TRANSITIVE HOLONOMY GROUP AND RIGIDITY IN NONNEGATIVE CURVATURE

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Abstract. In this note, we examine the relationship between the twisting of a vector bundle ξ over a manifold M and the action of the holonomy group of a Riemannian connection on ξ. For example, if there is a holonomy group which does not act transitively on each fiber of the corresponding unit sphere bundle, then for any $f : S^n \to M$, the pullback $f^*\xi$ of ξ admits a nowhere-zero cross section. These facts are then used to derive a rigidity result for complete metrics of nonnegative sectional curvature on noncompact manifolds.

Let ξ denote a rank k vector bundle over a simply connected manifold M^n. If ξ admits a Riemannian connection with trivial holonomy group, then ξ is a trivial bundle. On the other hand, it is easy to construct examples of trivial bundles with transitive holonomy (in the sense that the holonomy group acts transitively on each fiber of the unit sphere bundle of ξ). A natural question to ask is how twisted must the bundle be in order to guarantee that *every* connection will have transitive holonomy. Clearly, such a bundle cannot admit a nowhere-zero section. It turns out that when the base is a sphere, this necessary condition is also sufficient (see Example 2). For arbitrary base, this no longer holds, but a slightly stronger condition is sufficient: Let ξ^1 denote the unit sphere bundle of ξ; if ξ^1 admits a section, then for $q \geq 2$, one has a short exact sequence

$$0 \longrightarrow \pi_q(S^{k-1}) \xrightarrow{\imath_*} \pi_q(E^1) \xrightarrow{\pi_*} \pi_q(M) \longrightarrow 0 \tag{1}$$

which in fact splits. Here $\imath$ is the fiber inclusion, π the bundle projection, and E^1 the total space of ξ^1. We show that if ξ admits a connection with nontransitive holonomy, then (1) holds for all $q \geq 2$.

It easily follows that on bundles over spheres which admit no nonvanishing section, any connection has transitive holonomy. Other examples include the canonical line bundle over complex or quaternionic projective space P^n, the tangent bundle of P^n for even n, the canonical bundles over the Grassmannians $G_k(\mathbb{R}^n)$ and $G_k(\mathbb{C}^n)$ of k-planes in n-space for

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large n, and consequently the universal bundles over the classifying spaces $BSO(k)$ and $BU(k)$. All these examples underscore the general idea that the holonomy group increases as the bundle becomes more twisted.

This property has an interesting application to the geometry of open manifolds N with nonnegative sectional curvature: recall that such a space contains a compact totally geodesic submanifold S, called a soul of N. Many of the bundles mentioned above admit such a metric. The above result implies that actually *any* complete metric of nonnegative curvature on these bundles is quite rigid:

Theorem. *Let N denote a complete manifold with nonnegative sectional curvature and soul S. If the normal bundle ν of S is one of those mentioned above, i.e:*

1. *The unit sphere bundle of ν is a homotopy sphere,*
2. *S is a sphere and ν does not admit a nowhere-zero section,*
3. *$S = \mathbb{CP}^n$ or $\mathbb{HP}^n$ with n even, and ν is the tangent bundle,*
4. *$S = G_k(\mathbb{R}^n)$ with $n \geq 2k$, and ν is the canonical bundle,*

then the exponential map $\exp : \nu \to N$ *of the normal bundle of S in M is a diffeomorphism, the metric projection $N \to S$ onto the soul is a $\mathcal{C}^\infty$ Riemannian submersion, and the ideal boundary $N(\infty)$ of M consists of a single point.*

1. Nontransitive holonomy

Theorem. *Let E denote the total space of an oriented Euclidean rank k bundle over a manifold M. If this bundle admits a Riemannian connection whose holonomy does not act transitively on the unit sphere bundle E^1 of E, then for $q \geq 2$, there is a short exact sequence*

$$0 \longrightarrow \pi_q(S^{k-1}) \xrightarrow{\imath_*} \pi_q(E^1) \xrightarrow{\pi_*} \pi_q(M) \longrightarrow 0. \tag{1}$$

Proof. Denote by B the principal $SO(k)$ bundle of E^1, and consider a connection on E with nontransitive holonomy. Let G denote the holonomy group of the connection, and P the holonomy bundle through some $b \in B$; i.e., P consists of all points of B that can be joined to b by some horizontal curve. P is a principal subbundle of B [P], and E^1 is equivalent to $P \times_G S^{k-1}$. In other words, E^1 is the base of the principal G-fibration

$$\rho : P \times S^{k-1} \to E^1, \qquad \rho(pg, u) = \rho(p, gu), \quad p \in P, u \in S^{k-1}, g \in G.$$

Let us fix some $u \in S^{k-1}$, and define a map $h : P \to E^1$ by $h(p) = \rho(p, u)$. Notice that if G_p denotes the fiber of P through p and $F_{h(p)}$ the fiber of E^1 through $h(p)$, then $h : (P, G_p) \to (E_1, F_{h(p)})$: In fact, if $q = pg \in G_p$ for some $g \in G$, then $h(q) = \rho(pg, u) = \rho(p, gu) \in F_{h(b)}$.

Since the holonomy is not transitive, $h : G_p \to F_{h(p)}$ is not onto, and is then homotopically trivial because a sphere with a point deleted is contractible. Consider the commutative diagram

$$\begin{tikzcd}
\pi_l(M) \arrow[d, equal] & \pi_l(P, G_p) \arrow[l] \arrow[r, "\partial"] \arrow[d, "h_*"'] & \pi_{l-1}(G_p) \arrow[d, "h_*"] \\
\pi_l(M) & \pi_l(E^1, F_{h(p)}) \arrow[l] \arrow[r, "\partial"'] & \pi_{l-1}(F_{h(p)})
\end{tikzcd}$$

where the unlabeled arrows are the isomorphisms induced by the respective bundle projections, and ∂ is the boundary operator, cf. e.g. [S]. Now the h_* on the left is an isomorphism since it commutes with the bundle projections, whereas the h_* on the right is trivial. Thus, the lower ∂ is trivial. The theorem follows by looking at the homotopy sequence of the fibration $E^1 \to M$. □

2. Applications

We now consider several bundles satisfying the hypotheses of the Theorem, so that any Riemannian connection on them must have transitive holonomy:

Example 1. *A vector bundle whose unit sphere bundle is a homotopy-sphere.* Indeed, if the base has dimension n, then $\pi_{k-1}(E^1) = \pi_{k-1}(S^{n+k-1}) = 0$, so that one cannot have a short exact sequence as in (1) in dimension $k-1$. This result applies in particular to the *canonical line bundles over complex and quaternionic projective spaces* as well as to the vector bundle E over S^8 whose unit bundle corresponds to the Hopf fibration $S^7 \to S^{15} \to S^8$ (see also Example 4 below for further related results).

Before discussing our next example, we point out the following consequence of the Theorem:

Corollary 1. *Let ξ denote an oriented vector bundle over M. If ξ admits a connection with nontransitive holonomy, then for any map $f : S^n \to M$ from an n-sphere to M ($n \in \mathbb{N}$), the induced bundle $f^*\xi$ admits a nowhere-zero cross-section.*

Proof. By the Theorem, $p_* : \pi_n(E^1) \to \pi_n(M)$ is onto, where, as before, E_1 is the total space of the unit sphere bundle of ξ. Thus, there exists $\alpha \in \pi_n(E_1)$ such that $p_*(\alpha) = [f]$. By the covering homotopy property, there exists a representative $g : S^n \to E_1$ of α such that $p \circ g = f$; i.e., g induces a section of $f^*\xi$.

Example 2. (see also [GW]) *Let τ denote a vector bundle over a homotopy-sphere M. Then the following statements are equivalent:*

(1) *Any connection on τ has transitive holonomy*

(2) *τ does not admit a nowhere-zero section*
(3) *τ does not split as a Whitney sum $\tau_1 \oplus \tau_2$ of subbundles τ_1, τ_2.*

Proof.

(1) $\Rightarrow$ (3): If τ splits, then a connection on each subbundle induces a 'direct sum' connection on the bundle that leaves each factor invariant. The resulting holonomy group is clearly nontransitive.

(3) $\Rightarrow$ (2): Trivial.

(2) $\Rightarrow$ (1): This is an immediate consequence of Corollary 1 by taking $f : S^n \to M$ to be a homotopy equivalence.

Example 3. *The tangent bundles of the projective spaces $\mathbb{C}P^n$ and $\mathbb{H}P^n$ for even n.* The argument is quite similar in both cases, so we shall only outline the one for complex projective space: Let $\rho : S^{2n+1} \to \mathbb{C}P^n$ denote the Hopf fibration, and consider the pullback $\rho^*\tau$ of the tangent bundle of $\mathbb{C}P^n$ to the sphere. If τ admits a connection with nontransitive holonomy, then $\rho^*\tau$ admits a nowhere-zero section by Corollary 1. But $\rho^*\tau$ is canonically identified with a codimension 1 subbundle of TS^{2n+1}, and this would imply that S^{2n+1} admits two linearly independent vector fields. This is impossible for even n by [S, Appendix 7]. It should be noted that when $n > 1$ is odd, there do exist nontransitive connections on the tangent bundle of $\mathbb{C}P^n$ (this was pointed out by B. Wilking): The reason is that the Hopf fibration $S^{4k+3} \to \mathbb{H}P^k$ factors through $\mathbb{C}P^{2k+1}$, yielding a fibration $S^2 \to \mathbb{C}P^{2k+1} \to \mathbb{H}P^k$; the tangent bundle of $\mathbb{C}P^{2k+1}$ therefore splits as a Whitney sum, and any connection which leaves each factor invariant will have nontransitive holonomy.

Example 4. Let γ_k^n denote the canonical bundle over the Grassmannian $G_k(\mathbb{R}^n)$ of oriented k-planes in $\mathbb{R}^n$. Thus the fiber of γ_k^n over a k-dimensional subspace is the subspace itself. We claim that if n is large enough, then any connection on γ_k^n has transitive holonomy: To see this, it suffices to show that for any integer k, there exists a rank k bundle over some sphere which does not admit a nowhere-zero cross-section; indeed, such a bundle is the pullback of γ_k^{2k} for some map from the base into $G_k(\mathbb{R}^{2k})$ (see e.g. [M]), so if the latter were to admit a nontransitive connection, then the former would admit a nowhere-zero cross-section by Corollary 1. Now, for even k, one can just take the tangent bundle of S^k. For odd k, we argue as follows: Recall that oriented rank k bundles over S^l are classified by elements of $\pi_{l-1}SO(k)$. Since such a bundle admits a nonzero section iff. its structure group reduces to $SO(k-1)$, it suffices to show that the homomorphism $\imath_* : \pi_n SO(k-1) \to \pi_n SO(k)$ induced by the inclusion $\imath$ is not onto for some value of n. Consider the portion of the long exact homotopy sequence

$$\cdots \to \pi_{2k-3}SO(k-1) \xrightarrow{\imath_*} \pi_{2k-3}SO(k) \to \pi_{2k-3}(S^{k-1}) \xrightarrow{\partial} \pi_{2k-4}SO(k-1) \to \cdots$$

of the orthogonal fibration. It is a standard fact that for odd $k > 1$, the third group contains an infinite cyclic subgroup (see e.g. [W p. 671]). Thus if the inclusion homomorphism were onto, then the boundary homomorphism $\partial : \pi_{2k-3}(S^{k-1}) \to \pi_{2k-4}SO(k-1)$ would be injective, and $\pi_{2k-4}SO(k-1)$ would contain an infinite cyclic subgroup. This is impossible, since even-dimensional homotopy groups of compact semi-simple Lie groups are always finite by [Se, V3 Corollary 1].

Corollary 2. *Any $SO(k)$-connection on the universal bundle γ_k over the classifying space $BSO(k)$ has transitive holonomy.*

Proof. If γ_k were to admit a connection with nontransitive holonomy, then the pullback of this connection to γ_k^n via the inclusion $G_k(\mathbb{R}^n) \hookrightarrow BSO(k)$ would also have nontransitive holonomy by Example 4. Alternatively, one can directly apply the Theorem to the corresponding sphere bundle, since its total space is homotopy equivalent to $BSO(k-1)$ and the corresponding exact homotopy sequence is that of the orthogonal fibration, but shifted:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial} & \pi_q(S^{k-1}) & \xrightarrow{\imath_*} & \pi_q(BSO(k-1)) & \xrightarrow{\pi_*} & \pi_q(BSO(k)) & \longrightarrow & \cdots \\
 & & \| & & \| & & \| & & \\
\cdots & \longrightarrow & \pi_q(S^{k-1}) & \xrightarrow{\partial} & \pi_{q-1}SO(k-1) & \xrightarrow{\imath_*} & \pi_{q-1}SO(k) & \xrightarrow{\pi_*} & \cdots
\end{array}$$

A similar statement holds for $U(k)$-connections on the canonical bundles over complex Grassmannians (and on $BU(k)$), since the above argument can be applied to the fibration $U(k) \to U(k+1) \to S^{2n+1}$.

Corollary 3. *Let ξ denote an oriented vector bundle of rank k over M^n; suppose that M is $(k-1)$-connected and $\pi_k(M)$ is cyclic. If ξ admits a connection with nontransitive holonomy, then the Euler class of ξ is 0.*

Remark. We are implicitely assuming that $k \le n$ in the hypotheses of Corollary 3, since otherwise ξ admits a nowhere-zero section, and the conclusion is trivial. Notice also that the assumptions on M are satisfied if M is a homology k-sphere.

Proof. Let α denote a generator of $\pi_k(M)$, and $f : S^k \to M$ a representative of α. Then $f_* : \pi_i(S^k) \to \pi_i(M)$ is an isomorphism for $i < k$ and an epimorphism for $i = k$. By a result of J.H.C. Whitehead (see e.g. [B, page 481]), the same is true for $f_* : H_i(S^n) \to H_i(M)$. By the universal coefficient theorem, $f^* : H^k(M) \to H^k(S^n)$ is a monomorphism. But then the Euler class $e(\xi)$ must vanish, since $e(f^*(\xi)) = f^*(e(\xi))$, and $f^*(\xi)$ admits a nowhere-zero section by Corollary 2.

The bundles in Examples 1 and 3 all admit Riemannian connections with parallel curvature tensor, so that by [SW], they also admit complete metrics with nonnegative sectional curvature K and soul isometric to the zero section. It is still an open question [CG] whether every bundle over S^n carries such a metric. No counterexample has so far been found, and in fact, when $n \leq 4$, every bundle admits a metric with $K \geq 0$. In any case, for those that do, the metric must be quite rigid if the bundle is twisted enough:

Corollary 4. *Let N denote a complete manifold with nonnegative sectional curvature and soul S. If the normal bundle $\nu(S)$ is one of those listed in the above examples, then:*

1. *The exponential map* $\exp : \nu(S) \to N$ *is a diffeomorphism.*
2. *The metric projection $N \to S$ is a $\mathcal{C}^\infty$ Riemannian submersion.*
3. *The ideal boundary $N(\infty)$ consists of a single point.*

Proof. Given $u \in \nu(S)$, $|u| = 1$, let γ_u denote the geodesic in direction u. (1) is an immediate consequence of the fact that if γ_u is a ray (i.e., a geodesic with no conjugate points in $[0, \infty)$), then γ_v is again a ray for any parallel translate v of u [W]. For (2), it is known that the metric projection $N \to S$ is a $\mathcal{C}^2$ Riemannian submersion [G]. It follows from (1) that this projection equals $\pi_\nu \circ (\exp^{-1})$, and is therefore $\mathcal{C}^\infty$. Finally, to establish (3), it suffices to show that any 2 rays γ_u and γ_v remain at bounded distance from each other. But if c is a curve that parallel translates u to v, one obtains a flat totally geodesic rectangle by exponentiating the parallel section along c [CG]. Thus, the distance between $\gamma_u(t)$ and $\gamma_v(t)$ is always bounded above by the length of c.

As a concluding remark, we point out once again that if the base of the bundle is not a sphere, then nontransitive holonomy does not, in general, imply the existence of a nowhere-zero section. In fact, even the weaker assertion that nontransitive holonomy implies vanishing of the Euler class is not true (unless, of course, some additional assumptions such as those in Corollary 3 hold):

Example 5. Let M^3 be a compact oriented flat 3-manifold with $H^2(M; \mathbb{Z}) = \mathbb{Z}_4 \times \mathbb{Z}_4$ (see e.g. [Wo, page 122]). Any $\alpha \in H^2(M)$ represents the integral Euler class of a unique oriented plane bundle over M. Consider a plane bundle over M whose integral Euler class is non-zero. Since the rational Euler class vanishes, the bundle admits a flat connection by [K-T]. Thus the reduced holonomy group is trivial, and the holonomy group itself does not act transitively on the unit sphere bundle.

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