A METRIC CHARACTERIZATION OF RIEMANNIAN SUBMERSIONS

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Abstract. A map between metric spaces is called a submetry if it maps balls of radius R around a point onto balls of the same radius around the image point. We show that when the domain and target spaces are complete Riemannian manifolds, submetries correspond to $C^{1,1}$ Riemannian submersions. We also study some consequences of this fact, and introduce the notion of submetries with a soul.

1. Introduction.

In the last years it has become clear the advantage of translating concepts and ideas from Riemannian geometry to a purely metric setting. A big stimulus has come from the study of manifolds with restrictions on the curvature and/or other geometric invariants. It is common to consider sequences of manifolds converging under the Gromov–Hausdorff distance to an object which a priori is only a metric space; the Riemannian characteristics of the sequence persist in the limit in a weaker sense. It is in this moment that having the generalized concepts is useful.

For Riemannian submersions, the natural metric analogue, due to the first author, is that of a *submetry*: namely, a map ϕ between metric spaces is called a submetry if it sends closed balls of radius R around a point to closed balls of radius R around the image point [3]. Several applications of this idea appear in [1], [3] and [9]. For completeness we include in section 2 some of its main properties.

The definition of submetry does not impose any other restrictions on the domain and target spaces than being a metric space. In this paper we study the situation when both of them are smooth Riemannian manifolds. Our main result, which was announced by the first author in [4], is the following:

Theorem A. *Let $\phi : M \to N$ a submetry between smooth Riemannian manifolds. Then ϕ is a $C^{1,1}$ Riemannian submersion.*

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The proof appears in section 3. The $C^{1,1}$ regularity in Theorem A is optimal: already in dimension 2 there is an example due to Efremovich, Gorelik and Vainshtein of a submetry of the hyperbolic plane $\phi : \mathbb{H}^2 \to \mathbb{R}$ that is not C^2 (see section 4 and [7]).

However, the nonnegative curvature case seems to behave differently (see Theorem B). In fact, Theorem A has the following immediate corollary when combined with the results of [10] and Example 2.2(b):

Corollary 1.1. *Let M be an open manifold with nonnegative sectional curvature and soul S. Then the metric projection $\pi : M \to S$ (also known as the Sharafutdinov map* [11]*) is $C^{1,1}$*

In [8] this was improved to C^2, although for the proof, Corollary 1.1 is necessary. We still ignore whether the Sharafutdinov map is always smooth. But in this line it should be noticed that every known submetry between Riemannian manifolds with nonnegative curvature is smooth. This motivates us to formulate the following question:

Question 1. *Let M be a manifold with nonnegative sectional curvature, and $\phi : M \to N$ a submetry onto some complete Riemannian manifold. Is ϕ a C^∞ Riemannian submersion? Does the corresponding statement holds in the analytic category?*

In section 5 we study several concrete situations. First we characterize submetries onto $\mathbb{R}$ as those functions whose gradient vector field has constant unit norm. Then we examine two cases when the domain space M has nonnegative lower curvature bounds. Our results are the following three theorems:

Theorem B. *Let $\phi : M \to \mathbb{R}$ be a C^1 funtion where M is a complete Riemannian manifold. Then ϕ is a submetry if and only if $\|\nabla\phi\| = 1$. Moreover, if M has nonnegative Ricci curvature, then there is a totally geodesic hypersurface N on M such that $M = N \times \mathbb{R}$, and ϕ can be identified with projection onto the second factor.*

Theorem C. *Let $\phi : M \to N$ be a submetry of complete Riemannian manifolds where M has nonnegative sectional curvature. Then*

1. *N has also nonnegative sectional curvature.*
2. *If N is compact and M is flat, then N is flat.*

Theorem D. *Let $\phi : M \to N$ be a submetry of complete Riemannian manifolds where M and N have nonnegative Ricci curvature (these conditions are satisfied under the hypothesis of Theorem C). Then there is an integer $n_0 > 0$ and totally geodesic submanifolds M_0, N_0 of M and N respectively, such that M is isometric to $M_0 \times \mathbb{R}^{n_0}$, N is isometric to $N_0 \times \mathbb{R}^{n_0}$, N_0 does not contain any line, and ϕ can be identified with (ϕ_0, Id), where $\phi_0 : M_0 \to N_0$ is a submetry.*

Finally, in the last section we study *submetries with a soul.* These are those submetries $\phi : M \to N$ for which there is a distance nonincreasing map $\psi : N \to M$ with $\phi \circ \psi = Id_N$. The name is justified when M is an open nonnegative curvature manifold: the inclusion of a soul S in M is a map satisfying the conditions for ψ in our definition; thus the Sharafutdinov map is a submetry with soul. In similarity to this situation, we will show that $\psi(N)$ is a totally geodesic submanifold of M, and also extend Perelman's resolution of the soul conjecture of Cheeger and Gromoll to this case:

Theorem E. *Let M be a complete (possibly noncompact) manifold with nonnegative sectional curvature, $\phi : M \to N$ be a submetry with a soul, and N is compact. Let $N' = \psi(N)$,* *and denote by* $\nu(N')$ *the normal bundle of* $\psi(N)$ *in* M*, with* $\exp : \nu(N') \to M$ *its exponential map.*

(1) *If* $\gamma : \mathbb{R} \to N'$ *is a geodesic and* $U = U(t)$ *a parallel vector field of* $\nu(N')$ *along* $\gamma = \gamma(t)$*, then the surface* $R(t, s) = \exp(sU(t)); s, t \in \mathbb{R}$ *is a flat totally geodesically strip immersed in* M*.*

(2) *For any fixed* t_0*,* $\phi(R(t_0, s)) = \phi(R(t_0, 0))$ *for every* $s \in \mathbb{R}$*.*

Observe that this rules out the possibility of having nontrivial souls when M is a manifold with positive curvature, since $\exp : \nu(N') \to M$ is necessarily onto.

2. Some basic facts about submetries.

To facilitate the reading, we include in this section the main facts from [3] that are necessary for our proof of Theorem A. A reader familiarized with the concept of submetries could proceed safely to section 3.

2.1. **Definition:** If p is a point in a metric space X, we use $B(p, R)$ to denote a closed metric ball of radius R centered at p. As already mentioned, a map $\phi : M \to N$ between metric spaces is called a *submetry* if $\phi(B(p, R)) = B(\phi(p), R)$ for all $p \in M$, $R > 0$. To avoid unnecessary complications, we will always assume that M and N are Riemannian manifolds without boundary except if noted otherwise; nonetheless most of what is included in this section also applies to more general metric spaces [3].

2.2. **Examples:**

(a) If $p \in M$, then the distance function $f(q) = |pq|$ is a submetry onto an open interval in the open ball around q_0 of radius $|pq_0|$, where $|pq_0|$ is a small positive number.

(b) More generally, every C^1 Riemannian submersion of complete Riemannian manifolds is a submetry; Theorem A then shows that it is necessarily $C^{1,1}$.

(c) Two subsets of a metric space are called equidistant if for each $p \in F_1$ there exists a $q \in F_2$ with $|pq| = |F_1F_2|$. A *metric fibration* $\mathcal{F}$ of a complete metric space X is a decomposition into isometric disjoint mutually equidistant closed sets. The quotient space $X/\mathcal{F}$ inherits a natural metric for which the projection map is a submetry.

(d) However, submetries are not always metric fibrations, since its fibers are not necessarily isometric. This often happens when G is a group acting by isometries on a metric space X with $\phi : X \to X/G$ the quotient map. For example, choose a hyperplane $H \subset \mathbb{H}^3$ and let G be the set of oriented isometries of $\mathbb{H}^3$ leaving H invariant. H divides $\mathbb{H}^3$ in two components; the orbit passing through a point p is formed by all the points lying in the same component and at the same distance from H as p. None of them (except for H) is isometric to $\mathbb{H}^2$.

Nonetheless, if a submetry ϕ between Riemannian manifolds has totally geodesic fibers, then they form a metric fibration (see [5], theorem 9.56). But there are also metric fibrations with nontotally geodesic fibers, as the example in section 4 shows.

2.3. **Liftings:** Each fiber $\phi^{-1}(q)$ is a closed set. Let $F_i = \phi^{-1}(r_i)$, $i = 1, 2$, and suppose that between r_1 and r_2 there is a unique minimal geodesic segment $\bar{\alpha} : [0, l] \to N$ where $l = |r_1r_2|$. Let $p_1 \in F_1$; by the definition of submetry, $\phi(B(p_1, l)) = B(r_1, l)$, so there is a $p_2 \in F_2$ with $p_2 \in B(p_1, l)$. Let $\alpha : [0, l'] \to M$ a minimal geodesic with $\alpha(0) = p_1$, $\alpha(l') = p_2$.

Lemma 2.1. *In the above situation,*

(1) $l = l'$ *and* $\phi(\alpha(t)) = \bar{\alpha}(t)$.
(2) $\alpha : [0, l] \to M$ *is the unique geodesic with* $\phi(\alpha(t)) = \bar{\alpha}(t)$.

Proof: Since ϕ does not increase distances,

$$l = |\phi(p_1)\phi(p_2)| \leq |p_1 p_2| = l' \leq l$$

Thus $l = l'$ and $|p_1 p_2| = |\phi(p_1)\phi(p_2)|$. Also, $\phi \circ \alpha : [0, l] \to N$ is a curve between r_1 and r_2 whose length is not larger than l, thus it has to agree with $\bar{\alpha}$ everywhere.

To prove (2), extend $\bar{\alpha}$ to an interval $[-\varepsilon, l]$ so that $\bar{\alpha}$ is minimal in $[-\varepsilon, \varepsilon]$. If α_1, α_2 are such that $\phi(\alpha_i(t)) = \bar{\alpha}(t)$, let $\beta : [-\varepsilon, \varepsilon] \to M$ be $\alpha_1(t)$ for $t \leq 0$ and $\alpha_2(t)$ for $t \geq 0$. It is trivial to check that β is a lift of $\bar{\alpha}$ over $[-\varepsilon, \varepsilon]$; therefore, $l(\bar{\alpha}) = 2\varepsilon \leq l(\beta) \leq 2\varepsilon$. Thus β is a smooth geodesic, $\alpha_1'(0) = \alpha_2'(0)$ and by uniqueness, $\alpha_1 = \alpha_2$. □.

Definition 2.2. The α obtained in the above lemma is called *the lift of* $\bar{\alpha}$ *to* p.

3. Proof of Theorem A

There is an important reduction that allows us to simplificate the proofs considerably, thanks to an idea already appearing in [2]. Namely, let $p \in M$ and $r = \phi(p) \in N$ its image. We can choose points $r_1, \ldots, r_n \in N$ so that the map $r' \to (|r' r_1|, \ldots, |r' r_n|)$ is a smooth coordinate chart in a neigbourhood of $\phi(p)$. Each entry of the corresponding expression for ϕ is then given by a composition of two submetries, and will be a submetry onto some interval. Now it is enough to note that ϕ has the same regularity as its coordinates functions. Thus we can assume after centering that N is an interval of the form $(-R, R)$ for some $R > 0$. Furthermore, it is clear from the above that we can take R small enough so that the ball $B(p, R)$ is contained inside a totally normal neighbourhood of p.

Let F_s be the fiber $\phi^{-1}(s)$ for $s \in N$. For small $r > 0$, we start constructing a map $h_r : U \to M$ defined in some neighbourhood of p that preserves the fibers of ϕ: namely, if $q \in U$ with $\phi(q) = s$, lift the interval $[s, s + r]$ to q. This gives us a geodesic $\alpha : [0, r] \to M$ such that $\alpha(0) = q$, $\phi(\alpha(t)) = s + t$. Then define $h_r(q) = \alpha(r)$. Clearly, with a careful choice of U and r we can assume that h_r does not abandon the initial $B(p, R)$ that we set at the beginning. Moreover, from section (2.3), it follows that $|q h_r(q)| = r$ and that $h_r(q)$ is the closest point to q in F_{s+r}.

Lemma 3.1. h_r *is continuous in* U.

Proof: Suppose q_i is a sequence of points converging to $q \in U$. Since $|q_i h_r(q_i)| = r$, the sequence $h_r(q_i)$ stays in a compact set. Let Q be a limit point of some subsequence that we keep denoting by $\{q_i\}$. Clearly $d(q, Q) = \lim |q_i h_r(q_i)| = r$. On the other hand,

$$\phi(Q) = \lim \phi(h_r(q_i)) = \lim(\phi(q_i) + r) = \phi(q) + r$$

It follows that $h_r(q) = Q$, as desired. □

Therefore the unit vector field $X(q) = \frac{1}{r}\mathrm{Exp}^{-1}(q, h_r(q))$ (where $\mathrm{Exp} : TM \to M \times M$ is $\mathrm{Exp}(v) = (\pi(v), \exp v)$ and π is the tangent bundle projection), is well defined and continuous in some open set around p. Observe also that, if γ_q is the geodesic with $\gamma_q(0) = q$ and $\gamma'(0) = X(q)$, then $\phi(\gamma(t)) = \phi(q) + t$ for t small. We will see a little later that ϕ has a well defined continuous gradient vector field agreeing with X. The formula for X also shows that this vector field has the same order of regularity as the map h_r.

Next choose an open set U and a number $r > 0$ small enough so that any $\partial B(q, r)$ with center in U has second fundamental form bounded uniformly. The important fact is that every point of each fiber of ϕ is the unique common point of this fiber with two such balls. In fact, for any $q \in U$, lift the intervals $[\phi(q), \phi(q) + r]$ and $[\phi(q) - r, \phi(q)]$ to q. Thanks to section (2.3), they match well to give a geodesic segment centered at q. Let q_r^+ and q_r^- be the endpoints respectively of such lifts (obviously, $q_r^+ = h_r(q)$, and $h_r(q_r^-) = q$). Since q is the unique point in its fiber realizing the distance to q_r^+ and q_r^-, the balls of radius r centered at them are tangent to each other, and intersect the fiber through q only at q.

Lemma 3.2. *There is a constant $C > 0$ such that for any $q \in U$, there are C^∞ functions f_q^+ and f_q^- such that*

(i) $f_q^- \leq \phi \leq f_q^+$ *and equalities are attained only at the geodesic segment connecting q_r^+ to q_r^-.*
(ii) $\nabla f_q^-(q) = \nabla f_q^+(q) = X(q)$
(iii) $|\mathrm{Hess} f_q^-| \leq C, |\mathrm{Hess} f_q^+| \leq C$

Proof: We define

$$f_q^-(x) = \phi(q) + r - |q_r^+ x| \qquad f_q^+(x) = \phi(q) - r + |q_r^- x| \tag{3.1}$$

It is clear that (ii) and (iii) hold trivially from the choices made. To prove the first inequality of (i), observe that f_q^- is constant along metric spheres centered at q_r^+. If $|q_r^+ x| = s$, let q' be the only point in the geodesic segment $q_r^+ q_r^-$ at distance s from q_r^+. The ball $B(q_r^+, s)$ shares only the point q' with the fiber of ϕ through q' , thus $\phi(x) \geq \phi(q') = \phi(q) + (\phi(q') - \phi(q)) = \phi(q) + r - |q_r^+ q'| = f_q^-(x)$. The other inequality is similar. Finally, if $f_q^-(x) = \phi(x)$, then $\phi(q_r^+) = \phi(q) + r = \phi(x) + |q_r^+ x|$. This means that x is the only point in its fiber realizing the distance to q_r^+, and (i) follows. □

This lemma shows that $\phi(q) = f_q^+(q) = f_q^-(q)$ and also that f_q^+ and f_q^- have the same partial derivatives at q. It follows now from 3.2(i) that ϕ also has the same partial derivatives.Thus ϕ has a well defined continuous gradient vector field agreeing with the previously defined X. Since X was continuous together with h_r, ϕ is C^1.

From the previous considerations, it follows that for ϕ we have a Taylor formula

$$\phi(x_2) - \phi(x_1) = \frac{\partial \phi}{\partial x}(x_1) \cdot (x_2 - x_1) + O(\|x_2 - x_1\|^2) \tag{3.2}$$

for any two points x_1, x_2 sufficiently close in some open set U with coordinates (x^i). In the above formula, $\cdot$ means the usual euclidean scalar product of the vectors $\frac{\partial \phi}{\partial x}(x_1)$ (consisting of partial derivatives) and $(x_2 - x_1)$ with coordinates $(x_2^i - x_1^i)$, $\| * \|$ means euclidean norm, and $O(t)$ means a term which can be evaluated as $|O(t)| \leq Ct$, for $C, t \geq 0$ and C some universal constant.

Similarly to 3.2, we get the formula

$$\phi(x_2) - \phi(x_0) = \frac{\partial \phi}{\partial x}(x_0) \cdot (x_2 - x_0) + O(\|x_2 - x_0\|^2) \tag{3.3}$$

Taking the difference between 3.3 and 3.2, we get

$$\phi(x_1) - \phi(x_0) = \frac{\partial\phi}{\partial x}(x_0)\cdot(x_1 - x_0) + \left[\frac{\partial\phi}{\partial x}(x_0) - \frac{\partial\phi}{\partial x}(x_1)\right]\cdot(x_2 - x_1) + O(\|x_2 - x_0\|^2 + \|x_2 - x_1\|^2) \tag{3.4}$$

From this it follows that

$$\left[\frac{\partial\phi}{\partial x}(x_0) - \frac{\partial\phi}{\partial x}(x_1)\right]\cdot(x_2 - x_1) = O(\|x_2 - x_0\|^2 + \|x_2 - x_1\|^2 + \|x_1 - x_0\|^2)$$

For given x_0, x_1, we can certainly take x_2 such that the two vectors from the left side have the same directions, and so that $\|x_2 - x_1\| = \|x_1 - x_0\|$. Then

$$\left\|\frac{\partial\phi}{\partial x}(x_0) - \frac{\partial\phi}{\partial x}(x_1)\right\| \|x_2 - x_1\| = O(\|x_2 - x_0\|^2 + 2\|x_1 - x_0\|^2) = O(6\|x_1 - x_0\|^2)$$

where $\|x_2 - x_0\|^2 \leq 4\|x_1 - x_0\|^2$ by the triangular inequality. Hence

$$\left\|\frac{\partial\phi}{\partial x}(x_0) - \frac{\partial\phi}{\partial x}(x_1)\right\| = O(6\|x_1 - x_0\|)$$

and the conclusion of Theorem A follows. □

4. A $C^{1,1}$ submetry which is not C^2.

In [7] there is an example of a metric fibration of the hyperbolic plane whose fibers are not C^2 submanifolds. We describe it here to illustrate that the regularity obtained in Theorem A can not be improved without new restrictions.

Start by choosing a geodesic line $\gamma : (-\infty, \infty) \to \mathbb{H}^2$. There is an associated Busemann function $b_\gamma : \mathbb{H}^2 \to \mathbb{R}$ defined as

$$b_\gamma(x) = \lim_{t\to\infty}(t - |x, \gamma(t)|)$$

Its level sets are called horospheres, and they give a decomposition of $\mathbb{H}^2$ into equidistant sets. In the disk model, each horosphere corresponds to a circunference tangent to the unit circle at $\gamma(\infty)$. It is well known that b_γ is a C^∞ function with unit gradient. This implies that the horospheres are equidistant; furthermore, it is trivial to see that b_γ is a submetry over $\mathbb{R}$ (see Figure 1). Observe that $\alpha(t) = \gamma(-t)$ is also a geodesic line that we denote by $-\gamma$, and that its horospheres are the circles tangent to the oppposite point $\gamma(-\infty)$.

$\gamma(-\infty)$ $\gamma(\infty)$ $\gamma(-\infty)$ $\gamma(\infty)$

Figure 1. Figure 2.

In order to introduce some nonsmoothness in this metric fibration, notice that γ divides each horosphere into two halves. For each $\gamma(t)$, let the subset F_t be formed by taking the upper part of the horosphere for b_γ and the lower part of the horosphere corresponding to $b_{-\gamma}$. Clearly the F_t are equidistant, and the map ϕ sending each F_t to t is a submetry onto $\mathbb{R}$. However, its fibers are not C^2 since the horospheres do not match smoothly at the points of γ (see Figure 2).

In general, all the metric fibrations of $\mathbb{H}^2$ have been classified in [7]; in addition to the two mentioned above, there is a whole one parameter family of fibrations whose curves $(x(t), y(t))$ satisfy the equation

$$x' = y'\sqrt{y^2 + a}$$

When $a \to \infty$ the limiting fibration corresponds to the nonsmooth case.

5. Proof of Theorems B , C and D

Proof of Theorem B: If $\phi : M \to \mathbb{R}$ is a submetry, then its gradient vector field is the lift of the vector ∂_t of $\mathbb{R}$, as we have seen in the proof of Theorem A, and therefore it has unit norm. Conversely, assume that $\phi : M \to \mathbb{R}$ is a C^1 function with unit gradient. Let $p \in M$ and assume, without loss of generality that $\phi(p) = 0$. We need to show that for any $r > 0$, $\phi(B(p, r)) = (-r, r)$. Denote by $c(t)$ the integral curve of $\nabla\phi$ through p. Since $\|\nabla\phi\| = 1$, we have

$$\phi \circ (t) = \int_0^t \frac{d}{d\tau}\phi(c(\tau))\, d\tau = \int_0^t < \nabla\phi, c'(\tau) > d\tau = \int_0^t \|\nabla\phi\|^2\, d\tau = t$$

for any $t \in \mathbb{R}$. Obviously $|p\, c(t)| = t$, finishing the first part of the Theorem.

For the second part, recall that a line in a Riemannian manifold is a geodesic $c : \mathbb{R} \to M$ which is minimal between any two of its points, i.e, $|c(t_1)\, c(t_2)| = |t_1 - t_2|$ for any t_1, t_2. In our situation, it is easy to show that any horizontal lift of $\mathbb{R}$ is of such type: choose $p \in M$, and $c : \mathbb{R} \to M$ an integral curve for $\nabla\phi$ through p. Then $\phi(c(t)) = \phi(c(0)) + t$, and

$$|t_1 - t_2| = |\phi(c(t_1)) - \phi(c(t_2))| \le |c(t_1)\, c(t_2)| \le |t_1 - t_2|$$

thus proving that c is a line in M. When the ambient space has nonnegative Ricci curvature, any line splits isometrically by the splitting theorem of Cheeger and Gromoll ([6]). Thus we have an isometry from some $N \times \mathbb{R}$ to M where we can both assume that it sends $(n, 0)$ to p for some $n \in N$, as well as $\phi(n, t) = t$. The only point left is to show that under the identification $M \simeq N \times \mathbb{R}$, $\phi(n', t) = t$ for all $n' \in N$. However, this was already done in the proof of Theorem A, since following the gradient lines for $\nabla\phi$ preserves the fibers of ϕ. □

This theorem has the following pleasant consequence:

Corollary 5.1. *Let M be a complete Riemannian manifold with $Ric \ge 0$, and $\phi : M \to \mathbb{R}$ a C^1 function with unit gradient. Then ϕ is C^∞.*

As an example, consider a Busemann function corresponding to some ray γ. It is C^1 almost everywhere, and its gradient at a smooth point p is given by the vector tangent to the unique ray from p asymptotic to γ. If it extends to a C^1 function in a manifold with nonnegative Ricci curvature, then we are in the situation of the above theorem and can conclude that the original function was, in fact, smooth.

Proof of Theorem C: The first point follows from the fact that if $f : M \to N$ is a submetry and M has Alexandrov's curvature no less than K, then N also has this property. We present a proof of this in our situation. Let $q \in N$, $u_1, u_2 \in T_qN$ and $q_i = \gamma_{u_i}(r)$ for some r to be chosen later, where γ_{u_i} is the geodesic with initial condition u_i, $i = 1, 2$. Construct in $\mathbb{R}^2$ the geodesic biangle corresponding to u_1, u_2, and let $\bar{q}_i$ the points in $\mathbb{R}^2$ corresponding to q_1, q_2. If we choose $r > 0$ small enough, and if $K(u_1, u_2) < 0$, then $|q_1q_2| > |\bar{q}_1\bar{q}_2|_{\mathbb{R}^2}$. On the other hand, if we lift u_1, u_2 to vectors $\tilde{u_1}$, $\tilde{u_2}$ at a point $p \in \phi^{-1}(q)$, the results from Section 2 show that the angle between u_1, u_2 and the angle between the $\tilde{u}_i$ coincide. Since $K_M(\tilde{u_1}, \tilde{u_2}) \geq 0$, we get that $|\tilde{q_1}\tilde{q_2}| \leq |\bar{q}_1\bar{q}_2|_{\mathbb{R}^2}$. Now it is easy to see that $\tilde{q_1}$, $\tilde{q_2}$ are in the fibers over q_1 and q_2 respectively, which forces ϕ to increase distances. This provides the required contradiction.

(2) follows from Theorem 3.1 in [9] (see also [12]). □

Proof of Theorem D: This will be again the consequence of the Cheeger–Gromoll splitting theorem [6]. Clearly if N does not contain any lines, then the theorem is trivial. Otherwise, suppose that $\gamma : \mathbb{R} \to N$ is one. Applying the splitting theorem, we find a N_1 so that $N = N_1 \times \mathbb{R}$. Let $\pi_2 : N \to \mathbb{R}$ be the projection onto the second factor. The composition $\pi_2 \circ \phi$ is a submetry, and by the second part of Theorem B, we have a splitting $M = M_1 \times \mathbb{R}$ with $\pi_2 \circ \phi(m_1, t) = t$.

Denote by $\phi_1(m_1, t) = \pi_1 \circ \phi : M_1 \times \mathbb{R} \to N_1$ the other coordinate function for ϕ. Then ϕ_1 does not depend on t; in fact, since $\tilde{\gamma}(t) := (m_1, t)$ is the lift of γ to (m_1, t), we have

$$\pi_1\phi(m_1, t) = \pi_1(\gamma(t)) = \pi_1(n_1, t) = n_1$$

where $n_1 = \gamma(0)$. Furthermore, ϕ_1 can be seen as the restriction of ϕ to the totally geodesic submanifold $M_1 \times \{0\}$, followed by another submetry, and is then a submetry onto its image N_1.

This shows that $\phi = (\phi_1, t)$ with ϕ_1 a submetry. To complete the proof of the theorem, we just need to repeat this process while there are lines remaining in the base. After a finite number of steps, we obtain a decomposition of M, N and ϕ as that stated in the theorem. □

6. Submetries with a soul

Recall that a submetry $\phi : M \to N$ is said to have a soul if there is a distance nonincreasing map $\psi : N \to M$ with $\phi \circ \psi(q) = q$ for all $q \in N$. This is the case, for example, for the Sharafutdinov map $\pi : M \to S$ of a nonnegatively curved manifold onto one of its souls, since then we can take ψ as the inclusion. In fact, the main motivation behind this definition is to see what metric properties in the open case remain when the soul does not live inside the manifold. All through this section we will assume that N is a compact connected Riemannian manifold.

Lemma 6.1. *$\psi(N)$ is a totally geodesic submanifold of M.*

Proof: Let $\gamma : \mathbb{R} \to N$ be a geodesic parametrized by arclength, and denote $\alpha = \psi \circ \gamma$. It is well known that γ minimizes the distance locally, i.e, for any $t \in \mathbb{R}$, $|\gamma(t)\gamma(t + \varepsilon)| = \varepsilon$ for ε small enough. Since ψ does not increase distances,

$$\varepsilon \geq |\alpha(t)\,\alpha(t + \varepsilon)| \geq |\phi \circ \psi(\gamma(t))\,\phi \circ \psi(\gamma(t + \varepsilon))| = \varepsilon$$

Thus α also minimizes between t and $t+\varepsilon$, and since t was arbitrary, α is a geodesic.

If ψ was differentiable, this would be enough to conclude the proof; in that case, the above paragraph would show that the differential ψ_* is injective at every point, hence ψ would be a totally geodesic immersion. It would also be injective because of the condition $\phi\psi = Id$, hence an embedding.

In our situation, we can not assume that ψ is smooth, but since it does not increases distances it is differentiable almost everywhere. Thus the above argument shows that for any such point, there is a neighbourhood where ψ is actually a C^∞ map. Since the differentiability set had full measure, then ψ has to be C^∞ everywhere. ☐

Recall that for open manifolds with nonnegative curvature, one of the most useful tools is the rigidity theorem due to Perelman ([10]). Our next task is to show that such a result also holds for this type of submetries.

Proof of Theorem E: The proof is an exercise on adapting Perelman's original argument to our situation. As in [10], both (1) and (2) are proved simultaneously for small values of s by considering the function $f : [0,\infty) \to [0,\infty)$ given by

$$f(r) = \max\left\{ \, |q\,\phi(\exp_{\psi(q)} ru)| \; : \; q \in N, u \in \nu(N'), |u| = 1 \, \right\}$$

The arguments of [10] can be reproduced verbatim to ensure that $f(r) \equiv 0$ for some interval $[0,l]$. Then we change the definition of f to

$$f(r) = \max\left\{ \, |q\,\phi(\exp_{\psi(q)}(r+l)u)| \; : \; q \in N, u \in \nu(N'), |u| = 1 \, \right\}$$

Repeating this process for as long as needed, the theorem follows. ☐

This type of metric rigidity has several consequences:

Corollary 6.2. *Let $\phi : M \to N$ be a submetry with a soul, where M is a complete manifold with nonnegative sectional curvature. Then ϕ is a C^2 Riemannian submersion.*

Proof: This follows from the arguments appearing in [8] with ϕ replacing the role of the Sharafutdinov map. ☐

Corollary 6.3. *Let M be a Riemannian manifold with positive sectional curvature. If $\phi : M \to N$ is a submetry with a soul, then either N is a point, or $N = M$ and ϕ is an isometry.*

Proof: If N is not a point, then the last theorem rules out the possibility of N' having positive codimension in M, hence $N' = M$. By Lemma 6.1, ψ is then an isometry. ☐

We also have a different proof of the following result appearing in [12].

Corollary 6.4. *Let $M \times_f N$ be the warped product of two Riemannian manifolds using a function $f : M \to \mathbb{R}$. Then $M \times_f N$ does not have positive sectional curvature. Furthermore, if the warped product is nonnegatively curved, then it is a metric product.*

Proof: For any $n_0 \in N$, the map $\psi : M \to M \times N$ given by $\psi(m) = (m, n_0)$ shows that π_1 is a submetry with a soul. Hence, corollary 6.3 proves that the warped metric can not be of positive curvature. If it has nonnegative sectional curvature, then our extension of Perelman's rigidity theorem shows that through every point in $M \times N$ the fibers of ϕ are totally geodesic, and therefore the O'Neill tensors A and T vanish. This implies the metric splitting [5]. ☐

In fact, the argument also applies to generalized warped products (for a definition, see [5], section 9.10).

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