IMPROVING THE METRIC IN AN OPEN MANIFOLD WITH NONNEGATIVE CURVATURE

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1. INTRODUCTION

The existence of a metric with nonnegative curvature in an open Riemannian manifold sets important restrictions on its structure. For example, they contain a compact, totally geodesic submanifold S called the soul so that the total space is diffeomorphic to its normal bundle $\nu(S)$. In this note we study how, given such a metric, we can construct another one satisfying stronger conditions. In particular, we prove the following result:

Theorem A. *Let (M, g) be a compact convex manifold with smooth boundary, nonnegative sectional curvature and soul S. Then there exists a complete metric $\bar{g}$ in M with nonnegative curvature and that splits as a metric product $N \times [a, \infty)$ outside some tubular neighborhood of S, where N is diffeomorphic to the unit normal bundle of S in M.*

The following two facts are easy consequences:

Theorem 1.1. *Let M be an open manifold with nonnegative sectional curvature and soul S. Then there is a (possibly different) metric in M with nonnegative curvature, soul S and whose normal exponential map $\exp : \nu(S) \to M$ is a diffeomorphism. Moreover, the ideal boundary $M(\infty)$ of M with the new metric is only a point.*

Theorem 1.2. *Let M be an open manifold with nonnegative sectional curvature and soul S. If $D(S)$ is the normal disc bundle of S in M, then there is a metric with nonnegative sectional curvature in the double of $D(S)$.*

Recall that the double of $D(S)$ is the space obtained after gluing together two copies of $D(S)$ by the identity map of the boundary. Theorem 1.1 could help answering which vector bundles admit metrics with nonnegative curvature, since it allows us to assume that such metrics

are of a rather simple form. Theorem 1.2 should help constructing new compact manifolds with nonnegative curvature.

The proof of Theorem A uses some consequences of the main theorem in [5] that were indicated to us by G. Walschap, as well as a simple modification of a construction due to Kronwith ([3]).

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2. Convexity around the soul

Let M be a manifold with nonnegative sectional curvature and (possibly empty) smooth convex boundary. Recall that we denote its soul by S and its normal bundle by $\nu(S)$. There exists a distance non increasing retraction $sh : M \to S$ used by Perelman to prove the following result:

Theorem 2.1 (Perelman's rigidity theorem). [5]

(1) *Let $\alpha : [0, \infty] \to S$ be a geodesic, and $u \in \nu_p(S)$ a normal vector to the soul. Let U be the vector field along α obtained by parallel transport of u. Then*

$$R(t, s) = \exp_{\alpha(t)} sU(t) \qquad t \in [0, a], s \in [0, \infty)$$

is a flat rectangle totally geodesically immersed in M.

(2) $sh(\exp_p tu) = p \quad$ *for all $t \in [0, \infty)$*

(3) $sh : M \to S$ *is a C^1 Riemannian submersion.*

It follows from this that sh is C^∞ in a tubular neighborhood of S. We will use $\mathcal{H}$ and $\mathcal{V}$ for the horizontal and vertical distributions associated to this submersion.

We denote by D_r the set of points in M at distance not greater than r from S. When the soul is a point, D_r is convex for small r. It was pointed out to us by G.Walschap that Theorem 2.1 extends this to the case of non trivial souls:

Lemma 2.2. D_r *is convex for small $r > 0$*

The same result can be proved as in the paper [2], Lemma 1.2, part (a) which only differs from Lemma 2.2 because it includes some extra hypothesis that follows from Perelman's theorem.

3. Model metrics in tubular neighborhoods of the soul

In this section we will adapt the construction of [3] to prove Theorem A.

Let M, S be as in Theorem A and let $\nu_1(S)$ denote the unit normal bundle of the soul in M. By rescaling, we can always assume that $\exp : \nu_1(S) \times (0, 10) \to M$ is injective, $sh : D_{10} \to S$ is smooth, and D_r is convex for $r \leq 10$.

Define $\widetilde{\chi} : D_2 \to \mathbb{R}$ as

$$\widetilde{\chi}(x) = \begin{cases} \frac{1}{d(x, \partial D_2)} & x \in D_2 - D_{\frac{1}{2}} \\ \\ \chi(x) & x \in D_{\frac{1}{2}}, \end{cases}$$

where χ is any smooth extension of $d(x, \partial D_2)^{-1}$ to $D_{\frac{1}{2}}$ with $0 \leq \chi \leq \frac{2}{3}$.

Choose also a real function $g : (-\infty, 1] \to \mathbb{R}$ so that:

- g is smooth in $(-\infty, 1)$ and continuous at 1.
- $g = 0$ on $\left(-\infty, \frac{2}{3}\right)$, $g(1) = 1$.
- g' and g'' are positive on $\left(\frac{2}{3}, 1\right)$.
- $g^{-1} : [0, 1] \to \left[\frac{2}{3}, 1\right]$ has all left derivatives zero at 1.

The last condition assures that

$$G(y) = \begin{cases} g^{-1}(y) & \text{if } y \in [0, 1] \\ 1 & \text{if } y > 1 \end{cases}$$

is a smooth function, which is basically the only modification done to the argument of [3].

Finally, we define $\Gamma : D_1 \to \mathbb{R}$ by $\Gamma = g \circ \widetilde{\chi}$.

For $P = D_2 \times \mathbb{R}$, consider the subset $L = A \cup B$ where

$$\begin{aligned} A &= \text{Graph } \Gamma = \{(x, \Gamma(x)) : x \in D_1\} \\ B &= \partial D_1 \times [1, \infty) \end{aligned}$$

Obviously, $A \cap B = \partial D_1 \times \{1\}$.

Lemma 3.1. *L is a smooth convex hypersurface of P.*

Proof: To prove smoothness of $A \cap B$, define $\phi : \nu_1(S) \times (\frac{1}{2}, \frac{3}{2}) \to P$ as

$$\phi(u, s) = \left(\gamma_u \left[2 - \frac{1}{G(s)}\right], s\right).$$

Its image is clearly a neighborhood of $A \cap B$ in L. Furthermore, since the set of vectors tangent to curves of the form $\phi(u(t), 1)$ and $\phi(u, s)$ generate $T_{\phi(u,1)}L$, ϕ is an immersion.

For the convexity part, recall that for $p \in D_2$, we have an orthogonal decomposition
$$T_p D_2 = \mathcal{H}_p \oplus \mathcal{V}_p = \mathcal{H}_p \oplus \widetilde{\mathcal{V}}_p \oplus \mathbb{R}\gamma',$$
where γ is the unique minimizing geodesic joining the soul to p. This induces a decomposition of TL of the form
$$T_{(p,\Gamma(p))} L = \mathcal{H}_p \oplus \widetilde{\mathcal{V}}_p \oplus \mathbb{R}\left\{\gamma'(t) + (\Gamma \circ \gamma)'(t)\partial_t\right\}$$
for $(p, \Gamma(p)) \in A$ with $p = \gamma(t)$, and
$$T_{(p,s)} L = \mathcal{H}_p \oplus \widetilde{\mathcal{V}}_p \oplus \mathbb{R}\partial_s$$
for $(p, s) \in B$.

Suppose then that $(p, \Gamma(p)) \in A$ with $p = \gamma(t)$ where $t < 1$. For $F : P \to \mathbb{R}$ defined as $F(x, t) = t - \Gamma(x)$, we have $A = F^{-1}(0)$. Thus its second fundamental form is proportional to Hess Γ. A standard computation (see [3]) shows that
$$\text{Hess } \Gamma(v,v) = (g' \circ \tilde{\chi})\text{Hess } \tilde{\chi}(v,v) + (g'' \circ \tilde{\chi})(v(\tilde{\chi}))^2.$$
This is nonnegative because of the choice of g and the fact that $d(x, \partial D_2)$ is convex in D_2 [1].

On the other hand, if $(p, s) \in B$, $s > 1$, then
$$\nabla^P_{\partial_s} \gamma' = 0, \qquad \nabla^P_x \gamma' = \nabla^M_x \gamma'.$$
Since we chose D_1 to be convex, ∂D_1 has a positive semidefinite second fundamental form, and thus the same is true for B. By continuity, this is also valid for $A \cap B$. $\qquad\square$

Theorem A is now an easy consequence of Lemma 3.1 and the Gauss formula.

Remark 3.2. The metric in ∂D_1 at B is the same it receives from M. This is clear because the embedding of D_1 in P as the graph of Γ only modifies the original metric of D_1 in the radial direction.

4. PROOF OF THEOREMS 1.1 AND 1.2

Observe first that since the construction of the previous section does not modify the metric in a neighborhood of the soul, the set of vectors normal to S is the same in the original and the final metric. Hence we only need to check that if γ was a radial geodesic in M, then $\tilde{\gamma}(t) = (\gamma(t), \Gamma \circ \gamma(t))$ is a geodesic in the new metric once it is reparametrized by arc length. Writing $h(t) = \Gamma \circ \gamma(t)$ to simplify notation, it is easy to see that
$$\nabla^P_{\tilde{\gamma}'}\tilde{\gamma}' = h'h''\partial_t, \qquad \nabla^N_{\tilde{\gamma}'}\tilde{\gamma}' = \frac{h'^2 h''}{\|\tilde{\gamma}'\|^2}\tilde{\gamma}'.$$

Thus after reparametrizing, $\tilde{\gamma}$ becomes a geodesic for as long as it stays in A. In B it can be extended as $\gamma(1) \times [1, \infty)$, and therefore it is a geodesic for all its domain of definition.

Finally, thanks to the choices made at the start of section 3, we get that $\exp : \nu(S) \to M$ is injective, and therefore a diffeomorphism. The part about the ideal boundary follows trivially since clearly there is only one ray from each point not in S. This concludes the proof of Theorem 1.1.

Remark 4.1. The Sharafutdinov map ([6], [7]) of the metric of Theorem 1.1 is smooth, since this is the case when every normal direction from the soul is a ray.

Theorem 1.2 follows trivially by gluing together two "truncated" copies of the normal disc bundle $D(S)$ of the soul with the metrics constructed in Theorem A.

Remark 4.2. It was already known that these doubles admitted a metric with a nonnegative lower curvature bound in an Alexandrov sense: namely, we could always take the double of two sublevel sets of the Cheeger–Gromoll convex function (see [1] and [4]) However, such a metric is usually non smooth at the glued part. Furthermore, the spaces of directions at such points are not isometric to spheres except if the mentioned convex function has some extra regularity.

Remark 4.3. Theorem 1.2 shows that any open manifold with nonnegative curvature is diffeomorphic to the interior of a closed convex set in a compact manifold of nonnegative curvature. The converse is also true as follows using Theorems A, 2.2, as well as that the interior of such a set is diffeomorphic to an open tubular neighborhood of its soul.

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