

# Nonnegative Curvature and Normal Holonomy in Open Manifolds

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## 1. Basic facts and question.

In this note we are concerned with metrics of nonnegative sectional curvature in open (i.e, complete, noncompact) manifolds. We only try to give an overview of some of the points that have interested the author recently. For a survey of a more general character the reader is strongly encouraged to read [5]. Most of the topics treated in this paper arose from conversations with G. Walschap.

We should note that, for compact manifolds, the peculiarities of nonnegative curvature are still far from our understanding. As an example, there is almost no light at all on whether there exist any topological obstructions forbidding positive curvature in a nonnegatively curved simply connected space (e.g, the Hopf conjecture). Contrary to this state of affairs, the open case has benefited from two important structural results:

**THEOREM 1.1** (The Soul Theorem [3]). *Let  $M$  be an open manifold with nonnegative sectional curvature. There exists a compact, totally geodesic and totally convex submanifold  $S$  without boundary such that  $M$  is diffeomorphic to the normal bundle of  $S$ .*

The proof of this theorem is constructive, and the  $S$  that provides is called a *soul* of  $M$ . It does not need to be unique, as can be seen in the Euclidean space. The normal bundle of  $S$  is usually denoted by  $\nu(S)$ .

**THEOREM 1.2** (Perelman's Rigidity Theorem [14]). *Let  $M$  be an open manifold with nonnegative sectional curvature and soul  $S$ .*

- (1) *Let  $U$  be a parallel section of  $\nu(S)$  along a geodesic  $\alpha : I \rightarrow S$ . Then  $R(t, s) = \exp(tU(s))$  is a flat rectangle totally geodesically immersed in  $M$ .*
- (2) *The closest point projection  $\pi : M \rightarrow S$  is a well defined  $C^1$  Riemannian submersion. In fact,  $\pi$  coincides with the Sharafutdinov map ([16]).*
- (3) *Let  $\bar{\pi} : \nu(S) \rightarrow S$  be the bundle projection. Then  $\pi \circ \exp = \bar{\pi}$ .*

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In [9] we improved the differentiability of  $\pi : M \rightarrow S$  to  $C^2$ .

The Soul Theorem describes the differentiable structure of  $M$ , while Perelman's result describes some of the metric relations between  $M$  and  $S$ . In fact, it gives a strong geometric content to the delicate balance between the nonnegative lower curvature bound and the noncompactness of  $M$  (for a much better description of this tension, consult the first sections of [5]).

However, the relation between the topology of  $M$  and the existence of  $\text{sec} \geq 0$  metrics is still obscure. We only know that there are bundles over Bieberbach manifolds that do not admit such metrics ([13], [17]) but the simply connected case is still quite mysterious. The motivating question already appeared in [3]:

QUESTION (\*). Can every vector bundle over the standard sphere be realized as  $\nu(S)$  for some metric with  $\text{sec} \geq 0$ ?

The most direct approach "Can we put a metric with  $\text{sec} \geq 0$  in every vector bundle over the sphere?" is less satisfying since the answer could ignore the structure of the original bundle. As an example, for  $M = S^{11} \times \mathbb{R}^7$ , Haefliger showed that there is a second vector bundle structure  $M \rightarrow S^{11}$  different from the trivial. Thus for  $M$ , the answer to the more direct question is always yes (thanks to the product metric), but we still ignore whether there is a second metric with soul  $S$  whose normal bundle corresponds to Haefliger's bundle ([4]).

The following results could indicate that the answer to Question (\*) is affirmative:

- (1) Given any vector bundle  $E$  over  $S^n$  there is a representative in the stable class of  $E$  which admits a metric with  $\text{sec} \geq 0$  [15].
- (2) The answer to Question 1 is "yes" when  $n = 2$  ([20]) and when  $n = 4$  ([7]).

What type of information about  $E$  is necessary to solve (\*)? The results in [8] could be useful on answering this: Denote by  $D_r$  a tubular neighborhood of radius  $r$  about  $S$ . In [10] we showed that  $D_r$  is convex whenever  $r > 0$  is small enough. This, together with the existence of the submersion  $\pi$ , was used in [8] to show that the original metric in  $D_r$  can be deformed (by pushing radially the boundary of  $D_r$  to infinity) to produce another complete metric  $g_r$  with  $\text{sec} \geq 0$  in  $M$ . Moreover, this new metric coincides with the original metric in  $D_{r/2}$ . Thus, if we start with a nonnegatively curved metric in  $M$ , we can replace it by another sequence of metrics  $\{g_r\}$  (with  $r \rightarrow 0$ ) whose original input depends only on an  $r$ -tubular neighborhood of  $S$  for the original  $g$ . Naively said, the existence of a nonnegatively curved metric in  $M$  is only dependent on the germ of the original metric  $g$  around  $S$ . Or turning this around, if there is any obstruction in  $E$  forbidding us to put nonnegative curvature, it must show up already at the zero section of the bundle.

The natural approach to (\*) is then to examine properties at  $S$  of nonnegatively curved metrics. We focus in the normal holonomy group induced by the normal connection of  $S$  in  $\nu(S) \simeq M$ , and give three aspects of it in the next sections.

## 2. Normal curvature restrictions.

Let  $\Lambda^2(M)$  be the bundle of exterior 2-vectors of  $M$ . Its restriction to  $S$  splits into three subbundles corresponding to the splitting  $TM|_S = TS \oplus E$ . Namely,

$$\Lambda^2(M)|_S = \Lambda^2(S) \oplus \Lambda^2(E) \oplus (TS \wedge E)$$

where the fiber of the last summand is formed by all the bivectors  $\sum x_i \wedge u_i$ , where  $x_i \in T_p S$ ,  $u_i \in E_p$ . The curvature tensor splits accordingly; one of its parts is  $R^\nabla : TS \wedge E \rightarrow TS \wedge E$ , due to the identities of the curvature and the fact that  $S$  and  $\pi^{-1}(p)$  are totally geodesic at  $p$ .

Suppose that  $\dim S = n$ ,  $\text{codim} S = k$ . For each  $p \in S$ , there is a natural embedding  $\phi_p : S^{n-1} \times S^{k-1} \rightarrow T_p S \wedge E_p$  given by  $(x, u) \rightarrow x \wedge u$ . If  $(x, u)$  are orthonormal then  $\|x \wedge u\| = 1$  for the natural metric that  $T_p M$  induces on  $\Lambda_p^2 M$ . Hence the image  $G_p$  of each  $\phi_p$  is a submanifold of the unit sphere  $S^{n+k-1}$  in  $T_p S \wedge E_p$ , and we have a splitting  $TS^{n+k-1} = TG_p \oplus NG_p$  along the image of  $\phi_p$ . Moreover,  $NG_p$  at a given  $(x, u)$  is spanned by elements of the form  $y \wedge v$ , where  $y$  and  $v$  form orthonormal basis of the subspaces normal to  $x$  and  $u$  respectively in  $T_p S$  and  $E_p$ .

When  $R$  is an algebraic curvature operator with nonnegative sectional curvature, a zero curvature plane  $a \oplus b$  satisfies  $R(a, b)a = R(a, b)b = 0$  (see [19], for instance). In our situation, since  $S$  is totally geodesic and every vertical plane based at  $S$  is flat, we have that  $R^\nabla(x, u) \in NG_p$  for all  $x, u$ .

We can collect together this information for every  $p \in S$ . The final result is a vector bundle  $N$  over a compact manifold  $P$  which fibers itself over the soul  $S$  with fiber  $S^{n-1} \times S^{k-1}$ . As we have seen, a metric of nonnegative curvature in the original  $E$  forces  $R^\nabla$  to give a section of  $N$ . Since the Ambrose–Singer theorem assures that the image of  $R^\nabla$  (together with its parallel transports) generates the tangent space to the action of the holonomy group, it is clear that the shape of the bundle  $N$  could restrict it.

### 3. Holonomy and the geometry of $\pi$ .

In this section we present a few examples to show how some geometric aspects of  $M$  in different fibers of the map  $\pi$  are preserved by the holonomy. In order to compare fibers, we will introduce some useful notions [6]: given a smooth path  $\alpha : [0, 1] \rightarrow S$  and a point  $p$  in the fiber over  $\alpha(0)$  we can find another path  $\beta$  with  $\beta(0) = p$ ,  $\pi\beta = \alpha$  and with  $\beta' \in \mathcal{H}$ :  $\beta$  is called a *lift of  $\alpha$* . When we collect all such lifts for different points in the fiber, we obtain a smooth map  $h_\alpha : \pi^{-1}(\alpha(0)) \rightarrow \pi^{-1}(\alpha(1))$  defined as  $h_\alpha(q) = \beta(1)$ . If we repeat the construction with  $\alpha$  travelled "backwards", we obtain the inverse map to  $h_\alpha$ . For this reason,  $h_\alpha$  is known as the *holonomy diffeomorphism associated to  $\alpha$* . In our situation, they are at least  $C^1$  due to [9].

In the nonnegative curvature case, it is easy to visualize these maps thanks to Perelman's theorem: the lift of  $\alpha$  to any  $q$  is precisely the top part of the flat vertical rectangle determined by  $\alpha$  and any vertical vector  $u$  in  $\exp^{-1}(q)$  (once again, recall that  $\exp$  is the normal exponential map of the soul). In other words,  $h_\alpha(\exp(u)) = \exp(P_\alpha u)$ . If we consider only closed loops, the  $h_\alpha$  form a group of diffeomorphisms of the initial fiber. This group is the image, after composition with  $\exp$ , of the normal holonomy group  $\text{Hol}$  of the connexion  $\nabla$  that the metric with nonnegative curvature induces in  $\nu(S)$ .

Now we are ready to make more explicit the relation between some of the geometry of  $M$  and the Riemannian submersion  $\pi$ :

**THEOREM 3.1.** *For any path  $\alpha : I \rightarrow S$ ,  $h_\alpha$  preserves the following objects:*

- (1) *The ray structure of each fiber.*
- (2) *Level sets of convex functions, including the Cheeger–Gromoll compact convex exhaustion.*

- (3) *The pseudosoul structure of  $M$ .*
- (4) *The conjugate and cut locus of  $S$ .*

*Proof:* The key observation is that any horizontal geodesic remains in a compact set, and hence any convex function in  $M$  will be constant over it. This gives automatically the second part of the theorem. For the first, recall that a ray is a minimal geodesic  $\gamma : [0, \infty) \rightarrow M$ . Theorem 2.6 from [2] implies easily that a ray has vertical tangent vector, and hence it is entirely contained in a fiber of  $\pi$ . The rest of the argument for (1) appears in [20] for rays originating at the soul and in [9] for general rays. (4) is also proved in [9]. Finally, recall that pseudosouls are totally geodesic submanifolds of  $M$  that are homologous to the soul  $S$ . In [21] it was shown that they form a subset of the form  $S \times P$  embedded totally geodesically in  $M$  with  $P$  diffeomorphic to  $\mathbb{R}^l$  for some  $l$ . Furthermore, he also showed that if  $q = \exp(u)$  is in a pseudosoul, then  $\exp(P_\alpha u)$  remains in that pseudosoul. This proves the remaining statement.  $\square$

#### 4. Transitive Normal Holonomy

The fact that rays remain invariant under the action of the holonomy group is a noticeable feature of these manifolds. This is specially useful in the following situation:

LEMMA 4.1. *Suppose Hol acts transitively. Then*

- (1)  *$\exp : \nu(S) \rightarrow M$  is a diffeomorphism.*
- (2) *The ideal boundary of  $M$  is a point.*

Recall that the ideal boundary of  $M$  is formed by identifying rays starting at the same point with the equivalence relation  $\gamma_1 \sim \gamma_2$  if  $\lim d(\gamma_1(t), \gamma_2(t))/t = 0$ .

*Proof:* Since the holonomy group respects rays, all the normal directions originating at one point of the soul are ray directions, thus proving (1). The second part is a straightforward application of Perelman's theorem: the distance between two rays originating at one point is bounded above by the length of the loop needed to parallel transport one of the ray directions to the other.  $\square$

This lemma is useful when combined with topological conditions on the bundle forcing every Riemannian connection to have big holonomy:

EXAMPLE 4.2. Suppose that  $E$  is a rank  $n$  vector bundle over  $S^n$  with non-vanishing Euler number  $e(E)$  (this forces  $n$  to be even). When in presence of a Riemannian connection,  $e$  can be computed as follows: let  $u$  be a unit vector in the north pole. For each unit tangent vector  $x \in T_N S^n$ , parallel transport  $u$  along a meridian tangent to  $x$  to obtain a vector in the fiber of  $E$  at the south pole. Thus we obtain a map  $\Phi : S^{n-1} \rightarrow S^{n-1}$  whose degree equals  $e(E)$  [1]. Thus, if  $e(E) \neq 0$ ,  $\Phi$  needs to be onto. Therefore, given any two vectors  $v = \Phi(x)$  and  $w = \Phi(y)$  in  $E_S$ , the parallel transport of  $v$  to the north pole along the  $x$ -meridian (from south to north) composed with the parallel transport along the  $y$ -meridian (from north to south) maps  $v$  to  $w$ . Thus the holonomy acts transitively.

In [11] we have been able to generalize this example to the following theorem:

THEOREM 4.3. *Let  $E$  denote the total space of an oriented rank  $k$  vector bundle over  $S^n$ . If the bundle does not admit a nowhere-zero cross section, then the holonomy group of any Riemannian connection on the bundle acts transitively on  $E$ . Consequently any nonnegatively curved metric over such bundles with  $\nu(S) = E$  satisfies the conclusion of Lemma 4.1.*

For an even more general statement, see [12].

### 5. Some open questions.

Ultimately, we would like to understand better the action of the normal holonomy group when combined with the  $\sec \geq 0$  assumption. Thus, after the last section the following questions seem unavoidable:

- (1) If  $\text{Hol}$  acts only irreducibly, is the normal exponential still a diffeomorphism?
- (2) Suppose that  $E$  is a vector bundle with nonnegative curvature over a simply connected soul  $S$ . Suppose also that  $E$  does not split as a Whitney sum (except trivially). If  $e(E) \neq 0$ , does the normal holonomy acts transitively? The condition on  $E$  is necessary, as it is easy to construct counterexamples for the tangent bundle of  $S^{2k} \times S^{2k'}$ .
- (3) What is the right formulation of the above question when some other characteristic class of the bundle replaces the Euler class?

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