove existence of solutions and study properties of optimal configuration D. This is a free boundary problem which could be formulated as a two-sided unstable obstacle problem.

Moreover, we show that for some rotationally symmetric domains (thin annuli), the optimal configuration is not rotational symmetric, which implies the non-uniqueness of the optimal configuration D. On the other hand, we prove that for a convex domain Ω having reflection symmetries, the optimal configuration possesses the same symmetries, which implies uniqueness of the optimal configuration D in the ball case.

1. Introduction

We study an eigenvalue optimization in the composite membrane problem with fractional Laplacian. Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$-boundary and $D \subset \Omega$ be a measurable subset. For $0 < s < 1$ and $\alpha > 0$, we consider the following eigenvalue problem

$$\begin{aligned} (-\Delta)^s u + \alpha\chi_D u &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \mathbb{R}^n \setminus \Omega. \end{aligned} \tag{1.1}$$

Denote $\lambda_\Omega(\alpha, D)$ by the first eigenvalue of (1.1) and, for $0 \le A \le |\Omega|$, define

$$\Lambda_\Omega(\alpha, A) = \inf_{|D|=A} \lambda_\Omega(\alpha, D). \tag{1.2}$$

If D attains the minimum of (1.2), then we call D as an optimal configuration for the data (Ω, α, A) and (u, D) as an optimal pair. The main objective of this paper is to study optimal pairs of the nonlocal problem above.

Problem (N). *Investigate:*

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(i) the existence and regularity of optimal pairs (u, D)*,*
(ii) the shape of optimal configurations, D*,*
(iii) and the uniqueness of (u, D) *(up to a multiplicative constant for* u*).*

1.1. **History.** The composite membrane problem for the Laplacian operator was considered in [14] which shows existence and regularity of optimal pairs. Moreover, it was shown in the same paper that the optimal configuration D is given by a sublevel set of u whenever (u, D) is an optimal pair, i.e.,

$$D = \{x \in \Omega : u(x) \leq t\}$$

for some constant t satisfying $|D| = A$. In addition, they obtained symmetry and symmetry breaking phenomena of D, which imply uniqueness and non-uniqueness depending on the domain Ω.

Following this work, the optimal regularity of optimal pairs and the regularity and singularity of the free boundary ∂D have been studied by several authors in [15, 36, 16, 17]. In particular, the optimal regularity in dimension two was shown in [17]. We also refer to [34, 20, 3] for the p-Laplacian version of composite membrane problem.

In addition, the very recent works [19, 18] consider the bi-Laplacian case, which is related to the composite plate problem. They established a symmetry property which implies the uniqueness of the optimal pairs for the corresponding problem when the domain is a ball, and the existence of the optimal pairs. However, symmetry breaking phenomena, whose direct consequence is non-uniqueness, have not been considered except for the Laplacian case [14], even though some numerical evidence supports the occurrence of symmetry breaking phenomena (see [18] and [27]).

Finally, we recall the work [13], which relates the composite membrane problem to a certain eigenvalue minimization problem in two dimensions for the Laplace operator in conformal classes. They also provide the generalization to any even dimension n, where the Laplacian is replaced by the GJMS operators (these are conformally covariant operators which the same principal symbol as $(-\Delta)^{n/2}$). For odd dimensions, this equivalence should be also possible and the natural setting is that of fractional order operators.

1.2. **Main results.** Let us now go back to our question, Problem (**N**). We first note that λ_Ω is invariant under any change of D by a measure zero set, so we will ignore such differences. Also, we assume that

$$\alpha \leq \overline{\alpha}_\Omega(A), \tag{1.3}$$

where $\overline{\alpha}_\Omega(A)$ is the unique constant satisfying $\Lambda_\Omega(\overline{\alpha}_\Omega(A), A) = \overline{\alpha}_\Omega(A)$. The convenience of this notation will be clear in Lemma 3.3.

Our first result concerns the existence of an optimal pair and properties of optimal configurations, which give answers to (i) and (ii) in Problem (**N**). However, in contrast to the local case considered in [14], it is nontrivial to show that the optimal configuration D is given by a sublevel set of u due to nonlocal effects. The reason for such difficulty comes from the fact that $(-\Delta)^s u$ may not be zero at a point where u is locally constant. In any case, it is quite straightforward to prove that

$$\{x \in \Omega : u(x) < t\} \subset D \subset \{x \in \Omega : u(x) \leq t\},$$

where $t := \sup\{c : |\{u < c\}| < A\}$.

Now we state our first main result:

Theorem 1.1. *Let $\alpha > 0$ satisfying (1.3) and $A \in [0, |\Omega|]$. Then:*

(i) There exists an optimal pair (u, D).

(ii) Any optimal pair satisfies

$$u \in H^{2s}_{loc}(\Omega) \cap C^{\beta}(\Omega) \cap C^{s}(\mathbb{R}^n),$$

where β is $2s$ if $s \neq \frac{1}{2}$, and any constant in $(0, 2s)$ if $s = \frac{1}{2}$.

(iii) Let $\alpha < \overline{\alpha}_{\Omega}(A)$. If $s \leq \frac{1}{2}$, the optimal configuration D is a sublevel set of u, i.e.,

$$D = \{x \in \Omega : u(x) \leq t\}, \tag{1.4}$$

where $t := \sup\{c : |\{u < c\}| < A\}$.

Notice that $C^{2s}(\Omega)$ (resp. $C^{0,1}(\Omega)$) for $s \neq \frac{1}{2}$ (resp. $s = \frac{1}{2}$) is the optimal regularity for u since $(-\Delta)^s u$ is not continuous in Ω. We also remark that the sublevel set property (1.4) for local operators can be easily proved, if one has sufficient regularity, from the well-known fact that the weak derivative of u is zero a.e. on its constant set (see [14] for the Laplacian and [19] for the bi-Laplacian). But this property no longer holds for nonlocal operators. This is also the case in the p-Laplacian version of composite membrane problem because of the lack of regularity [34].

However, we can still expect the sublevel set property (1.4) even though $(-\Delta)^s u$ may not be zero on the locally constant points. Heuristically, this is because $(-\Delta)^s u$ is continuous at the locally constant points (Lemma 7.2) so that any connected component of the interior of $\{u = t\}$ should be contained in either $int(D)$ or $int(\Omega \setminus D)$, see Corollary 7.3.

In order to show the sublevel set property, we need to borrow some techniques coming from free boundary problems. The main idea is, first, to adapt the arguments in [24] on unique continuation properties for fractional Laplacian equations in order to show that the level set $\{u = t\}$ has measure zero. This involves looking at the structure of blowup limits at a free boundary point, proving that they are non-trivial in order to get a contradiction.

In particular, for every $s \in (0, 1)$ we prove that blowup sequences converge. Then, to show that their limits are non-trivial, we study optimal regularity and non-degeneracy. Our proof only works when $s \in (0, 1/2]$ because we are only able to control non-degeneracy in this case. More precisely, see the proof of Lemma 6.2, which is based on the arguments in [11]. Note also that the case $s = \frac{1}{2}$ is more involved due to the loss of regularity and it needs to be considered separately in the proof.

One of the crucial steps to obtain regularity is to transform our problem into a two-phase unstable obstacle problem for the fractional Laplacian. Defining $v = t - u$, $f = (\Lambda - \alpha)u$ and $g = -\Lambda u$, we consider

$$\begin{aligned} -(-\Delta)^s u &= f\chi_D - g\chi_{D^c} \quad \text{on } \Omega \subset \mathbb{R}^n, \\ v &= t \qquad \text{on } \mathbb{R}^n \setminus \Omega. \end{aligned} \tag{1.5}$$

In our case, f and g are functions with

$$f > 0, \quad g < 0, \quad f + g < 0, \tag{1.6}$$

which is referred as an unstable problem, see [31, 2].

The classical version of (1.5) has been studied earlier for various conditions on f and g. For instance, if $f > 0$ and $g > 0$, the corresponding two-phase membrane problem was considered in [39, 38, 37, 30]. In the case of $f > 0$ and $f + g > 0$, on the other hand, we refer to [43, 42]. The composite membrane problem, which corresponds to conditions (1.6), can be found in [14, 36, 16, 17] as stated above. However, to the best of authors knowledge, the nonlocal version (1.5), has not been studied so far except [1, 2] whose f and g are constants.

Our second result shows that if the domain has some geometric properties, then optimal pairs also have some geometric properties. The proof is based on the Steiner symmetrization method with a slight modification of the kernel in the singular integral.

Theorem 1.2. *Let Ω be a domain in $\mathbb{R}^n$. Assume that it has symmetry and convexity with respect to the hyperplane $\{x_1 = 0\}$, i.e., for each $x' \in \mathbb{R}^{n-1}$ the set $\{x_1 : (x_1, x') \in \Omega\}$ is either an interval of the form $(-b, b)$ or the empty set. Then, for any optimal pair (u, D) both u and D are symmetric with respect to $\{x_1 = 0\}$, and D^c is convex with respect to $\{x_1 = 0\}$. Moreover, u is decreasing in x_1 for $x_1 \geq 0$.*

Using this symmetry property, we obtain the first uniqueness result when the domain is a ball. Indeed, once we have that the optimal configuration is of the form (1.4), D is determined by A, which makes that our problem does not have a free boundary anymore. For this fixed boundary problem, we can see that u is a unique solution by the strong maximum principle and the fact that any solution has a sign, i.e., either u is always positive or u is always negative.

Corollary 1.3. *Assume that the domain Ω is the unit ball. Then there is a unique optimal pair (u, D) (up to multiplication by nonzero constants). Moreover, the solution u is rotationally symmetric and strictly decreasing in radial direction, and D is a shell region of the form*

$$D = \{x : r(A) \leq |x| < 1\}, \tag{1.7}$$

where $r(A)$ is the constant satisfying $|D| = A$.

Our last result in this paper is a symmetry breaking property when the domain is an annulus in $\mathbb{R}^2$, for $0 < s < \frac{1}{2}$. While the scheme of proof follows closely that of the local case in [14], the arguments in their paper use the fact that the Laplacian has a simple expression in polar coordinates. On the contrary, the fractional Laplacian has no easy decomposition in spherical harmonics (see, for instance, [4] and the references therein).

One of the main ingredients in our proof is the following decomposition formula: in polar coordinates, for any $x = (r, \theta_0) \in \Omega$ with fixed angle, given a function of the form $v = f(r)g(\theta)$,

$$(-\Delta)^s v(x) = g(\theta_0)(-\Delta)^s f(r) + c_{n,s} \int_0^\infty \int_{\partial B_\rho} \frac{f(\rho)(g(\theta_0) - g(\theta))\rho^{n-1}}{|x - (\rho, \theta)|^{n+2s}} \, \mathrm{d}\theta \mathrm{d}\rho.$$

Moreover, we relate the fractional Laplacian $(-\Delta)^s$ of a rotationally symmetric function to the one-dimensional fractional Laplacian $(-\Delta)_1^s$ of a function of radial variable. This connects some rotationally symmetric eigenvalue problem with a third eigenvalue problem which is not rotationally symmetric. See Lemma 9.3 and the equations above for details. We remark that the annulus is symmetric with respect to any axis but it is not convex, which violates the assumption in Theorem 1.2.

Theorem 1.4. *Let $0 < s < \frac{1}{2}$. For an annulus domain*

$$\Omega_b = \{x \in \mathbb{R}^2 : b < |x| < b+1\}$$

with sufficiently large $b \geq 1$, the optimal configuration D in Ω_b does not have rotational symmetry.

This yields non-uniqueness of the optimal pair since any rotation of a solution without symmetry generates a new solution. In conclusion, the uniqueness issue (iii) in Problem (**N**), is that the optimal pair is not unique in general, at least for $0 < s < \frac{1}{2}$, but it is so for some special domains. Other references where symmetry breaking phenomena for fractional Laplacian appear are [33] and [5].

We close this subsection with some remarks on the free boundary regularity. When $s > 1/2$, one can easily obtain that the free boundary near a regular point (i.e. $Du \neq 0$) is locally a C^1 graph by using the implicit function theorem. To obtain a similar result near a singular point (i.e. $Du = 0$), one may need singularity analysis. Nevertheless, if $s \leq 1/2$, one can not expect C^1 regularity near the free boundary, which means that we can not use the implicit function theorem. The expected regularity for free boundary is smooth or analytic, which is left for the future work.

1.3. **Equivalent problems.** Let us discuss two equivalent formulations of Problem (**N**). The first is a local expression via the well-known Caffarelli-Silvestre extension [10]. Let $a = 1 - 2s$ so that $-1 < a < 1$. For a bounded domain $\Omega \subset \mathbb{R}^n$ with $C^{1,1}$ boundary and numbers $\alpha > 0$, $A \in [0, |\Omega|]$, we consider the eigenvalue problem with mixed boundary conditions

$$\begin{aligned} L_a u &= 0 && \text{in } \mathbb{R}^{n+1}_+, \\ -M_a u + \alpha \chi_D u &= \lambda u && \text{on } \Omega \times \{0\} \subset \partial \mathbb{R}^{n+1}_+, \\ u &= 0 && \text{on } (\mathbb{R}^n \setminus \Omega) \times \{0\} \subset \partial \mathbb{R}^{n+1}_+, \end{aligned} \tag{1.8}$$

where $D \subset \Omega$ is any measurable subset in $\mathbb{R}^n$, and the operators L_a and M_a are the usual ones in the extension to $\mathbb{R}^{n+1}_+$ for the fractional Laplacian and are defined in (2.1).

Denote the first eigenvalue by $\lambda_\Omega(\alpha, D)$ and define

$$\Lambda_\Omega(\alpha, A) = \inf_{|D|=A} \lambda_\Omega(\alpha, D). \tag{1.9}$$

If D attains the minimum of (1.9), then we call it as an optimal configuration for the data (Ω, α, A) and (u, D) an optimal pair.

Problem (E). *Investigate the same questions as in Problem* (**N**) *for* (1.8) *and* (1.9).

From the well known extension theorem for the fractional Laplacian [10] we know that

$$M_a u := \lim_{y \to 0^+} y^a u_y = -C_{n,s}(-\Delta)^s \big(u(\cdot, 0)\big).$$

Thus we can identify Problem (**N**) and Problem (**E**). We shall use this equivalence in Section 9 about symmetry breaking phenomena.

We will consider two more equivalent forms of Problem (**E**). First, defining $v = t - u$, we can consider problem (5.1), which is the extension version of (1.5). This will be useful when we deal with blowups in Section 5.

We will also consider the formulation as a physical problem (Problem ($\mathbf{P_N}$) in Section 3). We will see that there is a unique constant $\overline{\alpha}_\Omega(A)$ satisfying $\Lambda_\Omega(\overline{\alpha}_\Omega(A), A) = \overline{\alpha}_\Omega(A)$ so that if $\alpha \leq \overline{\alpha}_\Omega(A)$, then Problem ($\mathbf{P_N}$) is equivalent to Problem (**N**). From this relation, we can give the physical interpretation of our problem, which is to optimize the basic frequency of an elastic membrane whose boundary has two parts; one is fixed and another is free but a prescribed mass is concentrated on the latter. This basic frequency is essentially given by the Steklov eigenvalue of the membrane.

1.4. **Outline.** In Section 2 we introduce notations, definition of weak solutions, and some properties for the solution to the extension problem. Also, in Section 3 we discuss the physical interpretation so that Problem (**N**) contains Problem ($\mathbf{P_N}$), while in Section 4 we prove the existence of optimal pairs and some regularity results, which yield some of the statements in Theorem 1.1. The non-triviality of blowup limits is considered in Section 5 and Section 6, and then we complete the proof of Theorem 1.1 in Section 7. The proofs of Theorem 1.2 and Corollary 1.3 are given in Section 8. Finally, we discuss the symmetry breaking phenomena in Section 9 and hence we prove Theorem 1.4.

2. Preliminaries

2.1. **Notations.** The following notations are used throughout the paper:

(1) Let $|\cdot|$ and $d\sigma$ denote, respectively, Lebesgue measure and the surface measure. The outer unit normal vector is denoted by ν.
(2) Let Ω be a bounded domain with $C^{1,1}$ boundary unless otherwise specified.
(3) Let μ_Ω be the first eigenvalue in Ω for the fractional Laplacian operator with zero Dirichlet boundary conditions on Ω^c.
(4) For a constant $1 < \gamma < 2$, the space C^γ is understood as $C^{1,\gamma-1}$.
(5) Denote $\mathcal{F}$ by the free boundary ∂D.
(6) For a point X in $\mathbb{R}^{n+1}$ we often use x to denote the first n coordinates and y for the last coordinate so that $X = (x, y)$.
(7) For balls and half balls:
$$\begin{aligned} &B_R(X_0) = \{X \in \mathbb{R}^{n+1} : |X - X_0| < R\},\\ &B_R^+(X_0) = B_R(X_0) \cap \{(x,y) \in \mathbb{R}^{n+1} : y > 0\},\\ &\Gamma_R^+ = \partial B_R \cap \{(x,y) \in \mathbb{R}^{n+1} : y \geq 0\},\\ &\Gamma_R^0 = \{(x,0) \in \mathbb{R}^{n+1} : |x| < R\}. \end{aligned}$$
(8) We define the operators on $\mathbb{R}^{n+1}_+$, for $a = 1 - 2s$,
$$\begin{aligned} L_a v &:= \operatorname{div}(y^a \nabla v),\\ M_a v &:= \lim_{y \to 0} y^a v_y. \end{aligned} \tag{2.1}$$

(9) The weighted Lebesgue space $L^2(B, y^a)$ is a Banach space with the norm

$$\|f\|_{L^2(B,y^a)} = \left(\int_B |f(x)|^2 y^a \, \mathrm{d}X < \infty\right)^{1/2}.$$

The weighted Sobolev space $H^1(B, y^a)$ is defined similarly. If $a = 0$, the spaces $L^2(B)$ and $H^1(B)$ are defined in the usual way.

2.2. **Fractional spaces and weak solutions.** Let $0 < s < 1$, and $\Omega \subset \mathbb{R}^n$ be a domain. The fractional Laplacian $(-\Delta)^s$ on $\mathbb{R}^n$ is defined as

$$(-\Delta)^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, \mathrm{d}y,$$

where $c_{n,s}$ is a normalization constant. We also introduce the classical fractional Sobolev space $H^s(\Omega)$ defined by

$$H^s(\Omega) = \left\{ u \in L^2(\Omega) : \frac{u(x) - u(y)}{|x-y|^{\frac{n+2s}{2}}} \in L^2(\Omega \times \Omega) \right\},$$

endowed with the norm

$$\|u\|_{H^s(\Omega)} = \|u\|_{L^2(\Omega)} + \left(\int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{2}}.$$

The localized version of $H^s(\Omega)$ is denoted by

$$H^s_{loc}(\Omega) = \left\{ u \in L^2(\Omega) : u\eta \in H^s(\Omega) \text{ for any test function } \eta \in \mathcal{D}(\Omega) \right\},$$

where $\mathcal{D}(\Omega)$ denotes the space of all continuously infinitely differentiable functions with compact support in Ω.

The admissible set for weak solutions for our non-local equation is given by

$$H^s_0(\Omega) = \{ u \in H^s(\mathbb{R}^n) : u \equiv 0 \text{ in } \mathbb{R}^n \setminus \Omega \}.$$

Weak solutions are then defined as follows: for any bounded function $\rho : \mathbb{R}^n \to \mathbb{R}$, a function u is called a weak solution of

$$\begin{aligned} (-\Delta)^s u + \rho u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega \end{aligned}$$

if $u \in H^s_0(\Omega)$, and

$$\frac{c_{n,s}}{2} \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y + \int_\Omega \rho(x) u(x) \varphi(x) \, \mathrm{d}x = 0.$$

for any $\varphi \in H^s_0(\Omega)$.

2.3. **Extension problem.** In this subsection, we list some properties for the extended function from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}_+$.

We recall the well-known Caffarelli-Silvestre extension (see [10] and also [9]). Given a function $u = u(x)$ on $\mathbb{R}^n$, its extension to $\mathbb{R}^{n+1}_+$ (still denoted by the same letter $u = u(x, y)$) is the solution to

$$\begin{aligned} L_a u = \Delta_x u + \frac{a}{y} u_y + u_{yy} &= 0 \quad \text{in} \quad \mathbb{R}^{n+1}_+, \\ u(x, 0) &= u(x) \quad \text{on} \quad \mathbb{R}^n. \end{aligned} \tag{2.2}$$

The extended function can be also written as

$$u(x,y) = (P(\cdot,y) * u)(x),$$

where P is the Poisson kernel

$$P(x,y) = C_{n,a} \frac{y^{1-a}}{(|x|^2+y^2)^{\frac{n+1-a}{2}}}, \tag{2.3}$$

with the constant $C_{n,a}$ chosen such that $\int_{\mathbb{R}^n} P(x,y)\,\mathrm{d}x = 1$. It is well known that

$$(-\Delta)^s u(x) = d_s M_a u \quad \text{in } \mathbb{R}^n = \partial\mathbb{R}^{n+1}_+,$$

where $d_s = 2^{2s-1}\Gamma(s)/\Gamma(1-s)$. Next, we give the definition of (localized) weak solutions for the extension problem:

Definition 2.1 (Weak solutions)**.** Let $-1 < a < 1$, $r > 0$, and $h \in L^1(\Gamma^0_r)$. A function $u : B^+_r \to \mathbb{R}$ is a weak solution of

$$\begin{aligned} L_a u &= 0 \quad \text{in } B^+_r, \\ M_a u &= h \quad \text{on } \Gamma^0_r, \end{aligned}$$

if $|\nabla u|^2 y^a \in L^1(B^+_r)$ and

$$\int_{B^+_r} (\nabla u \cdot \nabla \varphi) y^a \,\mathrm{d}X + \int_{\Gamma^0_r} h\varphi \,\mathrm{d}x = 0$$

for all $\varphi \in C^1(\overline{B}^+_r)$ such that $\varphi \equiv 0$ on Γ^+_r.

The next Lemma from [1, Theorem 6.4] will give us the optimal regularity when $a \neq 0$. (see also [2, Theorem 2.11]).

Lemma 2.2 (Optimal regularity for $a \neq 0$)**.** *Let $a \neq 0$ and $v \in W^{1,2}(B^+_1, y^a)$ be a bounded weak solution of*

$$\begin{aligned} L_a v &= 0 \quad \textit{in } B^+_1, \\ M_a v &= h \quad \textit{on } \Gamma^0_1. \end{aligned}$$

If $h \in L^\infty(\Gamma^0_1)$, then $v \in C^{1-a}(\overline{B}^+_{1/2})$. Moreover, we have

$$\|v\|_{C^{1-a}(\overline{B}^+_{1/2})} \le C(\|v\|_{L^\infty(B^+_1, y^a)} + \|h\|_{L^\infty(\Gamma^0_1)}),$$

where the constant C depends only on n and a.

We also recall the regularity result when $a = 0$ from [1, Section 5].

Lemma 2.3 (Regularity for $a = 0$)**.** *Let $v \in W^{1,2}(B^+_1)$ be a bounded weak solution of*

$$\begin{aligned} \Delta v &= 0 \quad \textit{in } B^+_1, \\ \partial_y v &= h \quad \textit{on } \Gamma^0_1. \end{aligned}$$

If $h \in L^\infty(\Gamma^0_1)$, then $v \in C^\gamma(\overline{B}^+_{1/2})$ for any $0 < \gamma < 1$. Moreover, we have

$$\|v\|_{C^\gamma(\overline{B}^+_{1/2})} \le C(\|v\|_{L^\infty(B^+_1)} + \|h\|_{L^\infty(\Gamma^0_1)}),$$

where the constant C depends only on n and a.

The following Liouville type theorem from [12, Lemma 2.7] will be used in Section 5.

Lemma 2.4 (Liouville type theorem). *Let v be a harmonic function in $\mathbb{R}^{n+1}$ such that $v(x,y) = v(x,-y)$ for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$. If v has a polynomial growth, i.e.*

$$|v(X)| \leq C(1 + |X|^k)$$

for some constant C and degree k, then v is a polynomial of degree at most k.

3. Physical Interpretation

In this section we shall show the physical interpretation of problem Problem (**N**) in terms of the basic fractional frequency. Fractional frequency is better understood when $s = \frac{1}{2}$, since it is related to the classical Steklov eigenvalue problem.

The local case ($s = 1$) for the composite membrane problem has been well studied (see [14] and related references). Our non-local optimization question is linked to the following:

Problem ($\mathbf{P_N}$). *Build a body of prescribed shape out of given materials of varying density, in such a way that the body has prescribed mass and so that the basic fractional frequency (with fixed boundary) is as small as possible.*

To give the exact mathematical formulation of this problem, we define the class of admissible densities by

$$\mathcal{P} = \left\{ \rho : \Omega \to [h, H] : \int_\Omega \rho(x)\,\mathrm{d}x = M \right\},$$

where h, H, and M are the given constants satisfying $0 \leq h < H$, $0 < M \in [h|\Omega|, H|\Omega|]$. Then Problem ($\mathbf{P_N}$) is to find a density ρ and a body u which achieve the double infimum in

$$\Theta(h,H,M) := \inf_{\rho \in \mathcal{P}} \inf_{u \in \mathcal{H}\setminus\{0\}} \frac{\frac{c_{n,s}}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y}{\int_\Omega \rho(x) u^2(x)\,\mathrm{d}x}. \tag{3.1}$$

The associated Euler-Lagrange equation is

$$\begin{aligned} (-\Delta)^s u &= \Theta \rho u \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega. \end{aligned} \tag{3.2}$$

Moreover, u has a sign in Ω if it is not a constant function.

Lemma 3.1. *Let u be a function achieving the infimum in (3.1). Then u has a sign, i.e., either $u > 0$ in Ω or $u < 0$ in Ω holds.*

Proof. Since (3.1) is invariant under the constant multiplication, we may assume that $\int_\Omega \rho(x)u^2(x)\,\mathrm{d}x = 1$. Now we consider the positive part $u_+ = \max\{u, 0\}$ and the negative part $u_- = \max\{-u, 0\}$, and define

$$J_+ = \int_\Omega \rho(x) u_+^2(x)\,\mathrm{d}x \quad \text{and} \quad J_- = \int_\Omega \rho(x) u_-^2(x)\,\mathrm{d}x$$

so that $J_+ + J_- = 1$. Observe that

$$|u(x) - u(y)|^2 \geq |u_+(x) - u_+(y)|^2 + |u_-(x) - u_-(y)|^2, \tag{3.3}$$

which yields

$$\begin{aligned} \Theta(h,H,M) &\geq \frac{c_{n,s}}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u_+(x) - u_+(y)|^2}{|x-y|^{n+2s}} + \frac{c_{n,s}}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u_-(x) - u_-(y)|^2}{|x-y|^{n+2s}} \\ &\geq \Theta(h,H,M)J_+ + \Theta(h,H,M)J_- = \Theta(h,H,M). \end{aligned}$$

Therefore the inequality in (3.3) should be an equality, that is, either $u_- \equiv 0$ or $u_+ \equiv 0$ holds. To finish the proof let us assume, without loss of generality, that the first case occurs so that $u \geq 0$ in Ω. By applying the strong maximum principle (e.g. [8, Theorem 2.3.3]) to (3.2), we obtain $u > 0$ in Ω. For the second case, we consider $-u$ instead of u. □

To see that Problem ($\mathbf{P_N}$) is contained in Problem ($\mathbf{N}$), we need the following density representation Lemma, which is essentially the "bathtub principle" (see Section 1.14 in [29]):

Lemma 3.2. *Assume that (u, ρ) is a minimizer for Problem ($\mathbf{P_N}$). For a set D such that $\{u < t\} \subset D \subset \{u \leq t\}$ where $t := \sup\{c : |\{u < c\}| < A\}$, let us define $\rho_D = h\chi_D + H\chi_{D^c}$. Then (u, ρ_D) is also a minimizer for Problem ($\mathbf{P_N}$).*

Proof. From Lemma 3.1, u has a sign so we may assume $u > 0$ in Ω. Now the conclusion follows from (3.1) and the inequality

$$
\begin{aligned}
\int_\Omega (\rho_D - \rho)u^2 &= \left(\int_{\{u<t\}} + \int_{\{u=t\}} + \int_{\{u>t\}} \right) (\rho_D - \rho)u^2 \\
&\geq \left(\int_{\{u<t\}} + \int_{\{u=t\}} + \int_{\{u>t\}} \right) (\rho_D - \rho)t^2 \\
&= 0.
\end{aligned}
$$

□

Unless otherwise stated, a density function for the minimizer is always of the form given by Lemma 3.2. Notice also that ρ_D is unique up to a measure zero set if and only if $|\{u = t\}| = 0$.

In the following lemma we see the relation between Problem ($\mathbf{P_N}$) and Problem ($\mathbf{N}$). Since the proof is identical to [14, Theorem 13], we omit it here.

Lemma 3.3. *Problem ($\mathbf{P_N}$) is solved by a pair (u, ρ_D) achieving $\Theta(h, H, M)$ if and only if Problem ($\mathbf{N}$) is solved by a pair (u, D) achieving $\Lambda(\alpha, A)$, where the parameters and the minimal eigenvalue are related by*

$$
\begin{aligned}
\alpha &= (H - h)\Theta(h, H, M), \\
A &= \frac{H|\Omega| - M}{H - h}, \\
\Lambda(\alpha, A) &= H\Theta(h, H, M).
\end{aligned}
$$

Moreover, for any $0 \leq h < H$, the possible value of the parameters is precisely $0 < \alpha \leq \overline{\alpha}_\Omega(A)$ if $0 \leq A < |\Omega|$, where the constant $\overline{\alpha}_\Omega(A)$ will be defined in (4.6), and $0 < \alpha < \infty$ if $A = |\Omega|$.

Until now, we saw that our nonlocal version of the composite membrane problem, Problem ($\mathbf{N}$), is a generalization of the physical problem, Problem ($\mathbf{P_N}$). Our next task is to explain the meaning of fractional frequency for $s = \frac{1}{2}$ by considering the optimization problem for the Steklov eigenvalue under the presence of a density ρ (this is "weighted").

Let $0 < h < H$ and $0 < M \in [h|\Omega|, H|\Omega|]$. As in Problem ($\mathbf{P_N}$), we define the class of admissible densities by

$$\mathcal{P}_S = \left\{ \rho : \partial\Omega \to [h, H] : \int_{\partial\Omega} \rho(x) \, \mathrm{d}\sigma(x) = M \right\},$$

for a C^2-boundary $\partial\Omega$ of a bounded domain $\Omega \subset \mathbb{R}^{n+1}$, and for each $\rho \in \mathcal{P}$, the class of admissible functions by

$$\mathcal{H}_S[\rho] = \left\{ u \in H^1(\Omega) : \int_{\partial\Omega} \rho u \, \mathrm{d}\sigma = 0 \right\}.$$

Now we state the optimization problem for the (weighted) Steklov eigenvalue:

Problem (S). *Find a density $\rho \in \mathcal{P}_S$ and a function $u \in \mathcal{H}_S[\rho]$ which realize the double infimum in*

$$\Theta_S(h, H, M) := \inf_{\rho \in \mathcal{P}_S} \inf_{u \in \mathcal{H}_S[\rho] \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 \, \mathrm{d}x}{\int_{\partial\Omega} \rho u^2 \, \mathrm{d}\sigma},$$

whose Euler-Lagrange equation is

$$\begin{aligned} \Delta u &= 0 && \textit{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= \Theta_S \rho u && \textit{on } \partial\Omega. \end{aligned}$$

According to [28], for a given density ρ and $\Omega \subset \mathbb{R}^2$, the physical meaning of the Steklov problem is constructing a free elastic membrane with prescribed mass concentrated at the boundary (see also [25].) Moreover, from the well known identification $(-\Delta)^{\frac{1}{2}} = \frac{\partial}{\partial \nu}$ when $\Omega = \mathbb{R}^{n+1}_+$ (cf. [10]), Problem ($\mathbf{P_N}$) can be thought as the mixed boundary version of Problem (S), i.e.,

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } A_0, \\ \frac{\partial u}{\partial \nu} &= \Theta_S \rho u && \text{on } A_1, \end{aligned}$$

where A_1 is a domain in $\partial\Omega$, and $A_0 = \partial\Omega \setminus A_1$ (see [6] for mixed Steklov problems). Keeping in mind this interpretation, we may think that the basic fractional frequency in Problem ($\mathbf{P_N}$) describes the basic frequency of a elastic membrane which is fixed on A_0 and free on A_1, with prescribed mass concentrated on A_1.

Notice that we can generalize Problem (S) in the same way as Problem ($\mathbf{P_N}$) does. In fact, the analogies of Lemma 3.2 and Lemma 3.3 can be established with minor modifications, for example, the set D should be of the form

$$\{-t < u < t\} \subset D \subset \{-t \le u \le t\},$$

since u is no longer a positive function in the Steklov problem.

4. Basic properties

In this section we establish existence for Problem (N) and investigate the parameter dependence on Λ.

Lemma 4.1. *For any $\alpha > 0$ and $A \in [0, |\Omega|]$ there exists an optimal pair (u, D) satisfying*

$$u \in H^{2s}_{loc}(\Omega) \cap C^\beta(\Omega) \cap C^s(\mathbb{R}^n),$$

where β is any constant in $(0, 2s)$.

Proof. As in [14], we first investigate a regularity of a weak solution to

$$(-\Delta)^s u + \rho u = 0 \quad \text{in } \Omega,$$

where ρ is a bounded function.

By the result of [7], u belongs to $H^{2s}_{loc}(\Omega)$. From Lemma 2.3 in [32], we can use a standard bootstrapping argument so that $u \in L^\infty(\Omega)$. Then by Proposition 1.1 in [35] gives that $u \in C^s(\mathbb{R}^n)$. Moreover, from Lemma 2.9 in the same paper, we have $u \in C^\beta(\Omega)$ for any $\beta \in (0, 2s)$.

Now we take a minimizing sequence $\{(u_j, D_j)\}$ such that u_j is a positive L^2-normalized first eigenfunction of $(-\Delta)^s + \alpha\chi_{D_j}$ in $H^s_0(\Omega)$, i.e., u_j minimizes the functional

$$\frac{c_{n,s}}{2}\left(\int_\Omega\int_\Omega \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y\right)^2 + \alpha\int_{D_j} w^2\,\mathrm{d}x$$

among all functions $w \in H^s_0(\Omega)$ that satisfy $\|w\|_{L^2(\Omega)} = 1$. The existence of such eigenfunctions is proved in [22]. Since $\lambda(D_j)$ is bounded, u_j is also bounded in $H^s_0(\Omega)$. Note that χ_{D_j} is bounded in $L^2(\Omega)$. Then, up to subsequence, we have

$$\begin{aligned} \chi_{D_j} &\rightharpoonup \eta \quad \text{in } L^2(\Omega) \\ u_j &\rightharpoonup u \quad \text{in } H^s_0(\Omega). \end{aligned}$$

By compactness (see [23]), we can find a strongly convergent subsequence $\{u_j\}$ in $L^2(\Omega)$. Thus we know that $\chi_{D_j} u_j \rightharpoonup \eta u$ in $L^2(\Omega)$. Therefore, u is a weak solution of

$$(-\Delta)^s u + \alpha\eta u = \Lambda u.$$

From the properties of weak convergence, we have

$$0 \le \eta \le 1, \quad \int_\Omega \eta = A.$$

This, and the following inequality

$$\begin{aligned} \int_\Omega (\eta - \chi_D)u^2 &= \left(\int_{\{u<t\}} + \int_{\{u=t\}} + \int_{\{u>t\}}\right)(\eta - \chi_D)u^2 \\ &\ge \left(\int_{\{u<t\}} + \int_{\{u=t\}} + \int_{\{u>t\}}\right)(\eta - \chi_D)t^2 \\ &= 0, \end{aligned} \tag{4.1}$$

where $t := \sup\{c : |\{u < c\}| < A\}$ and for any D satisfying

$$\{u < t\} \subset D \subset \{u \le t\}, \quad |D| = A, \tag{4.2}$$

imply that we may replace η by χ_D. □

Note that Lemma 3.1 implies that u has a sign. With loss of generality, assume in the following that $u > 0$. Now we remark that, from inequality (4.1), the optimal configuration D always has the form (4.2), and thus D contains a tubular neighborhood of $\partial\Omega$.

Lemma 4.2. *Let (u, D) be an optimal pair with $A > 0$. Then:*

(i) The optimal configuration satisfies

$$\{u < t\} \subset D \subset \{u \le t\}, \tag{4.3}$$

where $t := \sup\{c : |\{u < c\}| < A\} > 0$.

(ii) D *contains a tubular neighborhood of the boundary* $\partial\Omega$.

As indicated in the introduction, one of the main results in this paper is to show that D is exactly a sublevel set (Lemma 7.1), i.e.,

$$D = \{x \in \Omega : u(x) \le t\}. \tag{4.4}$$

The proof is very delicate and uses the blowup arguments in Section 5.

We close this section with the parameter dependence on Λ. The proof is standard with some necessary minor variations from [14, Proposition 10], but we include it here for convenience of the reader.

Lemma 4.3. *The optimal frequency* $\Lambda(\alpha, A)$ *is strictly increasing in each variable on* $\mathbb{R}^2_+$*, and* $\Lambda(\alpha, A) - \alpha$ *is strictly decreasing in* α *for fixed* $A > 0$*. Furthermore, the function* $(\alpha, A) \mapsto \Lambda(\alpha, A)$ *is Lipschitz continuous, uniformly on bounded sets, i.e., for any* $\alpha_1, \alpha_2 \ge 0$*, and* $A_1, A_2 \in [0, |\Omega|]$*,*

$$|\Lambda(\alpha_1, A_1) - \Lambda(\alpha_2, A_2)| \le \frac{\max(A_1, A_2)}{|\Omega|}|\alpha_1 - \alpha_2| + C|A_1 - A_2|, \tag{4.5}$$

where $C = C(\Omega, \max\{\alpha_1, \alpha_2\})$*. Consequently, there exists a unique value* $\overline{\alpha}_\Omega(A)$ *for* $A \in [0, |\Omega|)$ *satisfying*

$$\Lambda(\overline{\alpha}_\Omega(A), A) = \overline{\alpha}_\Omega(A), \tag{4.6}$$

and the function $A \mapsto \overline{\alpha}_\Omega(A)$ *is continuous and strictly increasing with* $\overline{\alpha}_\Omega(0) = \mu_\Omega$ *and* $\overline{\alpha}_\Omega(A) \to \infty$ *as* $A \to |\Omega|$*.*

Proof. Let $\Lambda_i = \Lambda(\alpha_i, A_i)$ and let (u_i, D_i) be a minimizer for Λ_i such that $\int_\Omega u_i^2 = 1$ for $i = 1, 2$. Then, we have

$$\Lambda_i = \int_{\mathbb{R}^n} |(-\Delta)^{s/2} u_i|^2 + \alpha_i \int_{D_i} u_i^2, \quad |D_i| = A_i.$$

We may assume $A_1 \le A_2$, and then take $D_1' \subset D_2$ with $|D_1'| = A_1$ and $D_2' \supset D_1$ with $|D_2'| = A_2$. From the optimality one obtains

$$\Lambda_i \le \int_{\mathbb{R}^n} |(-\Delta)^{s/2} u_j|^2 + \alpha_i \int_{D_j'} u_j^2 = \Lambda_j - \alpha_j \int_{D_j} u_j^2 + \alpha_i \int_{D_i'} u_j^2 \tag{4.7}$$

or, equivalently,

$$\Lambda_i - \alpha_i \le \Lambda_j - \alpha_j + \alpha_j \int_{\Omega \setminus D_j} u_j^2 - \alpha_i \int_{\Omega \setminus D_i'} u_j^2 \tag{4.8}$$

for all $i, j \in \{1, 2\}$. Then, (4.7) with $(i, j) = (1, 2)$ and $\alpha_1 = \alpha_2$ yields

$$\Lambda_2 - \Lambda_1 \ge \alpha_1 \int_{D_2 \setminus D_1'} u_2^2 \ge 0.$$

Moreover, equality in the above holds if and only if $u_2 \equiv 0$ on $D_2 \setminus D_1'$ or $\alpha_1 = 0$. By the global strong maximum principle, the former case cannot happen unless $A_1 = A_2$. In fact, for $A_2 > A_1$, $|D_2 \setminus D_1'| > 0$ so that $u_2 \equiv 0$ on $D_2 \setminus D_1'$, which cannot happen. This proves $\Lambda(\alpha, A)$ is strictly increasing in A.

Similarly, (4.7) with $(i,j)=(1,2)$ and $A_1=A_2$ gives

$$\Lambda_2-\Lambda_1 \geq (\alpha_2-\alpha_1)\int_{D_2} u_2^2>0 \tag{4.9}$$

for $\alpha_2>\alpha_1$, and (4.8) with $(i,j)=(2,1)$ and $\alpha_1=\alpha_2$ implies

$$(\Lambda_1-\alpha_1)-(\Lambda_2-\alpha_2)\geq \alpha_1\int_{D_2'\setminus D_1} u_1^2>0$$

for $A_2>A_1$.

For the second part, combining (4.7) with $(i,j)=(1,2)$ and $(i,j)=(2,1)$, we obtain

$$(\alpha_2-\alpha_1)\int_{D_2} u_2^2+\alpha_1\int_{D_2\setminus D_1'} u_2^2\leq \Lambda_2-\Lambda_1\leq (\alpha_2-\alpha_1)\int_{D_2'} u_1^2+\alpha_1\int_{D_2'\setminus D_1} u_1^2, \tag{4.10}$$

so that

$$|\Lambda_2-\Lambda_1|\leq|\alpha_2-\alpha_1|\max\left(\int_{D_2'} u_1^2,\int_{D_2} u_2^2\right)+\alpha_1\max\left(\int_{D_2'\setminus D_1} u_1^2,\int_{D_2\setminus D_1'} u_2^2\right). \tag{4.11}$$

From Lemma 4.2, D_2 satisfies (4.3) with u_2 and $t_2:=\sup\{s:|\{u_2<s\}<A_2\}$. Moreover, we may take D_2' of the form (4.3) for u_1 and some t_2' since (u_1,D_1) is also optimal pair. Now observe that for any $D\subset\Omega$ satisfying (4.3) with any $s>0$ we have

$$\frac{\int_D u^2}{|D|}\leq\frac{\int_\Omega u^2}{|\Omega|},$$

which comes from the fact that average on the whole space is greater than the average on the set $\{u\leq t\}$. Then

$$\max\left(\int_{D_2'} u_1^2,\int_{D_2} u_2^2\right)\leq\frac{A_2}{|\Omega|}\leq 1. \tag{4.12}$$

On the other hand, using (4.10) with $\alpha_2=0$, $A_1=A_2$, one has $\Lambda_i\leq\mu_\Omega+\alpha_i$ for $i=1,2$. Moreover, u_i solves

$$\begin{aligned}(-\Delta)^s u_i+(\alpha\chi_{D_i}-\Lambda_i)u_i=&0\quad\text{in }\Omega,\\ u=&0\quad\text{on }\mathbb{R}^n\setminus\Omega,\end{aligned}$$

whose coefficients are bounded if α is bounded. By global boundedness (c.f. Lemma 2.3 in [7]), we obtain

$$\max\left(\int_{D_2'\setminus D_1} u_1^2,\int_{D_2\setminus D_1'} u_2^2\right)\leq|A_2-A_1|\max\left(\sup_\Omega u_1^2,\sup_\Omega u_2^2\right)\leq C|A_2-A_1|,$$

where $C=C(\Omega,\alpha_1,\alpha_2)$, and hence (4.5) follows from (4.11), (4.12) and the estimate above.

Finally, we observe that $\Lambda(\alpha,A)-\alpha$ equals $\mu_\Omega>0$ for $\alpha=0$, and goes to $-\infty$ as $\alpha\to\infty$ since $\Lambda(\alpha,A)-\alpha=\Lambda-\mu_\Omega+\mu_\Omega-\alpha\leq-(1-\frac{A}{|\Omega|})\alpha+\mu_\Omega$ by taking $A_1=A_2=A$ and $\alpha_2=0$ in (4.5). Therefore, the function $\overline{\alpha}_\Omega$ is well-defined. Again, (4.5) implies the continuity assertion and the first inequality of (4.10) gives monotone assertion if we choose $\alpha_1=\overline{\alpha}_\Omega(A_1)$ and $\alpha_2=\overline{\alpha}_\Omega(A_2)$.

Then the remaining assertions are $\overline{\alpha}_\Omega(0) = 0$, which is trivial, and $\overline{\alpha}_\Omega(A) \to \infty$ as $A \to |\Omega|$. This follows at once by observing

$$\overline{\alpha}_\Omega \int_{\Omega\setminus D_2} u_2^2 \geq \mu_\Omega > 0$$

from (4.9) with $\alpha_2 = \overline{\alpha}_\Omega$ and $\alpha_1 = 0$. □

5. Blowups

Fix $-1 < a = 1-2s < 1$. In this section and the next one we will consider blowups for Problem (**E**) and its non-triviality on $\mathbb{R}^n \times \{0\}$. The results obtained will be used in Section 7 to show that an optimal configuration D is given by the sublevel set of the corresponding solution u, i.e., $D = \{u \leq t\}$ for some t, where (u, D) is an optimal pair. We first discuss the case of $a \neq 0$ ($s \neq \frac{1}{2}$), for which the optimal regularity is given by Lemma 2.2, and then the remaining case, $a = 0$ (this is, $s = \frac{1}{2}$), will be treated.

Throughout this section, Problem (**E**) is converted into a more general problem as in [16] by defining $v = t - u$, $f = (\Lambda - \alpha)u$, and $g = -\Lambda u$, namely

$$\begin{aligned} L_a v = \operatorname{div}(y^a \nabla v) &= 0 && \text{in } \mathbb{R}^{n+1}_+, \\ M_a v = \lim_{y\to 0}(y^a \partial_y v) &= f\chi_D - g\chi_{D^c} && \text{on } \Omega \subset \partial\mathbb{R}^{n+1}_+, \\ v &= t && \text{on } \partial\mathbb{R}^{n+1}_+ \setminus \Omega, \end{aligned} \tag{5.1}$$

where t is given by (4.3). Notice that $f > 0$, $g < 0$, $f + g < 0$ in a neighborhood of the free boundary $\mathcal{F}$, and $\{v > 0\} \subset D \subset \{v \geq 0\}$. Since most of the properties in this section use a local argument near the free boundary, we focus on a half-ball $B^+_{r_0}$ centered on the free boundary $\mathcal{F}$ with small radius $r_0 > 0$ so that $\Gamma^0_{r_0} \Subset \Omega$. By translation, we may assume that the center of this half-ball is the origin. We also assume that for some positive constant η_0,

$$f \geq \eta_0 > 0, \quad g \leq -\eta_0 < 0, \quad \text{and} \quad f + g \leq -\eta_0 < 0, \tag{5.2}$$

over a ball $\Gamma^0_{r_0}$, and that $f, g \in C^s(\overline{\Gamma}^0_{r_0})$. In the rest of this section v will always denote a weak solution of

$$L_a v = 0 \quad \text{in } B^+_{r_0} \quad \text{and} \quad M_a v = f\chi_D - g\chi_{D^c} \quad \text{on } \Gamma^0_{r_0} \tag{5.3}$$

in the sense of Definition 2.1.

From the standard argument of Caccioppoli's inequality (see for instance [26]) we obtain the following energy estimate:

Lemma 5.1 (Energy estimate)**.** *Let $-1 < a < 1$ and v be a weak solution of (5.3). Then for any $0 < r < r_0$ we have*

$$\int_{B^+_{r/2}} |\nabla v|^2 y^a \,\mathrm{d}X \leq \frac{32}{r^2}\int_{B^+_r} v^2 y^a \,\mathrm{d}X + 2\max\{\|f\|_{L^\infty}, \|g\|_{L^\infty}\}\int_{\Gamma^0_r} |v| \,\mathrm{d}x. \tag{5.4}$$

Let us now define the scaled function

$$v_r(X) = \frac{v(rX)}{r^{1-a}}, \quad f_r(x) = f(rx), \quad \text{and} \quad g_r(x) = g(rx),$$

and the scaled configuration set $D_r = \{x \in \mathbb{R}^n : rx \in D\}$ for $0 < r < r_0$. Notice that we assumed $0 \in \partial D$. If $a < 0$, we also assume that $Dv(0) = 0$. Observe that

$$\begin{aligned} L_a v_r &= \operatorname{div}(y^a \nabla v_r) = 0 && \text{in } B^+_{\frac{1}{r}}, \\ M_a v_r &= \lim_{y\to 0} y^a \partial_y v_r = f_r \chi_{D_r} - g_r \chi_{D_r^c} && \text{on } \Gamma^0_{\frac{1}{r}}. \end{aligned}$$

In terms of the scaled function above, inequality (5.4) becomes

$$\int_{B^+_{1/2}} |\nabla v_r|^2 y^a \,\mathrm{d}X \le 32 \int_{B_1^+} v_r^2 y^a \,\mathrm{d}X + 2\max\{\|f_r\|_{L^\infty}, \|g_r\|_{L^\infty}\} \int_{\Gamma_1^0} |v_r| \,\mathrm{d}x. \tag{5.5}$$

This, together with Lemma 2.2, yields:

Lemma 5.2. *Let $a \neq 0$. Assume that v is a bounded weak solution of (5.3). Then there exist a decreasing subsequence $\{r_j\}$ converging to zero and a function v_0 in $C^{1-a}_{loc}(\mathbb{R}^{n+1}_+)$ such that, for any $R > 0$, $v_{r_j} \to v_0$ in $C^\gamma(\overline{B}^+_R)$ for $\gamma < 1 - a$, and $v_{r_j} \rightharpoonup v_0$ in weakly in $H^1(B^+_R, y^a)$ as $j \to \infty$. Moreover, the function v_0 weakly solves the equation*

$$\begin{aligned} L_a v_0 &= 0 \quad \text{in } \mathbb{R}^{n+1}_+, \\ M_a v_0 &= h_0 \quad \text{on } \mathbb{R}^n, \end{aligned}$$

for a function $h_0 \in L^2_{loc}(\mathbb{R}^n)$ satisfying $h_0 \ge \lambda > 0$, where λ is the constant in (5.2).

Proof. The convergence in $C^\gamma(\overline{B}^+_R)$ and the limit $v_0 \in C^{1-a}_{loc}(\mathbb{R}^{n+1}_+)$ follow from Lemma 2.2 since $a \neq 0$, and the weak convergence is a consequence of the estimate (5.5). Thus it suffices to show the second part. Let $h_r := f_r \chi_{D_r} - g_r \chi_{D_r^c}$. Observe that $h_r \in L^2_{loc}(\mathbb{R}^n)$ and $h_r \ge \lambda > 0$. Then there exists a weakly convergent subsequence $\{h_{r_j}\}_{j\in\mathbb{N}}$ and a function $h_0 \in L^2_{loc}(\mathbb{R}^n)$ such that $h_{r_j} \rightharpoonup h_0$ and therefore $h_0 \ge \lambda > 0$. □

The function v_0 is called a *blowup* of v at the origin. For a later use, we also introduce blowups over a sequence. Let $\{x_j\}$ be a convergent sequence whose limit is x_0. We consider the limits $v_{r_j,x_j} \to v_0$ in $C^{1-a}_{loc}(\mathbb{R}^{n+1}_+)$ as $j \to \infty$, where $v_{r_j,x_j}(x) = v(x_j + r_j x)/r^{1-a}$. We call such v_0 a blowup over the sequence $x_j \to x_0$.

In the case of $a = 0$, the scaled functions $\{v_r\}$ may not be uniformly bounded in $L^\infty(B_1^+)$ so that we consider slightly different functions. For this, it will be useful to define the following quantity:

$$C_r := \sup_{B_1^+} \frac{v(rX)}{r}.$$

Lemma 5.3. *Let $a = 0$, $R > 0$, and v be a bounded weak solution of (5.3).*

(i) If $\sup_{0<r<r_0} C_r < \infty$, then there exist a decreasing subsequence $\{r_j\}$ converging to zero and a function v_0 in $C^{0,1}_{loc}(\mathbb{R}^{n+1}_+)$ such that for any $R > 0$, $v_{r_j} \to v_0$ in $C^\gamma(\overline{B}^+_R)$ for $\gamma < 1$ and $v_{r_j} \rightharpoonup v_0$ in weakly in $H^1(B^+_R)$ as $j \to \infty$. Moreover, the function v_0 weakly solves the equation

$$\begin{aligned} \Delta v_0 &= 0 \quad \text{in } \mathbb{R}^{n+1}_+, \\ \partial_y v_0 &= h \quad \text{on } \mathbb{R}^n, \end{aligned}$$

with the function $h \in L^2_{loc}(\mathbb{R}^n)$ satisfying $h \geq \lambda > 0$, where λ is the constant in (5.2).

(ii) If $\sup\limits_{0<r<r_0} C_r = \infty$, then there exist a decreasing subsequence $\{r_j\}$ converging to zero and a function $\tilde{v}_0$ such that for any $R > 0$, $\tilde{v}_{r_j} := v_{r_j}/C_{r_j} \to \tilde{v}_0$ in $C^\gamma(\overline{B}^+_R)$ for $\gamma < 1$ and $\tilde{v}_{r_j} \rightharpoonup \tilde{v}_0$ in weakly in $H^1(B^+_R)$ as $j \to \infty$. Moreover, there is a nonzero vector $(a_1, \cdots, a_n) \in \mathbb{R}^n$ so that

$$\tilde{v}_0(X) = (a_1, \cdots, a_n) \cdot x, \quad \textit{for } X = (x, y) \in \mathbb{R}^n. \tag{5.6}$$

Proof. The first case follows as in the argument in Lemma 5.2. For the second case, we can choose a subsequence $\{r_j\}$ such that $C_{r_j} \to \infty$ as $j \to \infty$ and

$$C_{r_j} \geq \sup_{r_j \leq r \leq r_0} C_r.$$

Then for any $0 < R < r_0/r_j$, we have

$$\int_{B^+_R} \nabla \tilde{v}_{r_j} \cdot \nabla \varphi \, \mathrm{d}X \leq \frac{\max\{\|f\|_{L^\infty(\Gamma^0_R)}, \|g\|_{L^\infty(\Gamma^0_R)}\}}{C_{r_j}} \int_{\Gamma^0_R} |\varphi| \, \mathrm{d}x$$

for all $\varphi \in C^1(\overline{B}^+_R)$ such that $\varphi \equiv 0$ on Γ^+_R. Moreover, we observe that $\tilde{v}_{r_j}(0) = 0$, $\sup_{B^+_1} \tilde{v}_{r_j} = 1$, and for $R \geq 1$,

$$\sup_{B^+_R} \tilde{v}_{r_j} = \frac{\sup_{B^+_R} v_{r_j}}{C_{r_j}} = \frac{\sup_{B^+_1} v_{r_j R}}{C_{r_j}} R = \frac{C_{r_j R}}{C_{r_j}} R \leq R.$$

From Lemma 2.3 and Lemma 5.1, there exist a subsequence, again denoted by $\{r_j\}$, and a function $\tilde{v}_0$ in $C^{0,1}_{loc}(\overline{\mathbb{R}}^{n+1}_+)$ such that for any $R > 0$, $v_{r_j} \to v_0$ in $C^\gamma(\overline{B}^+_R)$ for $\gamma < 1$ and $\tilde{v}_{r_j} \rightharpoonup \tilde{v}_0$ in weakly in $H^1(B^+_R)$ as $j \to \infty$. Thus we have $\tilde{v}_0(0) = 0$, $\sup_{B^+_1} \tilde{v}_0 = 1$, $\sup_{B^+_R} \tilde{v}_0 \leq R$, and

$$\int_{B^+_R} \nabla \tilde{v}_0 \cdot \nabla \varphi \, \mathrm{d}X = 0$$

for all $\varphi \in C^1(\overline{B}^+_R)$ such that $\varphi \equiv 0$ on Γ^+_R. Notice that the last equality follows by considering $-\varphi$ instead of φ. If we evenly reflect $\tilde{v}_0$ across $\{y = 0\}$, then the new function, still denoted by $\tilde{v}_0$, is harmonic in $\mathbb{R}^{n+1}$ satisfying

$$\tilde{v}_0(X) \leq 1 + |X| \quad \text{for all } X \in \mathbb{R}^{n+1}.$$

By the Liouville type result from Lemma 2.4, $\tilde{v}_0$ is a polynomial of degree at most one. Now (5.6) can be deduced from the fact $\tilde{v}_0(0) = 0$, $\sup_{B_1} \tilde{v}_0 = 1$, and $\tilde{v}_0$ is even in y-variable. □

6. Nondegeneracy

In this section we will show nondegeneracy, which will imply that blowups are not identically zero over $\mathbb{R}^n$. Notice that from the optimal regularity near ∂D, we see that $|t - u(x)| \leq C \operatorname{dist}(x, \partial D)^{2s}$ for some constant C. Nondegeneracy gives the opposite inequality. More precisely:

Lemma 6.1. *Let (u, D) be an optimal pair and $a \neq 0$. It holds:*

(i) *There exist positive constants* c_0 *and* C_0 *such that if* $x \in \{u < t\}$ *and* $\operatorname{dist}(x, \partial D) \le c_0$, *then*

$$u(x) \le t - C_0 \operatorname{dist}(x, \partial D)^{2s}. \tag{6.1}$$

(ii) *There exist positive constants* c_0 *and* C_0 *such that if* $x \in \{u > t\}$ *and* $\operatorname{dist}(x, \partial D) \le c_0$, *then*

$$u(x) \ge t + C_0 \operatorname{dist}(x, \partial D)^{2s}. \tag{6.2}$$

Proof. Fix a point x_0 in $\{u < t\}$, and let $d_0 = \operatorname{dist}(x_0, \partial D) > 0$ and $\beta = t - u(x_0) > 0$. We may assume x_0 is the origin. Denote by u_1 the extension of u to $\mathbb{R}^{n+1}_+$ through (1.8). We set $w := t - u_1(x,y) + t(\Lambda - \alpha)(1-a)^{-1} y^{1-a}$ with $1 - a = 2s$, which satisfies

$$L_a w = 0 \quad \text{and} \quad -M_a w = (\Lambda - \alpha) w.$$

Applying Harnack's inequality (see [41]) to w in a neighborhood of x_0, we have

$$\underline{c}\beta \le t - u(x) \le \overline{c}\beta \quad \text{in } B_{d_0/2}$$

for some positive constants $\underline{c}$ and $\overline{c}$. Now we define

$$\tilde{u}(x) = \begin{cases} \max\{u(x), t - \overline{c}\beta\psi(x)\} & \text{if } x \in B_{d_0/2}, \\ u(x) & \text{otherwise,} \end{cases}$$

where ψ is a radial cut-off function such that $\psi \equiv 0$ in $B_{d_0/4}$ and $\psi \equiv 1$ outside $B_{d_0/2}$.

We are going to use the following inequality: given $A', A'', B', B'' > 0$, if $B'/A' \le B''/A''$, then $(A'' - A')B'/A' \le B'' - B'$. From this inequality, together with the minimality of Λ, we have

$$\Lambda \int_\Omega \tilde{u}^2 \, \mathrm{d}x - \Lambda \int_\Omega u^2 \, \mathrm{d}x \le \|(-\Delta)^{s/2} \tilde{u}\|^2 - \|(-\Delta)^{s/2} u\|^2 + \alpha \int_D \tilde{u}^2 \, \mathrm{d}x - \alpha \int_D u^2 \, \mathrm{d}x.$$

Since $\tilde{u} \ge u \ge 0$ and $D \subset \Omega$, we arrive at

$$(\Lambda - \alpha) \left(\int_\Omega \tilde{u}^2 \, \mathrm{d}x - \int_\Omega u^2 \, \mathrm{d}x \right) \le \|(-\Delta)^{s/2} \tilde{u}\|^2 - \|(-\Delta)^{s/2} u\|^2. \tag{6.3}$$

To further proceed, we observe that

$$\int_\Omega \tilde{u}^2 \, \mathrm{d}x - \int_\Omega u^2 \, \mathrm{d}x \ge \int_{B_{d_0/4}} t^2 - (t - \underline{c}\beta)^2 = |B_{d_0/4}|(2t\underline{c}\beta - \underline{c}^2\beta^2),$$

and that for $K := \{x \in B_{d_0/2} : u(x) < t - \overline{c}\beta\psi(x)\}$ the following inequalities hold:

(i) if $x \in K$ and $y \in K^c \cap B_{d_0/2}$ then

$$\begin{aligned} (\tilde{u}(x) - \tilde{u}(y))^2 - (u(x) - u(y))^2 &= (t - \overline{c}\beta\psi(x) - u(x))(t - \overline{c}\beta\psi(x) + u(x) - 2u(y)) \\ &\le 2(1 - \psi(x))(\overline{c}\beta)^2(\psi(y) - \psi(x)), \end{aligned}$$

(ii) if $x \in K$ and $y \in K^c \cap B^c_{d_0/2} = B^c_{d_0/2}$ then

$$\begin{aligned} (\tilde{u}(x) - \tilde{u}(y))^2 - (u(x) - u(y))^2 &= (t - \overline{c}\beta\psi(x) - u(x))(t - \overline{c}\beta\psi(x) + u(x) - 2u(y)) \\ &\le (t - \overline{c}\beta\psi(x) - u(x))^2 + 2\overline{c}\beta(1 - \psi(x))(u(x) - u(y)) \\ &\le (\overline{c}\beta)^2(1 - \psi(x))^2 + 2\overline{c}\beta(1 - \psi(x))(u(x) - u(y)). \end{aligned}$$

Notice also that

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(\tilde{u}(x)-\tilde{u}(y))^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\le (\bar{c}\beta)^2 \int_K\int_K \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}y\,\mathrm{d}x \\
&+ 4(\bar{c}\beta)^2 \int_K\int_{K^c\cap B_{d_0/2}} \frac{(1-\psi(x))(\psi(y)-\psi(x))}{|x-y|^{n+2s}}\,\mathrm{d}y\,\mathrm{d}x \\
&\quad + 2(\bar{c}\beta)^2 \int_K\int_{B^c_{d_0/2}} \frac{(\psi(y)-\psi(x))^2}{|x-y|^{n+2s}}\,\mathrm{d}y\,\mathrm{d}x \\
&+ 4\bar{c}\beta \int_K\int_{B^c_{d_0/2}} \frac{(1-\psi(x))(u(x)-u(y))}{|x-y|^{n+2s}}\,\mathrm{d}y\,\mathrm{d}x \\
&=: I_1+I_2+I_3+I_4.
\end{aligned}
$$

By definition of ψ, we see that $\tilde{\psi}(x) := \psi(d_0x/2)$ is a radial cut-off function which is independent of d_0. Thus we have

$$
I_1+I_2+I_3 \le C\beta^2 d_0^{n-2s},
$$

where the constant C does not depend on β and d_0. To estimate I_4, we may assume that $1-\psi(x) = O((d_0/2-|x|)^2)$ and that $\|u\|_{L^\infty} = 1$, which implies

$$
\begin{aligned}
I_4 &\le C\beta \int_K\int_{B^c_{d_0/2}} \frac{(d_0/2-|x|)^2}{|x-y|^{n+2s}}\,\mathrm{d}y\,\mathrm{d}x \le C\beta \int_K\int_{B^c_{d_0/2-|x|}(x)} \frac{(d_0/2-|x|)^2}{|x-y|^{n+2s}}\,\mathrm{d}y\,\mathrm{d}x \\
&\le C\beta \int_K (d_0/2-|x|)^{2-2s}\,\mathrm{d}x \le C\beta d_0^{n+2-2s}.
\end{aligned}
$$

Combining these facts, together with (6.3), we obtain

$$
(\Lambda-\alpha)|B_{d_0/4}|(2t\underline{c}\beta - \underline{c}^2\beta^2) \le C\beta^2 d_0^{n-2s} + C\beta\,\|u\|_{L^\infty}\, d_0^{n+2-2s}.
$$

By the optimal regularity, we can take small c_0 so that $\beta \le \|u\|_{C^{2s}}\, d_0^{2s} \le \underline{c}t$, which gives

$$
(\Lambda-\alpha)t\underline{c} \le C\beta d_0^{-2s} + C\,\|u\|_{L^\infty}\, d_0^{2-2s}.
$$

Again, taking small c_0, we conclude that

$$
\beta \ge C_0 d_0^{2s}
$$

for some constant C_0. This completes the proof of (6.1). Since the proof of (6.2) is similar to that of (6.1), we omit the details here. $\square$

Using the previous Lemma, together with a blowup argument, we are able to show nondegeneracy. Let us denote the nearest point to x in ∂D by $\tilde{x}$ so that $\mathrm{dist}(x,\partial D) = \mathrm{dist}(x,\tilde{x})$. The argument follows as in [11]:

Lemma 6.2. *Let (u,D) be an optimal pair and $a>0$. Take x_0 to be a point in ∂D. Then there is a constant C, independent of u, such that*

$$
\sup_{B_r(x_0)} |t-u| \ge Cr^{2s}.
$$

Proof. Let $x_0 \in \partial D$ and $B_r(x_0) \subset \Omega$. Let $x_1 \in B_r(x_0)$ such that $u(x_1) < t$ and $d_1 := \text{dist}(x_1, \partial D) < c_0$ where the constant c_0 is defined in Lemma 6.1. By the same Lemma,

$$\tau := \frac{t - u(x_1)}{d_1^{2s}} \geq C_0.$$

We claim that there exist constants $\delta > 0$ and $M > 0$ which are independent of x_1 and such that

$$\sup_{B_{Md_1}(\tilde{x}_1)} (t - u(x)) \geq (1 + \delta)\tau d_1^{2s}.$$

If not, we can take a sequence x_k with $d_k := \text{dist}(x_k, \partial D)$ so that

$$\sup_{B_{kd_k}(\tilde{x}_k)} (t - u(x)) \leq \left(1 + \frac{1}{k}\right) \tau d_k^{2s}. \tag{6.4}$$

Now we define

$$v_{k,\tilde{x}_k}(x) = \frac{t - u(\tilde{x}_k + \text{dist}(x, \partial D)x)}{\text{dist}(x, \partial D)^{2s}}.$$

In terms of $v_{k,\tilde{x}_k}$, (6.4) becomes

$$\sup_{B_k(0)} v_{k,\tilde{x}_k} \leq \left(1 + \frac{1}{k}\right) \tau.$$

Then, passing to the limit, we have a limit v_0 satisfying $\sup_{\mathbb{R}^n} v_0 \leq \tau$, $v_0(0) = 0$, and $v_0(z) = \tau > 0$ for some $|z| = 1$, which is a contradiction.

Using the claim, we construct a sequence $\{x_j\}$ such that $|x_j - x_{j-1}| \leq (M+1)d_{j-1}$ and $t - u(x_j) \geq (1+\delta)(t - u(x_{j-1}))$. Since $\delta > 0$ does not depend on x_j, we deduce that there is an index j so that x_j exits from $B_r(x_0)$. Assume that j_0 is the first index such that $x_{j_0} \in B_r(x_0)$ and $x_{j_0+1} \notin B_r(x_0)$. Then we have

$$t - u(x_{j_0}) = \sum_{j \leq j_0} (t - u(x_j) - (t - u(x_{j-1}))) \geq \delta \sum_{j \leq j_0} (t - u(x_{j-1})).$$

By Lemma 6.1, we see that $t - u(x_{j-1}) \geq C_0 d_{j-1}^{2s} \geq C_0(M+1)^{-2s}|x_j - x_{j-1}|^{2s}$. If $2s \leq 1$, then we have

$$\sum_{j \leq j_0} |x_j - x_{j-1}|^{2s} \geq \left(\sum_{j \leq j_0} (x_j - x_{j-1})\right)^{2s} = |x_{j_0} - x_0|^{2s} \geq r^{2s}.$$

Combining these facts, we conclude that

$$\sup_{B_{Mr}} (t - u(x)) \geq t - u(x_{j_0}) \geq C r^{2s}$$

and, therefore, the desired estimate is obtained by replacing r by r/M. In a similar way, if we have a point $x_1 \in B_r(x_0)$ such that $u(x_1) > t$, we obtain $\sup_{B_r}(u(x) - t) \geq Cr^{2s}$. This completes the proof. □

We remark that in the case of $a = 0$ the proof of Lemma 6.2 also holds for the points at which the pointwise $C^{0,1}$ norm is bounded. In summary, we have the following results on non-triviality for blowups.

Corollary 6.3. *Let (u, D) be an optimal pair and $a > 0$. If a function v_0 is any blowup of $v := t - u$, then v_0 is not trivial, i.e., $v_0 \not\equiv 0$ on $\mathbb{R}^n$.*

Corollary 6.4. *Let (u, D) be an optimal pair and $a = 0$. Then either any convergent subsequence of the rescaled function v_r or that of v_r/C_r has a non-trivial limit, where C_r is the quantity in Lemma 5.3.*

7. Structure of optimal configuration

In this section, we shall prove the equation (1.4), i.e., the optimal configuration D is given by the sublevel set of u, for $s \leq 1/2$ ($a \geq 0$). The results from the previous section are the key ingredients. We follow the argument in [24].

Lemma 7.1. *Let (u, D) be an optimal pair and $a \geq 0$. The optimal configuration D is given by the sublevel set of optimal solution u, i.e.,*

$$D = \{x \in \Omega : u(x) \leq t\}.$$

Proof. Since D satisfies (4.3), it suffices to show that the t-level set Γ_t of u has measure zero. Assume that Γ_t has positive measure. By Lebesgue's density theorem,

$$\chi_{\Gamma_t}(x) = \lim_{r \to 0+} \frac{1}{|B_r(x)|} \int_{B_r(x)} \chi_{\Gamma_t}$$

for all $x \in \mathbb{R}^n \setminus N$, where $|N| = 0$. This implies that, for all $x \in \Gamma_t \setminus N$,

$$0 = \lim_{r \to 0+} \frac{|B_r(x) \cap (\mathbb{R}^n \setminus \Gamma_t)|}{|B_r(x)|}.$$

Now we fix a point $x_0 \in \Gamma_t \setminus N$. For any $\varepsilon > 0$ there exists r_0 such that, if $0 < r < r_0$, then

$$|B_r(x_0) \cap (\mathbb{R}^n \setminus \Gamma_t)| \leq \varepsilon |B_r(x_0)|.$$

We recall $v(x) = t - u(x)$ from the previous section and observe $v \equiv 0$ in Γ_t. Then we have

$$\begin{aligned}
\int_{B_r(x_0)} v^2 \,\mathrm{d}x &= \int_{B_r(x_0) \cap (\mathbb{R}^n \setminus \Gamma_t)} v^2 \,\mathrm{d}x \\
&\leq |B_r(x_0) \cap (\mathbb{R}^n \setminus \Gamma_t)|^{1-\frac{2}{2^*}} \left(\int_{B_r(x_0)} v^{2^*} \,\mathrm{d}x \right)^{2/2^*} \\
&\leq \varepsilon^{1-\frac{2}{2^*}} |B_r(x_0)|^{1-\frac{2}{2^*}} \left(\int_{B_r(x_0)} v^{2^*} \,\mathrm{d}x \right)^{2/2^*},
\end{aligned}$$

where $2^* = \frac{2n}{n-2s}$. In the notation of Section 5, recall that we have defined $v_r(x) = r^{-2s} v(x_0 + rx)$ so that the above inequality becomes

$$\int_{B_1(0)} v_r^2 \,\mathrm{d}x \leq \varepsilon^{1-\frac{2}{2^*}} |B_1(0)|^{1-\frac{2}{2^*}} \left(\int_{B_1(0)} v_r^{2^*} \,\mathrm{d}x \right)^{2/2^*}.$$

Note that this still holds for v_r/C_r instead of v_r. Then, Corollary 6.3 and Corollary 6.4 imply that we have a subsequence $\{r_k\}$ such that either $\{v_{r_k}\}$ or $\{\tilde{v}_{r_k}\}$

converge to a nonzero function v_0 or $\tilde{v}_0$ as $k \to \infty$, respectively. By taking $k \to \infty$ in the above inequality we obtain, either for $\overline{v} = v_0$ or for $v = \tilde{v}_0$,

$$\int_{B_1(0)} \overline{v}^2 \,\mathrm{d}x \le \varepsilon^{1-\frac{2}{2^*}} |B_1(0)|^{1-\frac{2}{2^*}} \left(\int_{B_1(0)} \overline{v}^{2^*} \,\mathrm{d}x \right)^{2/2^*},$$

and then, by taking $\varepsilon \to 0$, we finally have $v_0 \equiv 0$ or $\tilde{v}_0 \equiv 0$ in $B_1(0)$, respectively, which is a contradiction. Therefore, $|\Gamma_t| = 0$. □

We are now ready to prove our main Theorem:

Proof of Theorem 1.1. The regularity assertions follow from both Lemma 4.1 and Lemma 2.2. Now Lemma 4.2 and Lemma 7.1 give the sublevel set property.

□

Next, we investigate some properties of the optimal configuration D for general $\alpha > 0$. To do this, we begin with the following Lemma which implies the continuity of $(-\Delta)^s u$.

Lemma 7.2. *Let $u \in H^{2s}(\Omega) \cap L^\infty(\Omega)$. If u is locally constant near a point $x_0 \in \Omega$, then $(-\Delta)^s u(x)$ is continuous at x_0.*

Proof. Let u be constant in $B_r(x_0)$ for some $r > 0$, and take a sequence $\{x_k\} \subset B_{\frac{r}{2}}(x_0)$ converging to x_0. Notice that, for $\rho_k = r - |x_k - x_0|$, $B_{\rho_k}(x_k) \subset B_r(x_0)$, and thus

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{u(x_k) - u(y)}{|x_k - y|^{n+2s}} \,\mathrm{d}y &\le 2 \|u\|_{L^\infty} \int_{\mathbb{R}^n \setminus B_{\rho_k}(x_k)} \frac{1}{|x_k - y|^{n+2s}} \,\mathrm{d}y \\
&= 2 \|u\|_{L^\infty} \frac{n\omega_n \rho_k^{-2s}}{2s} \\
&\le \frac{n\omega_n 2^{2s} \|u\|_{L^\infty}}{s r^{2s}}.
\end{aligned}$$

Using this and Lebesgue dominated convergence theorem, we conclude that

$$\lim_{k \to \infty} (-\Delta)^s u(x_k) = c_{n,s} \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{u(x_0) - u(y)}{|x_0 - y|^{n+2s}} \,\mathrm{d}y = (-\Delta)^s u(x_0).$$

□

We remark here that our regularity statement in Lemma 4.1 does not imply the continuity of $(-\Delta)^s u$.

Corollary 7.3. *Let (u, D) be an optimal pair. If u is locally constant near a point $x_0 \in \Omega$, then either $x_0 \in int\ (D)$ or $x_0 \in int\ (\Omega \setminus D)$ holds.*

Proof. From Lemma 7.2, $(-\Delta)^s u$ is continuous at x_0, and therefore so is $(\Lambda - \alpha\chi_D)u$. Since u is a continuous function, there is a neighborhood U of x_0 such that $U \subset D$ or $U \subset \Omega \setminus D$ hold. This completes the proof. □

The last Lemma in this section asserts that the any level set $\{u = c\}$ does not have an interior point in Ω if $c > 0$. In particular, $\{u = t\}$ has no interior points.

Lemma 7.4. *Let (u, D) be an optimal pair. Then u is not locally constant near any point in Ω.*

Proof. Assume that u is a locally constant near a point x_0 in Ω. From Corollary 7.3, $(\Lambda - \alpha\chi_D)u$ is a locally constant function. Then using the unique continuation property (see [24]), we have $u \equiv 0$, which yields a contradiction. $\square$

8. Symmetry property

We devote this section in proving a symmetry property when the domain has a directional symmetry and convexity. The basic idea is to use Steiner symmetrization, but there is a technical issue when we consider the equality case. To overcome this, we slightly modify the kernel a little bit.

Proof of Theorem 1.2. As in the local case from [14], we apply Steiner symmetrization to the function $u(\cdot, x')$ and the set $\{x_1 : (x_1, x') \in D\}$ for each $x' = (x_2, \cdots, x_n)$. We refer the reader to Chapter 3 in [29] for the definition and various properties of Steiner symmetrization.

Let $u^*(\cdot, x')$ be the Steiner symmetrization of $u(\cdot, x')$ for each x', namely the function u^* is symmetric in x_1 and decreasing for $x_1 \geq 0$ with the same measure of super level set

$$|\{x_1 : u^*(x_1, x') > t\}| = |\{x_1 : u(x_1, x') > t\}|.$$

Using the integral representation, $\int f \,\mathrm{d}x = \int_0^\infty \{f > t\} \,\mathrm{d}t$, an easy consequence of the definition is that $\int_{\mathbb{R}} (u^*)^2 \,\mathrm{d}x_1 = \int_{\mathbb{R}} u^2 \,\mathrm{d}x_1$. Thus, integrating in x',

$$\int_\Omega (u^*)^2 \,\mathrm{d}x = \int_\Omega u^2 \,\mathrm{d}x. \tag{8.1}$$

It is also a well-known property that $\int_{\mathbb{R}} \chi_{D^c} u^2 \,\mathrm{d}x_1 \leq \int_{\mathbb{R}} (\chi_{D^c})^* (u^*)^2 \,\mathrm{d}x_1$, which is equivalent to

$$\int_{\mathbb{R}} (\chi_{D_*})(u^*)^2 \,\mathrm{d}x_1 \leq \int_{\mathbb{R}} \chi_D u^2 \,\mathrm{d}x_1, \tag{8.2}$$

where the set D_* is defined by $\chi_{D_*} = 1 - (\chi_{D^c})^*$. Again, we integrate (8.2) in x' to obtain

$$\int_\Omega (\chi_{D_*})(u^*)^2 \,\mathrm{d}x \leq \int_\Omega \chi_D u^2 \,\mathrm{d}x. \tag{8.3}$$

Now we claim that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u^*(x) - u^*(y))^2}{|x-y|^{n+2s}} \,\mathrm{d}x \,\mathrm{d}y \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x-y|^{n+2s}} \,\mathrm{d}x \,\mathrm{d}y, \tag{8.4}$$

and that this inequality holds with equality if and only if $u = u^*$. To see this, consider an approximate kernel

$$K_\varepsilon(x_1; x') = (|x_1|^2 + |x'|^2 + \varepsilon)^{-\frac{n+2s}{2}}$$

for $\varepsilon > 0$ and note that $\|K_\varepsilon(\cdot;x')\|_{L^1(\mathbb{R})} \le \|K_\varepsilon(\cdot;0)\|_{L^1(\mathbb{R})} \le C$. It follows from Theorem 3.7 in [29] that

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} (u(x)-u(y))^2 K_\varepsilon(x_1-y_1;x'-y')\,\mathrm{d}x_1\,\mathrm{d}y_1 \\
&\quad = 2\int_{\mathbb{R}} u(y)^2 \|K_\varepsilon(\cdot;x'-y')\|_{L^1(\mathbb{R})}\,\mathrm{d}x_1 \\
&\qquad - 2\int_{\mathbb{R}}\int_{\mathbb{R}} u(x)u(y)K_\varepsilon(x_1-y_1;x'-y')\,\mathrm{d}x_1\,\mathrm{d}y_1 \\
&\quad \ge 2\|K_\varepsilon(\cdot;x'-y')\|_{L^1(\mathbb{R})}\int_{\mathbb{R}}(u^*)^2\,\mathrm{d}x_1 \\
&\qquad - 2\int_{\mathbb{R}}\int_{\mathbb{R}} u^*(x)u^*(y)K_\varepsilon(x_1-y_1;x'-y')\,\mathrm{d}x_1\,\mathrm{d}y_1 \\
&\quad = \int_{\mathbb{R}}\int_{\mathbb{R}} (u^*(x)-u^*(y))^2 K_\varepsilon(x_1-y_1;x'-y')\,\mathrm{d}x_1\,\mathrm{d}y_1.
\end{aligned} \tag{8.5}$$

Since the both hand sides of this inequality converge to the claimed expressions as $\varepsilon \to 0$, the result follows by monotone convergence theorem and integrating in (x',y').

To verify the equality condition in (8.4), first note that if $x' \neq y'$, then the inequality (8.5) holds even for $\varepsilon = 0$ since $\|K_0(\cdot;x'-y')\|_{L^1(\mathbb{R})} \le C$. Thus, the equality in (8.4) implies that in (8.5) with $x' \neq y'$ and $\varepsilon = 0$. Then, from Theorem 3.9 in [29], $u(x) = u^*(x_1 - z, x')$, where $z \in \mathbb{R}$ depends on x'. The number z, however, must be zero because u is symmetric with respect to the hyperplane $\{x_1 = 0\}$. Thus $u \equiv u^*$, as claimed.

Now we are ready to prove assertions. Since the eigenvalue $\lambda(\alpha, D)$ is given by

$$\inf_u \frac{\left\|(-\Delta)^{s/2}u\right\|^2 + \alpha\int_D u^2\,\mathrm{d}x}{\int_\Omega u^2\,\mathrm{d}x},$$

we have from (8.1), (8.3), and (8.4) that $\lambda(\alpha, D_*) \le \lambda(\alpha, D)$. Therefore, if (u, D) is an optimal pair, then $\Lambda(\alpha, D) = \lambda(\alpha, D_*)$ and, by the equality condition in (8.4), $u \equiv u^*$. This proves the Theorem. □

Using this, we can show that the optimal configuration is an annulus when the domain is a ball.

Proof of Corollary 1.3. From Theorem 1.2, u is a rotationally symmetric function and decreases in the radial direction. Moreover, Lemma 7.1 implies (1.7) and then the strictly decreasing property follows from Lemma 7.4. To prove the uniqueness assertion, we first note that $r(A)$ does not depend on u so that the optimal configuration D is unique. Now we assume that there are two solutions u_1 and u_2 with t_1 and t_2 such that

$$D = \{u_1 \le t_1\} = \{u_2 \le t_2\}.$$

Define $v := u_1/t_1 - u_2/t_2$ and notice that v solves

$$\begin{aligned}
(-\Delta)^s v + \alpha\chi_D v &= \Lambda v \quad \text{in } \Omega, \\
v &= 0 \qquad \text{on } \mathbb{R}^n \setminus \Omega.
\end{aligned}$$

From the definition of Λ, together with Lemma 3.1 and Lemma 3.3, we can see that v has a sign in Ω. This is a contradiction since $v(x) = 0$ for $|x| = r(A)$. □

9. SYMMETRY BREAKING

In the previous section, we proved the symmetry property of the optimal pair (u, D) when the domain has directional symmetry and convexity. Here we construct an example which presents symmetry breaking when the domain has radial symmetry for the case $s < \frac{1}{2}$. An immediate consequence is non-uniqueness. Notice that the ball is the only case we can prove the uniqueness.

In [14], the authors give the symmetry breaking examples when the domains are an annulus and a dumbbell shape for the (local) composite membrane problem. For the nonlocal equation, Nápoli considered in [21] the symmetry breaking property for an elliptic equation involving the fractional Laplacian. In that work, the author proved that there are both a nontrivial radial solution and a non-radial one for a nonlocal elliptic problem. Note that we shall prove that, for some large annular domain, the nonlocal composite membrane problem admits only non-radially symmetric solutions.

We follow the argument in [14], considering the third eigenvalue problem connecting the radial symmetry eigenvalue problem to the non-radial one. However, some difficulties occur from the nonlocality; for example, it is not clear how to decompose the fractional Laplacian into radial and angular parts.

For an annulus

$$\Omega_b = \{x \in \mathbb{R}^2; b < |x| < b+1\}, \quad b > 0,$$

and a radial subset D in Ω_b such that

$$D = \{(r, \theta); r \in D_1, 0 \le \theta < 2\pi\}, \quad D_1 \subsetneq (b, b+1), \tag{9.1}$$

we consider the eigenvalue problem of the form

$$\begin{cases} (-\Delta)^s u + \alpha \chi_D u = \sigma u & \text{in } \Omega_b, \\ u = 0 \quad \text{on } \mathbb{R}^n \setminus \Omega_b, \end{cases} \tag{9.2}$$

where u and σ are the first eigenfunction and eigenvalue, respectively.

We shall construct a function $\tilde{u}$ and a domain $\tilde{D}$ with $|\tilde{D}| = |D|$, which satisfy

$$\frac{\int_{\Omega_b} \tilde{u}(-\Delta)^s \tilde{u}\,\mathrm{d}x + \alpha \int_{\Omega_b} \chi_{\tilde{D}} \tilde{u}^2\,\mathrm{d}x}{\int_{\Omega_b} \tilde{u}^2\,\mathrm{d}x} < \sigma.$$

This means that any domain D having symmetry is not an optimal configuration.

Let $\delta = |D|/|\Omega_b|$ and take a number $N = N(\delta)$ such that

$$\delta < 1 - \frac{1}{2N}.$$

To construct $(\tilde{u}, \tilde{D})$, we define the sector

$$E_+ = \Omega_b \cap \{(r, \theta); 0 \le \theta \le \pi/N\}.$$

Then we may choose $\tilde{D} \subset \Omega_b \setminus E_+$ since $|\tilde{D}| = \delta|\Omega_b| < (1 - \frac{1}{2N})|\Omega_b| = |\Omega_b \setminus E_+|$.

Let $\tilde{u}$ be the first Dirichlet eigenfunction of the fractional Laplacian on E_+ and $\lambda_1(E_+)$ be the first eigenvalue so that

$$\begin{aligned} (-\Delta)^s \tilde{u} =& \lambda_1(E_+)\tilde{u} \quad \text{in} \quad E_+, \\ \tilde{u} =& 0 \quad \text{on} \quad \mathbb{R}^n \setminus E_+. \end{aligned}$$

Note that since supp $\tilde{u} \cap \tilde{D} = \emptyset$, it is enough to show that

$$\lambda_1(E_+) < \sigma. \tag{9.3}$$

In order to prove this, we need to introduce an intermediate eigenvalue problem. Define v_0 to be the lowest eigenfunction for (9.2) among functions of the form

$$v(r,\theta) = h(r)\sin N\theta,$$

and let τ be the associated eigenvalue. Clearly, $\sigma \le \tau$. We claim that τ is close enough to σ when b is large.

Claim 1. $\tau \le \sigma + O(b^{-1-2s})$ as $b \to \infty$.

Notice that $-1-2s > -2$ since $s < \frac{1}{2}$. To prove (9.3), we also need to show that $\lambda_1(E_+)$ is strictly less than τ.

Claim 2. $\lambda_1(E_+) + c \le \tau$, where c does not depend on b.

We will prove these claims below. Before this, we show (9.3) under the assumption that Claims 1 and 2 hold. By the claims, we have

$$\lambda_1(E_1) + c \le \tau \le \sigma + O(b^{-1-2s}).$$

Taking large b, (9.3) follows.

Proof of Claim 1. Let h be the eigenfunction of (9.2) corresponding to the eigenvalue σ. Since D has radial symmetry, so does the eigenfunction h. Moreover, it is easy to see h has a sign. Let $h > 0$ in Ω_b.

Take $v(r,\theta) = h(r)\sin N\theta$ in $\mathbb{R}^2$, and consider its extension V to $\mathbb{R}^3_+$ given by (2.2). Recall that the extended function V is given by

$$V(x,y) = (P(\cdot,y) * v)(x),$$

where P is the Poisson kernel from (2.3). Since the Poisson kernel is a rotationally symmetric function and v has a special form, V also has such special form. More precisely, we have

$$\begin{aligned}
V(R_\varphi x, y) &= \int_{\mathbb{R}^2} P(R_\varphi x - \xi, y) v(\xi)\,\mathrm{d}\xi \\
&= \int_0^\infty \int_0^{2\pi} P(R_\varphi x - t(\cos\theta, \sin\theta), y) h(t) \sin(N\theta) t \,\mathrm{d}\theta\,\mathrm{d}t \\
&= \int_0^\infty \int_0^{2\pi} P(x - t(\cos(\theta-\varphi), \sin(\theta-\varphi)), y) h(t) \sin(N\theta) t \,\mathrm{d}\theta\,\mathrm{d}t \\
&= \int_0^\infty \int_0^{2\pi} P(x - t(\cos\theta, \sin\theta), y) h(t) \sin(N\theta + N\varphi) t \,\mathrm{d}\theta\,\mathrm{d}t \\
&= \cos(N\varphi) \int_0^\infty \int_0^{2\pi} P(x - t(\cos\theta, \sin\theta), y) h(t) \sin(N\theta) t \,\mathrm{d}\theta\,\mathrm{d}t \\
&\quad + \sin(N\varphi) \int_0^\infty \int_0^{2\pi} P(x - t(\cos\theta, \sin\theta), y) h(t) \cos(N\theta) t \,\mathrm{d}\theta\,\mathrm{d}t \\
&= \cos(N\varphi) V(x,y) + \sin(N\varphi) V(R_{\frac{\pi}{2N}} x, y),
\end{aligned}$$

where R_φ denotes a rotation and (t,θ) are polar coordinates for ξ. Notice that $V \equiv 0$ on $\{\theta = 0\}$. Therefore, the extended function V also has the form

$$V(x,y) = H(r,y)\sin(N\theta).$$

Using this, the extension problem (2.2) for V becomes

$$\begin{aligned} L_a H =& \frac{N^2}{r^2} H \quad \text{on} \quad \mathbb{R}\times\{y>0\}, \\ H(r,0) =& h(r) \quad \text{on} \quad \mathbb{R}. \end{aligned} \tag{9.4}$$

Moreover, we have

$$\begin{aligned} H(r,y) &= V(R_{\frac{\pi}{2N}}(r,0),y) \\ &= \int_0^\infty \int_0^{2\pi} P((r,0)-t(\cos\theta,\sin\theta),y)h(t)\cos(N\theta)t\,\mathrm{d}\theta\,\mathrm{d}t \\ &= C_{2,s}y^{2s}\int_b^{b+1}\int_0^{2\pi} \frac{th(t)\cos N\theta}{(r^2+t^2+y^2-2rt\cos\theta)^{1+s}}\,\mathrm{d}\theta\,\mathrm{d}t. \end{aligned} \tag{9.5}$$

These properties yield the following Lemma:

Lemma 9.1. *Let $\tilde{H}$ be the extended function of h. Then we have*

$$0 \le H \le \tilde{H}.$$

Proof. Since $\tilde{H}(\cdot,y) = P(\cdot,y) * h$, the second inequality follows from $h > 0$ on $(b,b+1)$ and $\cos(N\theta) \le 1$.

To see the first inequality, we observe that for any $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ such that

$$|H(r,y)| < \varepsilon \quad \text{on } \mathbb{R}^{n+1}_+ \setminus B^+_R. \tag{9.6}$$

In fact, if $\sqrt{r^2+y^2} \ge 2b+2$, then we see

$$(r-t)^2 + y^2 \ge (\sqrt{r^2+y^2} - t)^2 \ge \frac{1}{4}(r^2+y^2),$$

and hence, together with the Cauchy-Schwarz inequality and (9.5), we have that

$$\begin{aligned} |H(r,y)| &\le 2\pi C_{2,s}y^{2s}\int_b^{b+1} \frac{th(t)}{\big((r-t)^2+y^2\big)^{1+s}}\,\mathrm{d}t \\ &\le \frac{2^{3+2s}\pi C_{2,s}y^{2s}}{(r^2+y^2)^{1+s}}\sqrt{\left(\int_b^{b+1} h(t)^2 t\,\mathrm{d}t\right)\left(\int_b^{b+1} t\,\mathrm{d}t\right)} \\ &\le \frac{2^{2+2s}C_{2,s}\sqrt{\pi(2b+1)}\,\|h\|_{L^2(\Omega_b)}}{r^2+y^2} \to 0 \quad \text{as } r^2+y^2 \to \infty. \end{aligned}$$

This gives (9.6).

Now we assume, by contradiction, that $H(r,y) = -2\varepsilon$ for some point (r_0,y_0) where $\varepsilon > 0$. Take $R = R(\varepsilon)$ as in the above, and then

$$-2\varepsilon \ge \inf_{\mathbb{R}_+\times\mathbb{R}_+} H(r,y) = \inf_{B^+_R} H(r,y). \tag{9.7}$$

However, by a simple maximum principle argument for equation (9.4), we have

$$\inf_{B^+_R} H(r,y) = \inf_{\Gamma^+_R\cup\Gamma^0_R} H(r,y). \tag{9.8}$$

Since $H(r,0) = h(r) \geq 0$, (9.7) and (9.8) imply that

$$-2\varepsilon \geq \inf_{\Gamma_R^+} H(r,y) \geq -\varepsilon,$$

which is a contradiction. □

To finish the proof of Claim 1, we now compare the two eigenvalues τ and σ. From the definition of τ and σ, together with Lemma 9.1, we have

$$\tau = \frac{\int_{\mathbb{R}_+\times\mathbb{R}_+}(H_r^2 + H_y^2)y^a r\,\mathrm{d}r\,\mathrm{d}y + \alpha\int_{D_1} h^2 r\,\mathrm{d}r}{\int_b^{b+1} h^2 r\,\mathrm{d}r} + \frac{\int_{\mathbb{R}_+\times\mathbb{R}_+}\frac{N^2}{r^2}H^2 y^a r\,\mathrm{d}r\,\mathrm{d}y}{\int_b^{b+1} h^2 r\,\mathrm{d}r}$$

$$\leq \sigma + \frac{\int_{\mathbb{R}_+\times\mathbb{R}_+}\frac{N^2}{r}H^2 y^a\,\mathrm{d}r\,\mathrm{d}y}{\int_b^{b+1} h^2 r\,\mathrm{d}r}. \tag{9.9}$$

Hence, it only remains to prove

$$\frac{\int_{\mathbb{R}_+\times\mathbb{R}_+}\frac{N^2}{r}H^2 y^a\,\mathrm{d}r\,\mathrm{d}y}{\int_b^{b+1} h^2 r\,\mathrm{d}r} = O(b^{-1-2s}) \quad \text{as } b\to\infty. \tag{9.10}$$

To see this, we consider the following two quantities:

$$I_1 := \frac{\int_0^\infty \mathrm{d}y\int_0^{\frac{b}{2}}\mathrm{d}r\frac{N^2}{r}H^2y^a}{\int_b^{b+1} h^2 r\,\mathrm{d}r}, \quad I_2 := \frac{\int_0^\infty \mathrm{d}y\int_{\frac{b}{2}}^\infty\mathrm{d}r\frac{N^2}{r}H^2y^a}{\int_b^{b+1} h^2 r\,\mathrm{d}r}.$$

We first estimate I_2. Recall that $\sin\theta \geq \frac{2}{\pi}\theta$ if $0\leq\theta\leq\frac{\pi}{2}$. Using this and expression (9.5), we have

$$\begin{aligned} H(r,y) &= C_{2,s}y^{2s}\int_b^{b+1}\int_{-\pi}^{\pi}\frac{th(t)\cos N\theta}{\left((r-t)^2+y^2+4rt\sin^2\frac{\theta}{2}\right)^{1+s}}\,\mathrm{d}\theta\,\mathrm{d}t \\ &\leq C_{2,s}y^{2s}\int_b^{b+1}\int_{-\pi}^{\pi}\frac{th(t)}{\left((r-t)^2+y^2+\frac{4}{\pi^2}rt\theta^2\right)^{1+s}}\,\mathrm{d}\theta\,\mathrm{d}t. \end{aligned}$$

Let $K = \sqrt{\frac{(r-t)^2+y^2}{\frac{4}{\pi^2}rt}}$ and $\theta = K\theta'$. Then we obtain

$$\begin{aligned} H(r,y) \leq& 2C_{2,s}y^{2s}\int_b^{b+1}\int_0^{\frac{\pi}{K}}\frac{Kth(t)}{((r-t)^2+y^2)^{1+s}(1+\theta'^2)^{1+s}}\,\mathrm{d}\theta'\,\mathrm{d}t \\ \leq& 4C_{2,s}y^{2s}\int_b^{b+1}\frac{Kth(t)}{((r-t)^2+y^2)^{1+s}}\,\mathrm{d}t, \end{aligned}$$

since $\int_0^\infty \frac{\mathrm{d}\theta'}{(1+\theta'^2)^{1+s}} < 2$. By the Cauchy-Schwarz inequality,

$$\begin{aligned} H(r,y)^2 &\leq C(s)y^{4s}\int_b^{b+1}\frac{K^2t}{((r-t)^2+y^2)^{2+2s}}\,\mathrm{d}t\int_b^{b+1}h^2(t)t\,\mathrm{d}t \\ &\leq C(s)y^{4s}\int_b^{b+1}\frac{1}{r\left((r-t)^2+y^2\right)^{1+2s}}\,\mathrm{d}t\int_b^{b+1}h^2(t)t\,\mathrm{d}t. \end{aligned}$$

Therefore, we see that

$$I_2 \leq C(N,s)\int_0^\infty\mathrm{d}y\int_{\frac{b}{2}}^\infty\mathrm{d}r\int_b^{b+1}\mathrm{d}t\frac{y^{1+2s}}{r^2\left((r-t)^2+y^2\right)^{1+2s}},$$

Now take $y = |r-t|y'$. Using the property $\int_0^\infty \frac{y'^{1+2s}}{(1+y'^2)^{1+2s}}\,\mathrm{d}y' \le 1 + \frac{1}{2s}$, we arrive to

$$I_2 \le C(N,s) \int_b^{b+1} \mathrm{d}t \left(\int_{\frac{b}{2}}^t + \int_t^{2b+1} + \int_{2b+1}^\infty \right) \mathrm{d}r \frac{1}{r^2 (r-t)^{2s}}.$$

We conclude that $I_2 = O(b^{-1-2s})$ as $b \to \infty$ by a direct computation.

Our next task is estimating I_1. Again, recall (9.5) so that we observe

$$\begin{aligned}
\frac{H(r,y)}{C_{2,s} y^{2s}} &= \sum_{i=0}^{2N-1} \int_b^{b+1} \int_{\pi(2i-1)/2N}^{\pi(2i+1)/2N} \frac{th(t)\cos(N\theta)}{(r^2+t^2+y^2-2rt\cos\theta)^{1+s}}\,\mathrm{d}\theta\,\mathrm{d}t \\
&= \frac{1}{N} \sum_{i=0}^{2N-1} \int_b^{b+1} \int_0^\pi \frac{th(t)\cos(\pi(2i-1)/2+\varphi)}{(r^2+t^2+y^2-2rt\cos(\varphi/N+\pi(2i-1)/2N))^{1+s}}\,\mathrm{d}\varphi\,\mathrm{d}t \\
&= \frac{1}{N} \sum_{i=0}^{2N-1} \int_b^{b+1} \int_0^\pi \frac{(-1)^i th(t)\sin\varphi}{\big(r^2+t^2+y^2-2rt\cos(\varphi/N+\pi(2i-1)/2N)\big)^{1+s}}\,\mathrm{d}\varphi\,\mathrm{d}t.
\end{aligned}$$

By the mean value theorem, we see that

$$\begin{aligned}
\frac{H(r,y)}{C_{2,s} y^{2s}} &\le \frac{1}{N} \sum_{i=0}^{2N-1} \int_b^{b+1} \int_0^\pi \frac{(-1)^i th(t)\sin\varphi}{(r^2+t^2+y^2-2(-1)^i rt)^{1+s}}\,\mathrm{d}\varphi\,\mathrm{d}t \\
&= 2\int_b^{b+1} \left[\frac{th(t)}{\big((t-r)^2+y^2\big)^{1+s}} - \frac{th(t)}{\big((t+r)^2+y^2\big)^{1+s}} \right] \mathrm{d}t \\
&\le \frac{4(1+s)(3b+2)r}{\big(b^2/4+y^2\big)^{2+s}} \int_b^{b+1} th(t)\,\mathrm{d}t
\end{aligned}$$

for $r \in (0, b/2)$, which implies, using Cauchy-Schwarz as in the estimate for I_2, that

$$I_1 \le C(N,s) \int_0^{\frac{b}{2}} \mathrm{d}r \int_0^\infty \mathrm{d}y \frac{y^{1+2s}(b+1)^3 r}{(b^2+y^2)^{4+2s}}.$$

We finally take $y = by'$ and use $\int_0^\infty \frac{y'^{1+2s}}{(1+y'^2)^{4+2s}}\,\mathrm{d}y' \le 2$ to conclude $I_1 = O(b^{-1-2s})$ as $b \to \infty$. This completes the proof of Claim 1. □

In order to prove Claim 2 we need the following Lemma. Although it is true for any dimension n, we just consider the two-dimensional case for simplicity.

Lemma 9.2. *Let N be any positive integer. Let v be a function of the form $v(r,\theta) = h(r)\sin(N\theta)$ in Ω_b with $v \equiv 0$ in $\mathbb{R}^2 \setminus \Omega_b$ and $h(r) \ge 0$ for $r \in [b, b+1]$. Then we have*

$$\left\| (-\Delta)^{s/2} v \right\|_{L^2(\mathbb{R}^2)}^2 \ge 2N \left\| (-\Delta)^{s/2}(v\chi_E) \right\|_{L^2(\mathbb{R}^2)}^2,$$

where $E = \Omega_b \cap \{(r,\theta) : 0 \le \theta < \pi/N\}$.

Proof. In order to prove this, we first define

$$E_i = \Omega_b \cap \{(r,\theta) : (i-1)\pi/N \le \theta < i\pi/N\}$$

for $i = 1, \cdots, 2N$, and note that $E_1 = E$. Since v is defined on $\Omega_b = \cup_{i=1}^{2N} E_i$, we can decompose v as $\sum_{i=1}^{2N} v_i$ where $v_i = v\chi_{E_i}$. Observe that

$$\begin{aligned}|v(x) - v(y)|^2 &= \left|\sum_{i=1}^{2N} \big(v_i(x) - v_i(y)\big)\right|^2 \\ &= \sum_{i=1}^{2N} \big(v_i(x) - v_i(y)\big)^2 + \sum_{i\neq j} \big(v_i(x) - v_i(y)\big)\big(v_j(x) - v_j(y)\big)\end{aligned}$$

and $v(R_{k\pi/N}(x_1, x_2)) = (-1)^k v(x_1, x_2)$. Using these, we have

$$\begin{aligned}\left\|(-\Delta)^{s/2} v\right\|^2_{L^2(\mathbb{R}^2)} &= 2N \left\|(-\Delta)^{s/2} (v\chi_{E_1})\right\|^2_{L^2(\mathbb{R}^2)} \\ &\quad + Nc_{2,s} \sum_{i\neq 1} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{(v_1(x) - v_1(y))(v_i(x) - v_i(y))}{|x-y|^{2+2s}}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

We claim that the last term in the right hand side above is nonnegative. In fact, if one of x and y is contained in $\mathbb{R}^2 \setminus \Omega_b$ or both x and y are contained in the same E_j for some j, then

$$(v_1(x) - v_1(y))\,(v_i(x) - v_i(y)) = 0 \tag{9.11}$$

since $i \neq 1$. Moreover, (9.11) also holds unless $(x, y) \in E_1 \times E_i$ or $(x, y) \in E_i \times E_1$. Thus, to have the conclusion, it suffices to show that

$$-\sum_{i\neq 1} \int_{E_1}\int_{E_i} \frac{v_1(x)v_i(y)}{|x-y|^{2+2s}}\,\mathrm{d}x\,\mathrm{d}y \geq 0. \tag{9.12}$$

To simplify the notation, let us define

$$I_i := -\int_{E_1}\int_{E_i} \frac{v_1(x)v_i(y)}{|x-y|^{2+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

Assume that $N = 2k + 1$. Notice that

$$\begin{aligned}I_{2k+2} &= -\int_{E_1}\int_{E_{2k+2}} \frac{v_1(x)v_{2k+2}(y)}{|x-y|^{2+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{E_1}\int_{E_1} \frac{v_1(x)v_1(y)}{|x-R_\pi y|^{2+2s}}\,\mathrm{d}x\,\mathrm{d}y \geq 0,\end{aligned}$$

and

$$\begin{aligned}&I_2 \geq -I_3 \geq \cdots \geq -I_{2k+1}, \\ &I_{4k+2} \geq -I_{4k+1} \geq \cdots \geq -I_{2k+3}.\end{aligned}$$

Then we have

$$\sum_{i=2}^{2N} I_i = \sum_{i=1}^{k} (I_{2i} + I_{2i+1}) + \sum_{i=1}^{k} (I_{4k+4-2i} + I_{4k+3-2i}) + I_{2k+2} \geq 0.$$

Now assume that $N = 2k$. In this case, we see that

$$\begin{aligned}&I_2 \geq -I_3 \geq \cdots \geq I_{2k}, \\ &I_{4k} \geq -I_{4k-1} \geq \cdots \geq I_{2k+2}, \\ &I_{2k} + I_{2k+1} + I_{2k+2} \geq 0,\end{aligned}$$

which implies

$$\sum_{i=2}^{2N} I_i = \sum_{i=1}^{k-1}(I_{2i}+I_{2i+1}) + \sum_{i=1}^{k-1}(I_{4k+2-2i}+I_{4k+1-2i}) + I_{2k}+I_{2k+1}+I_{2k+2}$$
$$\geq 0.$$

In any case, we have (9.12), which completes the proof. □

To further proceed, we focus on the equation satisfied by the radial part h_0 of v_0. With some abuse of notation, we write $h_0 = h_0(|x|) = h_0(x)$, and then $(-\Delta)^s h_0$ is understood as fractional Laplacian of the function h_0 defined on $\mathbb{R}^2$. Now we observe that for any r, by taking the point x such that $|x| = r$ and the angle of x is $\frac{\pi}{2N}$, we have

$$(-\Delta)^s v_0(x) = (-\Delta)^s h_0(r) + c_{n,s}\int_0^{2\pi}\int_b^{b+1} \frac{h_0(t)(1-\sin(N\theta))t}{(r^2+t^2-2rt\cos(\theta-\frac{\pi}{2N}))^{1+s}}\,\mathrm{d}t\,\mathrm{d}\theta.$$

Moreover, for this x, the eigenfunction v_1 satisfies

$$(-\Delta)^s v_0(x) = (\tau - \alpha\chi_{D_1}(r))h_0(r).$$

Therefore, the equation satisfied by h_0 is given by

$$(-\Delta)^s h_0(r) = (\tau - \alpha\chi_{D_1}(r))h_0(r) - B[h_0], \tag{9.13}$$

where

$$B[h] = c_{n,s}\int_0^{2\pi}\int_b^{b+1} \frac{h(t)(1-\sin(N\theta))t}{(r^2+t^2-2rt\cos(\theta-\frac{\pi}{2N}))^{1+s}}\,\mathrm{d}t\,\mathrm{d}\theta.$$

From now on, we estimate the coefficients in the right hand side of (9.13).

Lemma 9.3. *Let $0 < s < \frac{1}{2}$. Then we have, for $r \in [b, b+1]$,*

$$B[h] \leq C(s,N)b^{-1-2s}\,\|h\|_{L^2(\Omega_b)},$$

where $C(s,N)$ is a constant.

Proof. We notice that

$$\begin{aligned} B[h] &= c_{2,s}\int_{-\pi}^{\pi}\int_b^{b+1} \frac{h(t)(1-\cos(N\theta))t}{(r^2+t^2-2rt\cos\theta)^{1+s}}\,\mathrm{d}t\,\mathrm{d}\theta \\ &= 4c_{2,s}\int_0^{\pi}\int_b^{b+1} \frac{h(t)\sin^2(\frac{N\theta}{2})t}{((r-t)^2+4rt\sin^2(\frac{\theta}{2}))^{1+s}}\,\mathrm{d}t\,\mathrm{d}\theta. \end{aligned}$$

Since $\frac{2}{\pi}\theta \leq \sin\theta$ for $0 \leq \theta \leq \frac{\pi}{2}$ and $\sin\theta \leq \theta$ for any θ, we have

$$B[h] \leq C(s,N)\int_0^{\pi}\int_b^{b+1} \frac{h(t)\theta^2 t}{((r-t)^2+\frac{4}{\pi^2}rt\theta^2)^{1+s}}\,\mathrm{d}t\,\mathrm{d}\theta.$$

Now set $\theta = \kappa\theta'$ with $\kappa = \frac{|r-t|}{\sqrt{4rt/\pi^2}}$ so that

$$\begin{aligned} B[h] &\leq C(s,N)\int_0^{\frac{\pi}{\kappa}}\int_b^{b+1} \frac{h(t)(\theta')^2 t\kappa^3}{(r-t)^{2+2s}(1+(\theta')^2)^{1+s}}\,\mathrm{d}t\,\mathrm{d}\theta' \\ &\leq C(s,N)\int_0^{\frac{\pi}{\kappa}}\int_b^{b+1} \frac{h(t)(\theta')^2|r-t|^{1-2s}}{b^2(1+(\theta')^2)^{1+s}}\,\mathrm{d}t\,\mathrm{d}\theta', \end{aligned}$$

where $C(s,N)$ is a constant depending only on s,N. Observe that $\int_0^\infty \frac{\theta^2}{(1+\theta^2)^{1+s}}\,\mathrm{d}\theta$ is the finite constant depending on s if $s<\frac{1}{2}$. Using the Cauchy-Schwarz inequality, we therefore obtain

$$B[h] \le C(s,N)b^{-1-2s}\left\|h\right\|_{L^2(\Omega_b)}.$$

□

To estimate τ, we need to estimate the energy of h_0 according to the dimension.

Lemma 9.4. *Assume that $b\ge 1$. Then the eigenvalue τ is bounded by some constant which is independent of b.*

Proof. From (9.9) and (9.10), we have

$$\tau \le \lambda_1 + \alpha + O(b^{-1-2s}),$$

where λ_1 is the first eigenvalue of $(-\Delta)^s$ in Ω_b. It suffices to show that λ_1 has a uniform bound independent of b.

Let h be a function defined in $\mathbb{R}^2$ with $h(x)=h(y)$ for any $|x|=|y|$ and $h(x)=0$ unless $x\in\Omega_b$. Writing $(-\Delta)_1^s$ for the 1-dimensional fractional Laplacian, we shall compare to $(-\Delta)_1^s h$ and its first Dirichlet eigenvalue.

First, we observe that

$$2\pi b\left\|h\right\|^2_{L^2(\mathbb{R})} \le \left\|h\right\|^2_{L^2(\mathbb{R}^2)} \le 2\pi(b+1)\left\|h\right\|^2_{L^2(\mathbb{R})}, \tag{9.14}$$

and

(9.15)
$$\begin{aligned}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|h(x)-h(y)|^2}{|x-y|^{2+2s}}\,\mathrm{d}x\,\mathrm{d}y &= \int_0^\infty\int_0^\infty\int_{-\pi}^{\pi}\frac{2\pi(h(r)-h(t))^2rt}{((r-t)^2+4rt\sin^2(\frac{\theta}{2}))^{1+s}}\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}t\\ &=: I_{2,1}+2I_{2,2},\end{aligned}$$

where

$$I_{2,1} = \int_0^{2b+1}\int_0^{2b+1}\int_{-\pi}^{\pi}\frac{2\pi(h(r)-h(t))^2rt}{((r-t)^2+4rt\sin^2(\frac{\theta}{2}))^{1+s}}\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}t$$

and

$$I_{2,2} = \int_{2b+1}^\infty\int_0^{2b+1}\int_{-\pi}^{\pi}\frac{2\pi h(r)^2rt}{((r-t)^2+4rt\sin^2(\frac{\theta}{2}))^{1+s}}\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}t.$$

We also notice that

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|h(x)-h(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y &= \int_{-\infty}^\infty\int_{-\infty}^\infty\frac{(h(r)-h(t))^2}{|r-t|^{1+2s}}\,\mathrm{d}r\,\mathrm{d}t\\ &\le 4I_{1,1}+8I_{1,2},\end{aligned} \tag{9.16}$$

where

$$I_{1,1} = \int_0^{2b+1}\int_0^{2b+1}\frac{(h(r)-h(t))^2}{|r-t|^{1+2s}}\,\mathrm{d}r\,\mathrm{d}t$$

and

$$I_{1,2} = \int_{2b+1}^\infty\int_0^{2b+1}\frac{h(r)^2}{|r-t|^{1+2s}}\,\mathrm{d}r\,\mathrm{d}t$$

We have

$$I_{2,1} \leq C \int_0^{2b+1} \int_0^{2b+1} \int_0^{\pi} \frac{(h(r)-h(t))^2}{((r-t)^2+(4rt/\pi^2)\theta^2)^{1+s}} rt \, \mathrm{d}\theta \, \mathrm{d}r \, \mathrm{d}t$$

and for $K = \sqrt{\frac{(r-t)^2}{4\pi rt/\pi^2}}$, we substitute $\theta = K\theta'$ so that

$$\begin{aligned} I_{2,1} &\leq C \int_0^{\pi/K} \frac{1}{(1+\theta^2)^{1+s}} d\theta \int_0^{2b+1} \int_0^{2b+1} \frac{(h(r)-h(t))^2}{|r-t|^{2+2s}} Krt \, \mathrm{d}r \, \mathrm{d}t \\ &\leq C \int_0^{2b+1} \int_0^{2b+1} \frac{(h(r)-h(t))^2}{|r-t|^{1+2s}} \sqrt{rt} \, \mathrm{d}r \, \mathrm{d}t. \end{aligned}$$

This implies $I_{2,1} \leq CbI_{1,1}$.

Now we estimate $I_{2,2}$. Since $\operatorname{supp} h \subset [b, b+1]$, we have

$$\begin{aligned} I_{2,2} &= \int_{2b+1}^{\infty} \int_0^{b+1} \int_{-\pi}^{\pi} \frac{2\pi h(r)^2}{((r-t)^2 + 4rt\sin^2(\frac{\theta}{2}))^{1+s}} rt \, \mathrm{d}\theta \, \mathrm{d}r \, \mathrm{d}t \\ &\leq C \int_{2b+1}^{\infty} \int_0^{b+1} \frac{h(r)^2}{|r-t|^{1+2s}} \frac{rt}{t-r} \, \mathrm{d}r \, \mathrm{d}t \\ &\leq CbI_{1,2}, \end{aligned}$$

where we have used $\frac{t}{t-r} \leq 3$.

From (9.14), (9.15), and (9.16), together with the above estimate, we have

$$\lambda_1 \leq \frac{I_{2,1} + 2I_{2,2}}{\|h\|^2_{L^2(\mathbb{R}^2)}} \leq C \frac{4I_{1,1} + 8I_{1,2}}{\|h\|^2_{L^2(\mathbb{R})}} \leq C \frac{\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|h(x)-h(y)|^2}{|x-y|^{1+2s}} \, \mathrm{d}x \, \mathrm{d}y}{\|h\|^2_{L^2(\mathbb{R})}}.$$

If we take h to be the first eigenfunction defined on $(b, b+1)$, then the last quantity exists. Now the conclusion follows from the fact that this quantity does not depend on b. $\square$

Now we are ready to prove the Lemma below in analogy to Lemma 15 in [14].

Lemma 9.5. *Assume that* $b \geq 1$ *and let* $v_0 = h_0(r)\sin(N\theta)$ *be the first eigenfunction corresponding to the eigenvalue* τ *in* Ω_b. *Let* $\delta := |D|/|\Omega_b|$. *Then*

$$\int_{\Omega_b} \chi_D v_0^2 \, \mathrm{d}x \geq c \int_{\Omega_b} v_0^2 \, \mathrm{d}x,$$

where c *does not depend on* b.

Proof. From (9.1), since we can take $|D_1| = \delta$, we have

$$|[b + \delta/4, b + 1 - \delta/4] \cap D_1| \geq \frac{\delta}{2}.$$

Then we have

$$\int_{\Omega_b} \chi_D v_0^2 \, \mathrm{d}x = \pi \int_b^{b+1} \chi_{D_1} h_0^2 r \, \mathrm{d}r \geq \frac{\pi\delta b}{2} \inf_{[b+\delta/4, b+1-\delta/4]} h_0^2 \tag{9.17}$$

and

$$\int_{\Omega_b} v_0^2 \, \mathrm{d}x = \pi \int_b^{b+1} h_0^2 r \, \mathrm{d}r \leq 2\pi b \int_b^{b+1} h_0^2 \, \mathrm{d}r. \tag{9.18}$$

Denote by K the one dimensional compact subset $[b+\varepsilon, b+1-\varepsilon]$ of the interval $[b, b+1]$, where ε is a small positive number. Now use the Harnack's inequality from [40] applied to the equation (9.4) in order to estimate

$$\sup_K h_0 \le C \inf_K h_0.$$

Moreover, using Lemma 2.3 in [7] for equation (9.13), with the estimates from Lemma 9.3 and Lemma 9.4,

$$\|h_0\|_{L^\infty((b,b+1))} \le C \|h_0\|_{L^2((b,b+1))},$$

for some C independent of b. Therefore, we have

$$\begin{aligned}\int_b^{b+1} h_0^2 \, \mathrm{d}r &= \int_K h_0^2 \, \mathrm{d}r + \int_{[b,b+1]\setminus K} h_0^2 \, \mathrm{d}r \le |K| \sup_K h_0^2 + (1-|K|) \sup_{(b,b+1)} h_0^2 \\ &\le C \left(\inf_K h_0^2 + (1-|K|) \int_b^{b+1} h_0^2 \, \mathrm{d}r \right).\end{aligned}$$

By taking sufficiently small ε, we can have $C(1-|K|) \le \frac{1}{2}$ so that we finally arrive to

$$\int_b^{b+1} h_0^2 \, \mathrm{d}r \le C \inf_K h_0^2. \tag{9.19}$$

Again, we may take small ε satisfying $[b+\delta/4, b+1-\delta/4] \subset K$. Now the conclusion follows from (9.17), (9.18), and (9.19). □

Proof of Claim 2. The conclusion follows directly from Lemma 9.2 and Lemma 9.5. □

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