PARALLEL CONNECTIONS OVER SYMMETRIC SPACES

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Abstract. Let M be a simply connected Riemannian symmetric space, with at most one flat direction. We show that every Riemannian (or unitary) vector bundle with parallel curvature over M is an associated vector bundle of a canonical principal bundle, with the connection inherited from the principal bundle. The problem of finding Riemannian (or unitary) vector bundles with parallel curvature then reduces to finding representations of the structure group of the canonical principal bundle.

This paper concerns connections on Riemannian vector bundles over simply connected symmetric spaces. Given a hypothesis about the Riemannian curvature of a symmetric space, we show that every Riemannian vector bundle with parallel curvature over that space is an associated vector bundle of a canonical principal bundle, with the connection inherited from the principal bundle. The problem of finding Riemannian vector bundles with parallel curvature then reduces to finding representations of the structure group of the canonical principal bundle. Our results apply also to unitary vector bundles, since a $\mathbb{C}^k$ bundle can be viewed as an $\mathbb{R}^{2k}$ bundle together with an additional structure.

The hypothesis we put on a symmetric space is quite mild, and is satisfied by all simply connected irreducible symmetric spaces, and by simply connected reducible symmetric spaces that do not contain an $\mathbb{R}^2$ factor. If a symmetric space does contain such a factor, then the theorem does not hold, and we give explicit counterexamples.

Recall that a Riemannian connection on a vector bundle $E \to M$ is said to be *Yang-Mills* if its curvature tensor R^E is harmonic; i.e., if $d_A^* R^E = 0$, where d_A^* is the covariant divergence operator. Yang-Mills connections have generated substantial interest, and much of our current knowledge of the topology of smooth 4-manifolds comes from their study [DK]. In the special case of the Levi-Civita connection on the tangent bundle of a Riemannian manifold, there is a stronger concept which has been of central importance in Riemannian geometry. A Riemannian manifold M, with Riemann curvature tensor R^M, is said to be *locally symmetric* if all covariant derivatives of R^M vanish. This is equivalent to $R^M(X,Y)Z$ being a parallel vector field along every path γ for any X, Y, Z parallel along γ. This concept generalizes naturally to arbitrary Riemannian vector bundles $E \to M$:

Definition. A connection on E is said to be *parallel* if $\nabla R^E = 0$; equivalently, if for any smooth path γ of M, parallel vector fields X, Y along γ, and parallel section U of E along γ, the section $R^E(X,Y)U$ is parallel along γ.

1991 *Mathematics Subject Classification.* 53C05, 53C07, 53C20, 53C35.

This condition involves, of course, the Riemannian metric on the base manifold M. (By a Riemannian vector bundle we mean a vector bundle with an inner product on each fiber. A connection on such a bundle is required to respect the inner product, meaning that the inner product is parallel. In this paper all vector bundles are assumed to be real and Riemannian, except where stated otherwise.)

Recall that a symmetric space is a Riemannian manifold M for which the geodesic symmetry at any point is a global isometry of M. Every locally symmetric space has a globally symmetric universal cover. Conversely, every symmetric space is locally symmetric. If M is a symmetric space, then M can be written as G_0/H_0, where G_0 and H_0 are groups, with $H_0 = \mathrm{Hol}(M)$, the holonomy group of M. For example, $S^n = SO(n+1)/SO(n)$ and $\mathbb{CP}^n = [U(n+1)/U(1)]/U(n)$. By lifting to the universal cover of G_0 we can write $M = G/H$, where G is simply connected and H is a cover of H_0. If M is simply connected, then H is connected. For example, $S^n = Spin(n+1)/Spin(n)$ and $\mathbb{CP}^n = SU(n+1)/U(n)$. (In the $\mathbb{CP}^n$ example, H_0 and H are both isomorphic to $U(n)$ as groups, but H is still the $n+1$-fold cover of H_0.)

Let M_p denote the tangent space of M at p. The space $\Lambda^2(M_p)$ is naturally identified with $\mathfrak{so}(M_p)$, the Lie algebra of skew-adjoint endomorphisms of M_p, by requiring that

$$u \wedge v\,(w) = \langle u, w\rangle v - \langle v, w\rangle u, \qquad u, v, w \in M_p. \tag{1.1}$$

The holonomy of M acts on Λ^2 in a natural way via $h(u \wedge v) = hu \wedge hv$, for $h \in \mathrm{Hol}(M)$. Under the above identification, the action of Hol is by conjugation on $\mathfrak{so}(M_p)$. This is also true for $\Lambda^2 E$ and the holonomy group in any vector bundle E with a Riemannian connection. Furthermore, the Riemannian curvature R^M at p is an equivariant map from $\mathfrak{so}(M_p)$ to itself. It is easy to see that in general $[\ker R^M, \operatorname{Im} R^M] \subset \ker R^M$, and therefore the same is valid for the linear subspace spanned by $[\ker R^M, \operatorname{Im} R^M]$. For our purposes, we will consistently assume that R^M satisfies a stronger condition:

Condition A: $\operatorname{span}[\ker R^M, \operatorname{Im} R^M] = \ker R^M$.

This condition is well understood; in fact, in §4 we will prove:

Theorem 1.1. *Let M be a simply connected Riemannian symmetric space. Condition A holds unless M is the product of $\mathbb{R}^2$ and another symmetric space, in which case Condition A fails.*

Put another way, Condition A holds as long as the dimension of the center of G does not exceed 1. This is discussed in detail in section 4.

Since the Levi-Civita connection on the tangent bundle of a symmetric space M is parallel, any bundle built from the tangent bundle will also admit a parallel connection, naturally induced from the Levi-Civita connection on TM. In particular $\bigotimes_k TM$ and $\Lambda^k(TM)$ admit parallel connections, as does the (principal) bundle $\mathrm{Fr}(TM)$ of orthonormal frames of TM. Sub-bundles of $\mathrm{Fr}(TM)$, and finite covers of $\mathrm{Fr}(TM)$ or of sub-bundles of $Fr(TM)$, also inherit parallel connections, whenever such subbundles are invariant by parallel transport.

If B is a principal H-bundle over M and $\rho : H \to SO(k)$ is a representation of H, then the associated vector bundle $E := B \times_\rho \mathbb{R}^k$ is the quotient of $B \times \mathbb{R}^k$ by the equivalence $(bh, v) \sim (b, \rho(h)v)$. The equivalence class of (b, v) is denoted $[(b, v)]$. The vector bundle E inherits a connection from that of B, which will be parallel if the connection on B is parallel. Thus vector bundles with parallel connections naturally arise from representations of the holonomy group of a symmetric space M, or of its finite covers.

Parallel connections are also related to group actions. Let G be a group that acts orthogonally and transitively on a vector bundle E, covering a group of isometries of M, and suppose that E has a connection that is invariant under this G-action. Then that connection is known to be parallel. Furthermore, for any fixed bundle E and G-action, Wang's theorem [W] classifies the G-invariant connections.

In this paper we show that, on symmetric spaces satisfying Condition A, these two constructions are equivalent, and are the only source of parallel connections:

Main Theorem. *Let $M = G/H$ be a simply connected symmetric space written as a canonical group quotient, with G simply connected, and suppose that the canonical metric on M satisfies Condition A. Let E be any rank-k real vector bundle with parallel connection over M. Then there exists a representation $\rho : H \to SO(k)$ such that E is isomorphic to the vector bundle $E' = G \times_\rho \mathbb{R}^k$, with the isomorphism taking the natural connection on E' to the given connection on E.*

In [L], Laquer classified left invariant affine connections on compact irreducible Riemannian symmetric spaces. In the case of Riemannian connections, this is a special case of our Main Theorem when E is the tangent bundle of M, since as we will see in section 4, all irreducible symmetric spaces satisfy condition A.

Corollary 1. *Every parallel connection over M is invariant under a G-action on the bundle that covers the natural G-action on M.*

Corollary 2. *Every unitary rank-k vector bundle with parallel connection over M is isomorphic to $G \times_{\rho'} \mathbb{C}^k$, where $\rho' : H \to U(k)$ is a rank-k unitary representation of H.*

Several authors have investigated the question of when the action of a given transitive group of isometries on the base M can be lifted to a bundle over M, see for instance [B-H]. Since such a lifted action implies the existence of a parallel connection, Corollary 1 provides a complete answer: Over a symmetric space satisfying the hypotheses of the main theorem, the vector bundles that admit lifts of transitive group actions are exactly the bundles that admit parallel connections, which are exactly the associated vector bundles of G itself. Notice, though, that G is usually a cover of the isometry group G_0 of the base, and that the action of the latter will not, in general, lift to the bundle: For example, the action of $SO(5)$ on the 4-sphere does not lift to the spinor bundle, even though that of $Spin(5)$ does.

The main theorem tells us that a local condition on a bundle with connection, namely that its curvature be parallel, implies a rigid global structure. However, for this to be true we need assumptions about the underlying manifold, namely that it be a simply connected symmetric space whose metric connection satisfies Condition A. Without these conditions the conclusions of the theorem can be false, as the following counterexamples show.

Let $M = T^2 = G/H$, where $G = \mathbb{R}^2$ and $H = \mathbb{Z}^2$. M is not simply connected and Condition A fails, as $\ker R^M$ is all of $\Lambda^2(M_p)$ while $\operatorname{Im} R^M$ is zero. Since

$R^M = 0$, any bundle constructed from a representation of H is necessarily flat (although possibly with nontrivial holonomy). However, there exist bundles and connections over T^2 with parallel nonzero curvature. For example, consider the trivial complex line bundle over $\mathbb{R}^2$. Taking the quotient of this by the relation $f(x+n, y+m) = \exp(2\pi i n y) f(x,y)$ gives a complex line bundle of Chern class $+1$ over T^2 that we denote E. E admits many connections of constant nonzero curvature; one has connection form $A = -2\pi i x dy$. Not only does this line bundle not come from a representation of H, but the connection is not invariant under translation. Indeed, the curvature itself measures the extent to which holonomy changes when a homologically nontrivial loop is translated. There is a 2-parameter family of parallel connections on E, indexed by T^2 itself, and translation takes one such connection into another.

Now let $M = G = \mathbb{R}^2$, with H trivial. M is simply connected but does not satisfy Condition A. Once again, any bundle built from the tangent bundle with its canonical structure has a flat connection, but there are connections over $\mathbb{R}^2$ that are parallel but not flat. For example, one can again look at a complex line bundle with connection form $A = -2\pi i x dy$ and constant curvature $-2\pi i dx \wedge dy$. Unlike the T^2 example, this connection is invariant under translation, so Wang's theorem does apply.

The proof of the main theorem is based on the following observations. Let $M = G/H$ be a simply connected symmetric space, with G simply connected. H is a covering space of $\mathrm{Hol}(M)$, the holonomy group of M. Let $\pi : H \to \mathrm{Hol}(M)$ be the covering map, so that $TM = G \times_\pi \mathbb{R}^n$. Suppose $E = G \times_\rho R^k$, for a representation $\rho : H \to SO(k)$, and endow E with the connection induced by the one on G. Then the holonomy around a loop γ for the three bundles TM, G, and E are related by the commutative diagram

$$\begin{tikzcd} & \gamma \arrow[dl, "\mathbb{P}^M_\gamma"'] \arrow[d, "\mathbb{P}^G_\gamma"] \arrow[dr, "\mathbb{P}^E_\gamma"] & \\ \mathrm{Hol}(M_p) & H \arrow[l, "\pi"'] \arrow[r, "\rho"] & \mathrm{Hol}(E_p), \end{tikzcd} \tag{1.2}$$

where $\mathbb{P}^E_\gamma$ denotes parallel translation in E along the curve γ in M, etc. One cannot compute $\mathbb{P}^E_\gamma$ directly from $\mathbb{P}^{TM}_\gamma$, since π is not 1–1. However, looking at infinitesimal loops we have the commutative diagram

$$\begin{tikzcd} & \Lambda^2(M_p) \arrow[dl, "R^M"'] \arrow[d, "R^G"] \arrow[dr, "R^E"] & \\ \mathfrak{hol}(M_p) & \mathfrak{h} \arrow[l, "\hat\pi"'] \arrow[r, "\hat\rho"] & \mathfrak{hol}(E_p), \end{tikzcd} \tag{1.3}$$

where $\hat\pi : \mathfrak{h} \to \mathfrak{so}(M_p)$ and $\hat\rho : \mathfrak{h} \to \mathfrak{so}(E_p)$ are the Lie algebra homomorphisms corresponding to the group homomorphisms π and ρ. Since π is a covering map, $\hat\pi$ is an algebra isomorphism, and we have

$$R^E = \hat\rho \circ (\hat\pi)^{-1} \circ R^M. \tag{1.4}$$

Since R^M is parallel and $\hat\rho$ and $\hat\pi$ are fixed, R^E is parallel. Notice that $\ker R^M \subset \ker R^E$, so that $R^E \circ (R^M)^{-1}$ is well-defined (although $(R^M)^{-1}$ may not be), and is a Lie algebra homomorphism from $\mathfrak{hol}(M_p)$ to $\mathfrak{hol}(E_p)$. Indeed, it is equal to $\hat{\rho} \circ \hat{\pi}^{-1}$.

Now suppose that E is a vector bundle over $M = G/H$ with parallel curvature. To show that E is an associated vector bundle of G, we reverse the above argument. First we show that $\ker R^M \subset \ker R^E$. This is the step where condition A is needed. Then, with $\mathfrak{h}$ identified with $\mathfrak{hol}(M_p)$, we define $\hat{\rho} = R^E \circ (R^M)^{-1}$ and show that it is a Lie algebra homomorphism from $\mathfrak{h} = \mathfrak{hol}(M_p)$ to $\mathfrak{hol}(E_p)$. Next, we exponentiate this algebra homomorphism to get a group homomorphism $\rho : H \to \mathrm{Hol}(E_p) \subset SO(k)$. This is somewhat delicate, since H may not be simply connected. To get around this, we extend $\hat{\rho}$ to an algebra homomorphism from $\mathfrak{g}$ to the Lie algebra of the extended gauge group, exponentiate that to a group homomorphism from G to the extended gauge group, and then restrict the group homomorphism to H. This is the step that requires the simple connectivity of G.

Having reconstructed ρ from the curvature of E, we define $E' = G \times_\rho \mathbb{R}^k$. Identifying the fibers of E and E' at our base point p, we note that the curvatures of E and E' are equal, and indeed the two vector bundles have the same holonomy around any given loop. We then use parallel transport to identify the fibers of E and E' at every point, completing the proof.

In §2 we establish the local properties of R^M and R^E, where E is any vector bundle with parallel curvature. In particular, we show (Theorem 2.4) that $R^E \circ (R^M)^{-1}$ is a Lie algebra homomorphism. In §3 we construct the bundle E' and show that it is isomorphic to E. In §4 we consider Condition A, and show that it holds on all simply connected symmetric spaces that do not have an $\mathbb{R}^2$ factor. Finally, in §5 we discuss some of the implications of the Main Theorem.

2. Local properties of parallel connections

We briefly recall some basic facts about holonomy and curvature in bundles. The reader is referred to [P] for further details and other facts that will be freely used here. It will often be convenient to work with principal bundles instead of vector bundles. Recall that an oriented Riemannian vector bundle E of rank k over M is isomorphic to $\mathrm{Fr}(E) \times_{\rho_0} \mathbb{R}^k$, where $\mathrm{Fr}(E)$ is the principal $SO(k)$ bundle of oriented orthonormal frames of E and ρ_0 is the fundamental representation of $SO(k)$. A sub-bundle of $\mathrm{Fr}(E)$ is obtained by taking all parallel translates of a given frame. This is a principal bundle over M, whose fiber is the holonomy group $\mathrm{Hol}(E)$ of E.

A connection on E induces one on the corresponding principal bundle $\mathrm{Fr}(E)$, on all sub-bundles of $\mathrm{Fr}(E)$, and on all covers of sub-bundles of $\mathrm{Fr}(E)$. Conversely, if B is a principal H-bundle over M and ρ is an orthogonal k dimensional representation of H, then a connection on B induces one on the vector bundle $E = B \times_\rho \mathbb{R}^k$, which will be parallel if the connection on B is parallel.

The curvature tensor R^E of a connection on E is a bundle map $\Lambda^2 TM \to \mathfrak{hol}(E) \subset \mathfrak{so}(E)$. A result of Ambrose and Singer says that the fiber of $\mathfrak{hol}(E)$ over $p \in M$ is spanned by the parallel translates of all curvature transformations $R^E(x \wedge y)$, $x \wedge y \in \Lambda^2 M_q$, $q \in M$, along smooth curves from q to p [P]. When the connection is parallel, the curvature is invariant under parallel translation, and we obtain:

Lemma 2.1. *Let E be a vector bundle with parallel connection over a manifold*

M, and let p be a point on M, then
(1) *$R^E : \Lambda^2(M_p) \to \mathfrak{hol}(E_p)$ is onto.*
(2) *If γ is a curve in M with $\gamma(0) = \gamma(1) = p$, and $A \in \Lambda^2(M_p) = \mathfrak{so}(M_p)$, then*

$$R^E \circ \mathbb{P}_\gamma^{\mathfrak{so}(TM)}(A) = R^E(\mathbb{P}_\gamma^{TM} A\, \mathbb{P}_{-\gamma}^{TM}) = \mathbb{P}_\gamma^E R^E(A)\, \mathbb{P}_{-\gamma}^E = \mathbb{P}_\gamma^{\mathfrak{so}(E)} \circ R^E(A).$$

(3) *For $A, B \in \Lambda^2(M_p)$, $R^E[R^M A, B] = [R^E A, R^E B]$.*

Proof. (1) and (2) are clear from the above discussion, together with the fact that the holonomy acts by conjugation on both $\mathfrak{so}(TM)$ and $\mathfrak{so}(E)$. It suffices to establish (3) for decomposable elements $A = x \wedge y$, $B = z \wedge w$, $x, y, z, w \in M_p$. So consider vector fields X, Y on M such that $X_p = x$, $Y_p = y$, and $[X, Y] = 0$ in a neighborhood of p. If $\{\phi_s\}$ and $\{\psi_s\}$ denote the local 1-parameter groups of X and Y, then for small t, the product γ_t of the curves $s \mapsto \phi_{\sqrt{s}}(p)$, $s \mapsto \psi_{\sqrt{s}}\phi_{\sqrt{t}}(p)$, $s \mapsto \phi_{-\sqrt{s}}\psi_{\sqrt{t}}\phi_{\sqrt{t}}(p)$, and $s \mapsto \psi_{-\sqrt{s}}\phi_{-\sqrt{t}}\psi_{\sqrt{t}}\phi_{\sqrt{t}}(p)$, $0 \le s \le t$, is a piecewise smooth loop at p, such that

$$\frac{d}{dt}\Big|_0 \mathbb{P}_{\gamma_t}^{TM} = R^M(A), \qquad \text{and } \frac{d}{dt}\Big|_0 \mathbb{P}_{\gamma_t}^{E} = R^E(A), \tag{2.2}$$

cf. also [P]. If β denotes the curve $t \mapsto R^E \mathbb{P}_{\gamma_t}^{\mathfrak{so}(TM)}(B)$ in $\Lambda^2(M_p)$, we have that $\beta(t) = R^E \mathbb{P}_{\gamma_t}^{TM} B (\mathbb{P}_{\gamma_t}^{TM})^{-1}$, and by the first identity in (2.2), $\dot\beta(0) = R^E[R^M A, B]$. On the other hand, $\beta(t) = \mathbb{P}_{\gamma_t}^{\mathfrak{so}(E)} R^E(B) = \mathbb{P}_{\gamma_t}^E R^E(B)(\mathbb{P}_{\gamma_t}^E)^{-1}$, and the second identity in (2.2) yields $\dot\beta(0) = [R^E(A), R^E(B)]$, which establishes the claim. □

We now consider the case when the base is a locally symmetric space. The curvature tensor of the Levi-Civita connection then satisfies the following property:

Proposition 2.2. *Let M be a locally symmetric space, $E_\lambda \subset \mathfrak{so}(M_p)$ the λ-eigenspace of the curvature operator $R^M : \mathfrak{so}(M_p) \to \mathfrak{so}(M_p)$. If $\lambda \neq 0$, then E_λ is an ideal of the algebra bundle $\mathfrak{hol}(M_p) \subset \mathfrak{so}(M_p)$. Thus, $\mathfrak{hol}(M_p)$ is a Whitney sum $\oplus E_i$ of parallel eigenspace bundles, which are fiberwise ideals.*

Remark. Adding the 0-eigenbundle of the curvature decomposes $\mathfrak{so}(TM)$ as a Whitney sum of parallel subbundles. In general, however, the kernel of R^M is not closed under the Lie bracket.

Proof of Proposition 2.2. Let A and B denote λ-eigenvectors. Then Lemma 2.1(3) with $E = TM$ implies

$$\lambda R^M[A, B] = R^M[R^M A, B] = [R^M A, R^M B] = \lambda^2 [A, B], \tag{2.3}$$

so that E_λ is a subbalgebra bundle if $\lambda \neq 0$. Similarly, for nonzero $\mu \neq \lambda$, and λ-eigenvector A, μ-eigenvector B, we have

$$\begin{aligned} \lambda R^M[A, B] = R^M[R^M A, B] =& [R^M A, R^M B] \\ =& R^M[A, R^M B] = \mu R^M[A, B], \end{aligned} \tag{2.4}$$

where the identity $[R^M A, R^M B] = R^M[A, R^M B]$ follows by interchanging A and B in Lemma 2.1(3). Thus, $[A, B] \in \ker R^M \cap \mathfrak{hol}(M_p) = \{0\}$. □

Lemma 2.3. *If M is a locally symmetric space satisfying Condition A and R^E is the curvature of a parallel connection on a bundle E over M, then* $\ker R^M \subset \ker R^E$.

Proof. Suppose that $A \in E_\lambda$ and that $B \in \ker R^M$, with $\lambda \neq 0$. Then

$$\lambda R^E[A, B] = R^E[R^M A, B] = [R^E A, R^E B] = R^E[A, R^M B] = 0. \tag{2.5}$$

Thus $R^E[E_\lambda, \ker R^M] = 0$. Summing over λ we have that $R^E[\operatorname{Im} R^M, \ker R^M] = 0$. But by Condition A, $\operatorname{span}[\operatorname{Im} R^M, \ker R^M] = \ker R^M$, so $R^E(\ker R^M) = 0$. □

Since R^M is onto $\mathfrak{hol}(M_p)$, and $R^E(\ker R^M) = 0$, there exists a map $\hat\rho : \mathfrak{hol}(M_p) \to \mathfrak{so}(E_p)$ such that $R^E = \hat\rho \circ R^M$. Formally we write $\hat\rho = R^E \circ (R^M)^{-1}$.

Theorem 2.4. *Let M denote a symmetric space, R^E the curvature at a point p of a parallel connection on a bundle E over M. Then $\hat\rho := R^E \circ (R^M)^{-1} : \mathfrak{hol}(M_p) \to \mathfrak{so}(E_p)$ is a Lie algebra bundle homomorphism whose image is all of $\mathfrak{hol}(E_p)$.*

Proof. Since $R^E = \hat\rho \circ R^M$, the image of $\hat\rho$ is the image of R^E, namely $\mathfrak{hol}(E_p)$. To see that $\hat\rho$ is a Lie algebra homomorphism, we need to check that $\hat\rho[A, B] = [\hat\rho(A), \hat\rho(B)]$ when A and B belong to a common eigenspace E_λ of R^M, and when $A \in E_\mu$, $B \in E_\lambda$, $\lambda \neq \mu$. Let $b = \frac{1}{\lambda} B \in (R^M)^{-1} B$. In the former case,

$$[\hat\rho(A), \hat\rho(B)] = R^E[A, b] = \tfrac{1}{\lambda} R^E[A, B] = \hat\rho[A, B], \tag{2.6}$$

since $[A, B]$ belongs to E_λ. In the latter case, we have

$$[\hat\rho(A), \hat\rho(B)] = R^E[A, b] = \tfrac{1}{\lambda} R^E[A, B]. \tag{2.7}$$

After interchanging the roles of A and B, we see that the left side of (2.7) is also equal to $(1/\mu) R^E[A, B]$, and therefore vanishes. However, $[A, B] = 0$ by Proposition 2.2, so $[\hat\rho(A), \hat\rho(B)] = 0 = \hat\rho[A, B]$. □

3. The global structure of E

In this section we complete the proof of the main theorem, using the local structure of §2 to construct a bundle E', and then showing that E and E' are isomorphic with the isomorphism respecting the connections. Our base manifold is a Riemannian symmetric space M. Recall that $M = G_0/H_0$, where G_0 is the largest connected group of isometries of M, and H_0 is the isotropy group at some fixed point p. If $\lambda : H_0 \to SO(M_p)$ denotes the (faithful) linear isotropy representation $\lambda(h) = h_{*p}$, then the tangent bundle TM is isomorphic to $G_0 \times_\lambda M_p$ via $[(g, u)] \mapsto g_{*p} u$, and the holonomy group of M at p is $\lambda(H_0)$. An equivalent description of the tangent bundle is obtained by considering the involutive automorphism σ of G_0 given by $\sigma(g) = s_p \circ g \circ s_p^{-1}$, where s_p denotes the geodesic symmetry at p. Its derivative at the identity squares to 1 and decomposes the Lie algebra $\mathfrak{g}$ into a sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ of the plus and minus 1 eigenspaces of σ_{*e}. The derivative of the projection $\pi : G_0 \to M$ (which sends g to $g(p)$) has kernel $\mathfrak{h}$ and thus identifies $\mathfrak{m}$ with the tangent space of M at p. A straightforward computation shows that for $h \in H_0$, the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{Ad_h} & \mathfrak{g} \\ {\scriptstyle \pi_{*e}}\downarrow & & \downarrow{\scriptstyle \pi_{*e}} \\ M_p & \xrightarrow{h_{*p}} & M_p \end{array} \tag{3.1}$$

commutes, so that TM is also identified with $G_0 \times_{Ad} \mathfrak{m}$, with Ad denoting the adjoint representation of H_0.

We work throughout with $M = G/H$ where G is the universal cover of G_0, and identify the Lie algebra $\mathfrak{h}$ of H with $\mathfrak{hol}(M_p)$. Our first task is to promote the Lie algebra homomorphism $\hat{\rho} : \mathfrak{hol}(M_p) \to \mathfrak{so}(E_p)$ into a group homomorphism $\rho : H \to SO(E_p)$. Since in general H is not simply connected, this promotion is not automatic. We must extend $\hat{\rho}$ to a homomorphism from $\mathfrak{g}$ to an appropriate Lie algebra, promote that extension to a group homomorphism, and then restrict the homomorphism to H.

Recall that if B is a principal K-bundle over M, the *gauge group* $\mathcal{G}_B$ of B is the group of K-equivariant diffeomorphisms of B that cover the identity on M, cf. e.g. [BL]. It is canonically identified with the space of sections of the bundle of groups $B \times_c K$, where c denotes conjugation; i.e., $(bk_1, k_2) \sim (b, k_1 k_2 k_1^{-1})$. The gauge algebra $\mathfrak{G}_B$ of B is similarly defined as the Lie algebra of sections of the Lie algebra bundle $B \times_{\mathrm{Ad}} \mathfrak{K}$, and there is a fiberwise exponential map

$$\exp : \mathfrak{G}_B \to \mathcal{G}_B. \tag{3.2}$$

In the case of a Riemannian vector bundle E, the gauge group $\mathcal{G}_E$ and algebra $\mathfrak{G}_E$ are defined to be the gauge group and algebra of the principal bundle $\mathrm{Fr}(E)$ of oriented orthonormal frames. Equivalently, $\mathcal{G}_E$ is the group of orthogonal bundle maps $E \to E$ that cover the identity on M. An element of $\mathfrak{G}_E$ can also be viewed as a (vertical) vector field on E.

Let $M = G/H$ be a symmetric space with G simply connected, and E a vector bundle over M. Define the *enlarged gauge group* of E to be the group $\tilde{\mathcal{G}}_E$ of all orthogonal bundle maps $E \to E$ that cover the natural action of G on M. The projection $\alpha : E \to M$ determines an exact sequence

$$1 \to \mathcal{G}_E \to \tilde{\mathcal{G}}_E \xrightarrow{\alpha} G \to 1, \tag{3.3}$$

which, when differentiated at the identity, yields the exact sequence of vector spaces

$$0 \to \mathfrak{G}_E \to \tilde{\mathfrak{G}}_E \xrightarrow{\alpha_*} \mathfrak{g} \to 0. \tag{3.4}$$

A connection on E determines a splitting of the sequence (3.4). An element $X \in \tilde{\mathfrak{G}}_E$, viewed as a vector field on E, splits accordingly as $X = X^h + X^v$, where X^v is a section of $\mathfrak{hol}(E)$, and $\alpha_* X^h = \alpha_* X \in \mathfrak{g}$.

When $E = TM$, any $X \in \mathfrak{g}$ induces a vector field $\tilde{X} \in \tilde{\mathfrak{G}}_M$ on TM: Identify X with the Killing field on M induced by the G-action. If ϕ_t denotes the one-parameter group generated by X, then ϕ_{t*} is a bundle map that covers ϕ_t, and we set $\tilde{X} := \frac{d}{dt}\big|_0 (\phi_{t*})$. By standard properties of connections (see e.g. [P]), one has

$$[\tilde{X}^v, \tilde{Y}^h] = 0, \quad [\tilde{X}^h, \tilde{Y}^h]^v = R^M(X,Y), \qquad X, Y \in \mathfrak{g}. \tag{3.5}$$

Define a map $\phi : \mathfrak{g} \to \tilde{\mathfrak{G}}_E$ by $\phi(X) = X^h + \hat{\rho}(\tilde{X}^v)$, where $\hat{\rho} = R^E \circ (R^M)^{-1} : \mathfrak{hol}(TM) \to \mathfrak{hol}(E)$ is the Lie algebra bundle homomorphism from Theorem 2.4, and X^h is the horizontal lift of X (viewed as a Killing field on M) to E.

Lemma 3.1. *$\phi : \mathfrak{g} \to \tilde{\mathfrak{G}}_E$ is a Lie algebra homomorphism.*

Proof. We compute

$$\begin{aligned}\widetilde{[X,Y]} &= [\tilde{X},\tilde{Y}] = [\tilde{X},\tilde{Y}]^h + [\tilde{X},\tilde{Y}]^v = [\tilde{X}^h,\tilde{Y}^h]^h + [\tilde{X},\tilde{Y}]^v \\ &= [\tilde{X}^h,\tilde{Y}^h]^h + [\tilde{X}^h,\tilde{Y}^h]^v + [\tilde{X}^v,\tilde{Y}^v] \\ &= [\tilde{X}^h,\tilde{Y}^h]^h + R^M(X\wedge Y) + [\tilde{X}^v,\tilde{Y}^v].\end{aligned} \tag{3.6}$$

Thus,

$$\begin{aligned}\phi[X,Y] &= [X^h,Y^h]^h + R^E(X\wedge Y) + \hat{\rho}[\tilde{X}^v,\tilde{Y}^v] \\ &= [X^h,Y^h]^h + R^E(X\wedge Y) + [\hat{\rho}(\tilde{X}^v),\hat{\rho}(\tilde{Y}^v)],\end{aligned} \tag{3.7}$$

since $\hat{\rho}$ is an algebra homomorphism. On the other hand,

$$\begin{aligned}[\phi X,\phi Y] &= [X^h + \hat{\rho}(\tilde{X}^v), Y^h + \hat{\rho}(\tilde{Y}^v)] \\ &= [X^h,Y^h]^h + [X^h,Y^h]^v + [\hat{\rho}(\tilde{X}^v),\hat{\rho}(\tilde{Y}^v)] \\ &= [X^h,Y^h]^h + R^E(X\wedge Y) + [\hat{\rho}(\tilde{X}^v),\hat{\rho}(\tilde{Y}^v)]. \quad \square\end{aligned} \tag{3.8}$$

Proposition 3.2. *Let p denote the coset $eH \in G/H = M$. There exists a group homomorphism $\rho : H \to SO(E_p)$ whose derivative at the identity is $\hat{\rho}|_p : \mathfrak{h} \to \mathfrak{so}(E_p)$.*

Proof. Since G is simply connected, the algebra homomorphism ϕ can be promoted to a group homomorphism $\Phi : G \to \tilde{\mathcal{G}}_E$. Restricting Φ to H we get a map $H \to \tilde{\mathcal{G}}_E$. Now, if $X \in \mathfrak{h}$, then X, viewed as a Killing field on M, has a zero at p because H_0 is the isotropy group of p. Therefore $\Phi(\exp(tX))$, and generally $\Phi(h)$ for any $h \in H$, preserves the fiber E_p. We define $\rho(h)$ to be the restriction of $\Phi(h)$ to that fiber. ρ is a group homomorphism since Φ is, and the derivative of ρ is $ev_p \circ \phi$, where ev_p is evaluation at the base point p. But for $X \in \mathfrak{h}$,

$$ev_p \circ \phi(X) = \tilde{X}^h(p) + \hat{\rho}(\tilde{X}^v)(p) = 0 + \hat{\rho}_p(X). \quad \square \tag{3.9}$$

We now define the bundle E' to be $G \times_\rho E_p$. E' inherits a natural parallel connection from G. The Main Theorem boils down to

Theorem 3.3. *E and E' are isomorphic vector bundles, with the isomorphism preserving the connection.*

Proof. The fibers of E and E' over p are canonically identified. Denote the isomorphism $I_p : E'_p \to E_p$. I_p is an element of $Hom(E', E)$. We must show that I_p can be extended in a parallel fashion to all of M.

Let F be the vector bundle $Hom(E', E)$. F inherits a parallel structure from those of E' and E. In particular, the holonomy of F is generated by the curvature of F at p, and this curvature is constructed from the curvatures of E' and E. By (1.4) the curvature of E' at p is

$$R_p^{E'} = \hat{\rho}_p \circ R_p^M = R_p^E. \tag{3.10}$$

Thus, for any $A \in \Lambda^2(M_p)$, $R^{E'}(A) = R^E(A)$, and so

$$R^F(A)(I_p) = 0. \tag{3.11}$$

Since I_p is in the kernel of $R^F(A)$ for every A, $\mathrm{Hol}(F)$ acts trivially on I_p, so I_p can be extended, by parallel transport, to a parallel section of F. This section is our global isomorphism between E' and E. □

This completes the proof of the main theorem. Corollary 1 is an obvious consequence. For Corollary 2, note that a rank-k complex vector bundle is just a real rank-$2k$ vector bundle with an additional feature, namely the complex structure J. If E is a unitary bundle with parallel connection, then the holonomy of E commutes with J, and so $\hat{\rho}_p$ commutes with J. Thus ρ is actually a rank-k unitary representation of H, not just a rank $2k$ orthogonal representation.

In the proof of the Main Theorem we have constructed the representation ρ as linear operators on the fixed vector space E_p rather than as matrices acting on $\mathbb{R}^k$, and the definition of E' is similar. The difference between working with E_p and $\mathbb{R}^k$ is of course nothing more than a choice of orthonormal basis for E_p.

4. The curvature condition

In this section we prove Theorem 1.1, showing that Condition A holds on all simply connected symmetric spaces that do not have an $\mathbb{R}^2$ factor. We begin with a special case:

Theorem 4.1. *Condition A applies on all simply connected irreducible Riemannian symmetric spaces.*

Proof. Let M be such a space of dimension n, let $p \in M$ be a point, and suppose that $W := \mathrm{span}[\ker R_p^M, \mathrm{Im}\, R_p^M] \subset \mathfrak{so}(M_p)$ is a proper subspace of $V := \ker R_p^M$. Let A be an element of V, orthogonal to W. A is then orthogonal to all of $[\mathfrak{so}(M_p), \mathrm{Im}\, R^M]$, since $[\mathrm{Im}\, R^M, \mathrm{Im}\, R^M]$ is contained in $\mathrm{Im}\, R^M$, and so is orthogonal to A. Since Hol acts orthogonally on $\mathfrak{so}(M_p)$, A is invariant under Hol.

Since A is a skew endomorphism of M_p, it can be expressed as a block diagonal $n \times n$ matrix, where each block is either zero or a skew $2{\times}2$ matrix $L_\lambda = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$, with $\lambda > 0$. For $\lambda \neq 0$, let $E_\lambda \subset M_p$ be the direct sum of the 2-planes with block L_λ. Since A is invariant under Hol, so is E_λ, and so is the orthogonal complement of E_λ.

By de Rham's holonomy theorem (see e.g. [B]), M is then the product of spaces M_1 and M_2 with tangent spaces E_λ and $E_\lambda^\perp$. Since M is irreducible, one of these spaces must be trivial. Since $A \neq 0$, we can find a λ such that the vanishing subspace is $E_\lambda^\perp$, so E_λ is all of M_p. But then A is λ times a complex structure on M_p. Furthermore, since Hol acts trivially on A, $\lambda^{-1}A$ can be extended to a global integrable complex structure. M is thus a complex symmetric space, and Hol is a subgroup of $U(k)$, with $2k = n$.

Since $A \in \ker R^M$, and since R^M is a self-adjoint map from $\mathfrak{so}(TM)$ to itself, A is orthogonal to the image of R^M. Thus, for any $B \in \Lambda^2(TM)$, $0 = \langle A, R^M(B)\rangle = \frac{1}{2}\,\mathrm{tr}(A^t \circ R^M(B))$. But A is a (multiple of a) complex structure, so that each

$R^M(B) \in \mathfrak{u}(k)$ has complex trace 0, and the holonomy of M lies in $SU(k)$, not just in $U(k)$. The determinant of the tangent bundle (viewed as a complex k-dimensional vector bundle) is trivial, with a flat connection. In other words, M is a Calabi-Yau manifold.

Calabi-Yau manifolds have Ricci-flat metrics (see Ch.9, Theorem 4.6 in [K-N]). However, Ricci-flat homogeneous spaces (and in particular Ricci-flat symmetric spaces) are flat (see section 7.61 in [B]). Since M is simply connected, it must equal $\mathbb{C}^k$. However, $\mathbb{C}^k = \mathbb{R}^k \times \mathbb{R}^k$ is reducible, and we have a contradiction. □

Remark. The simple connectivity assumption in the previous theorem was necessary to ensure that Hol is connected. However, a version of the theorem holds in the non-simply connected case. If M is not simply connected, we work on the universal cover $\tilde{M}$. If $\tilde{M}$ is irreducible, then Condition A holds on $\tilde{M}$, and therefore holds on M.

Remark. Condition A is considerably weaker than the nonexistence of a parallel 2-form, which is a common condition applied to symmetric spaces. We have just seen that a parallel section of $\mathfrak{so}(TM)$ (or equivalently a parallel 2-form) on an irreducible symmetric space must be a complex structure, but many irreducible symmetric spaces are indeed complex, and their complex structures are indeed parallel. Condition A allows such structures, just not in the kernel of R^M.

To understand the general case, we will examine first how Condition A behaves under products with irreducible symmetric spaces:

Lemma 4.2. *Let $M \neq \mathbb{R}$ be a simply connected irreducible symmetric space and N a simply connected symmetric space, not necessarily irreducible. Then $M \times N$ satisfies condition A if and only if N does.*

Proof. The curvature of $M \times N$ is the direct sum of the curvature of M and the curvature of N, so the kernel of $R^{M\times N}$ at a point (p,q) is the kernel of R^M at p plus the kernel of R^N at q plus the linear subspace spanned by the wedge of M_p and N_q. Similarly, the image of $R^{M\times N}$ is the direct sum of the images of R^M and R^N under the diagonal inclusion $\mathfrak{so}(M_p) \oplus \mathfrak{so}(N_q) \subset \mathfrak{so}(M_p \oplus N_q)$.

It is clear then that $[\ker R^M, \operatorname{Im} R^{M\times N}] = [\ker R^M, \operatorname{Im} R^M]$ which spans $\ker R^M$. Similarly, $[\ker R^N, \operatorname{Im} R^{M\times N}] = [\ker R^N, \operatorname{Im} R^N]$. Since M is irreducible and $M \neq \mathbb{R}$, it is easy to see that $\{R^M(\mathfrak{so}(M_p))M_p\}$ spans all of M_p. Thus for any $x \in M_p$ we can find $a_i \in \mathfrak{so}(M_p)$ and $x_i \in M_p$ with $\sum_i R^M(a_i)x_i = x$. This implies that

$$x \wedge y = \sum_i R^M(a_i)x_i \wedge y = -\sum_i [x_i \wedge y, R^M(a_i)], \tag{4.1}$$

and therefore $\operatorname{span}(M_p \wedge N_q) = \operatorname{span}([M_p \wedge N_q, \operatorname{Im} R^M])$. It follows from this that the span of $[\ker R^{M\times N}, \operatorname{Im} R^{M\times N}]$ equals $\ker R^{M\times N}$ if and only if $\operatorname{span}([\ker R^N, \operatorname{Im} R^N])$ equals $\ker R^N$, giving us the conclusion of the lemma. □

Theorem 1.1. *Let M be a simply connected symmetric space with de Rham decomposition given by $M = \mathbb{R}^{n_0} \times M_1 \times \cdots \times M_l$. Then M satisfies condition A if and only if $n_0 \leq 1$. Equivalently, M satisfies condition A if and only if the dimension of the center of G does not exceed 1.*

Proof. From the last two lemmas it is clear that $M_1 \times \cdots \times M_l$ satisfies condition A. On the other hand, the condition holds on $\mathbb{R}^{n_0}$ precisely when $n_0 \leq 1$. Thus, the first part of the theorem follows trivially from Lemma 4.2. The statement about the center follows, since the center of G generates the flat factors in the de Rham decomposition. □

5. Examples and applications

A. Bundles over spheres.

The sphere S^n is a symmetric space with $G_0 = SO(n+1)$, $H_0 = SO(n)$, $G = Spin(n+1)$ and $H = Spin(n)$. By the main theorem, a parallel vector bundle over S^n corresponds to a representation of $Spin(n)$, and is thus the Whitney sum of vector bundles corresponding to irreducible representations of $Spin(n)$. Our analysis simplifies to listing the irreducible representations of $Spin(n)$.

If $n = 2$, then $SO(2)$ and $Spin(2)$ are both circle groups. We identify $SO(2)$ with $\mathbb{R}/(2\pi\mathbb{Z})$ and $Spin(2)$ with $\mathbb{R}/(4\pi\mathbb{Z})$, so that the Lie algebras of $SO(2)$ and $Spin(2)$ are naturally identified. Besides the trivial 1-dimensional representation, the irreducible representations of $Spin(2)$ are all 2-dimensional, namely

$$\rho_k(x) = \begin{pmatrix} \cos(kx/2) & \sin(kx/2) \\ -\sin(kx/2) & \cos(kx/2) \end{pmatrix}. \tag{5.1}$$

The corresponding vector bundle is a plane bundle of Euler class k, or equivalently a complex line bundle of first Chern class k. Important special cases are the Hopf fibrations $k = \pm 1$ and the tangent bundle $k = 2$.

Before analyzing the higher dimensional cases, we recall some general concepts and facts about orthogonal representations, cf. also [BD]. If V is an inner product space, then an irreducible orthogonal representation $\rho : H \to SO(V)$ is either unitary for some complex structure, or else its complexification is irreducible, and the latter is of *real type*; i.e., it admits a *real structure*, or in other words, a complex conjugate linear map $J : \mathbb{C} \otimes V \to \mathbb{C} \otimes V$ that squares to the identity and commutes with the action of H. Its restriction to the $+1$ eigenspace of J is the original real representation. A unitary representation is said to be of *quaternionic type* if it admits a *quaternionic structure*, or in other words, a complex conjugate linear map that squares to minus the identity and commutes with H.

If $n = 3$, then $Spin(3) = SU(2)$, and the irreducible representations are well known, see for example [BD]. The fundamental representation, corresponding to matrix multiplication, has complex dimension 2, and has a quaternionic structure given by $J(z_1, z_2) = (-\bar{z}_2, \bar{z}_1)$ for $(z_1, z_2) \in \mathbb{C}^2$. All other complex irreducible representations are obtained from symmetric products of the fundamental representation. Explicitly, let V_k denote the space of homogeneous polynomials p of degree k in two complex variables,

$$p(z_1, z_2) = \sum_{r=0}^{k} a_r z_1^r z_2^{k-r}, \tag{5.2}$$

of complex dimension $k+1$. The action of $SU(2)$ on V_k is given by

$$gp(z) = p(gz), \qquad g \in SU(2), \quad z \in \mathbb{C}^2, \quad p \in V_k. \tag{5.3}$$

In particular, the structure map J on $\mathbb{C}^2 = V_1$ induces a structure map J on V_k satisfying $J^2 p = (-1)^k p$. Thus, the odd-dimensional representations have an underlying real structure, while the (complex) even-dimensional representations have a quaternionic structure. Complex irreducible representations of odd dimension $2\ell+1$ therefore correspond to real irreducible representations of dimension $2\ell+1$, while complex irreducible representations of dimension 2ℓ correspond to real irreducible representations of dimension 4ℓ. The odd-dimensional irreducible representations also are representations of $SO(3)$ (since $-1 \in SU(2)$ acts trivially on V_{2l}, so that the action of $SU(2)$ induces one of $SO(3) = SU(2)/\pm 1$), whereas the even-dimensional irreducible representations are not.

Over S^3, therefore, there exists a unique irreducible parallel unitary bundle in each (complex) dimension. Irreducible Riemannian parallel vector bundles exist in odd dimensions, and in dimensions divisible by 4. The first few examples are:

1. The trivial bundle, of real dimension 1.
2. The tangent bundle, of dimension 3. This corresponds to the fundamental representation of $SO(3)$. $\Lambda^2(TM)$ is also 3-dimensional, and is isomorphic to TM via the Hodge star operator.
3. The spinor bundle, of complex dimension 2 and real dimension 4, corresponding to the fundamental representation of $SU(2)$.
4. The quadrupole bundle of traceless symmetric endomorphisms from TM to itself, of real dimension 5.

If $n = 4$, then $Spin(n) = SU(2) \times SU(2)$. The irreducible representations are tensor products of the representations of the first and second factors. If we take the k_1+1-dimensional representation of the first factor and the k_2+1-dimensional representations of the second, we obtain a representation of complex dimension $(k_1+1)(k_2+1)$ with a structure map J that squares to $(-1)^{k_1+k_2}$. We label this representation (k_1, k_2). The complex dimension of (k_1, k_2) is always $(k_1+1)(k_2+1)$, while the real dimension is $(k_1+1)(k_2+1)$ if k_1+k_2 is even and $2(k_1+1)(k_2+1)$ if k_1+k_2 is odd. Once again, we see that real irreducible representations come only in odd dimensions and in dimensions that are multiples of 4.

The first few examples of irreducible parallel Riemannian bundles are:

1. The trivial bundle, of dimension 1, corresponding to the representation $(0,0)$.
2. The bundle Λ^+ of self-dual 2-forms on S^4, of real dimension 3, corresponding to the representation $(2,0)$.
3. The bundle Λ^- of anti-self-dual 2-forms on S^4, of real dimension 3, corresponding to the representation $(0,2)$.
4. The tangent bundle, of real dimension 4, corresponding to the representation $(1,1)$.
5. The bundle S^+ of positive spinors, of complex dimension 2 and real dimension 4, originating from the representation $(1,0)$. The parallel connection is the standard round self-dual $SU(2)$ instanton of second Chern class -1. If we identify $S^4 = \mathbb{HP}^1$, this is the tautological quaternionic line bundle, and the unit sphere bundle is a Hopf fibration $S^7 \to S^4$.
6. The bundle S^- of negative spinors, corresponding to the representation $(0,1)$. The parallel connection, with structure group $SU(2)$, is the round anti-self-dual

anti-instanton, of second Chern class 1.

7. There are bundles of real dimension 5, corresponding to the representations $(4,0)$ and $(0,4)$. These are the traceless symmetric endomorphisms of Λ^+ and Λ^-, respectively.

Combining these low-dimensional irreducible representations, we obtain a list of all the rank-4 Riemannian vector bundles over S^4 with parallel curvature.

Proposition 5.1. *There are exactly six rank-4 Riemannian vector bundles over S^4 with parallel curvature. Three of these, namely S^+, S^- and the trivial bundle, admit a complex structure. The corresponding rank-2 complex vector bundles have second Chern class -1, $+1$ and 0, respectively. The remaining three Riemannian vector bundles, namely TM, $\mathbb{R} \oplus \Lambda^+$ and $\mathbb{R} \oplus \Lambda^-$, are not complex.*

Proof. These examples are constructed from representations of $Spin(4)$, and there are no other real 4-dimensional representations. All that remains is to show that these bundles are distinct. This can be seen from their characteristic classes. TM and $S^{\pm}$ all have nontrivial Euler classes (2 and ∓ 1 respectively), and their Pontryagin classes are 0, ± 2. The bundles $\mathbb{R}^4$, $\mathbb{R} \oplus \Lambda^+$ and $\mathbb{R} \oplus \Lambda^-$ all have trivial Euler class, but their Pontryagin classes are all distinct (0 and ± 4, respectively). $\square$

If $n > 4$, then $SO(n)$ is simple, and $spin(n)$ is a simple Lie Algebra. The smallest nontrivial representation is the fundamental representation of $SO(n)$. Hence there are no nontrivial parallel bundles of dimension less than n, and the only nontrivial parallel bundle of dimension n is the tangent bundle. Other noteworthy representations, and their corresponding bundles, are

1. The k-th exterior power of the fundamental representation of $SO(n)$ has dimension $\binom{n}{k}$, and corresponds to the bundle of k-forms on S^n.
2. The symmetric product of the fundamental representation of $SO(n)$ with itself splits as a trivial factor plus an $[n(n+1)-2]/2$ irreducible representation. The latter corresponds to the traceless symmetric endomorphisms of TM.
3. The fundamental representation of $Spin(n)$ has complex dimension $2^{n/2}$ if n is even and $2^{(n+1)/2}$ if n is odd. If n is even the spin bundle splits as the sum of positive and negative spinor bundles, each of complex dimension $2^{(n-2)/2}$.

B. Bundles over complex projective spaces.

The complex projective space $M = \mathbb{CP}^n$ is naturally the quotient of $U(n+1)$ by $U(1) \times U(n)$. However, this quotient does not describe the structure of $\mathbb{CP}^n$ as a symmetric space, since the center of $U(n+1)$ is contained in $U(1) \times U(n)$. We must take the quotient of both $U(n+1)$ and $U(1) \times U(n)$ by $U(1)$. The question is, which $U(1)$?

The most natural choice of $U(1)$ is the center Z of $U(n+1)$. Let $G_0 = U(n+1)/Z$, and let $H_0 = (U(1) \times U(n))/Z \sim U(n)$. G_0/H_0 is indeed $\mathbb{CP}^n$, H_0 is the holonomy group of $\mathbb{CP}^n$, and G_0 is the bundle of complex orthonormal frames of TM.

G_0 is not simply connected, however. Its universal cover is $G = SU(n+1)$, and the covering map is n+1-to-1. G can be viewed as the quotient of $U(n+1)$ by $Z' = U(1) \times I_n$, where I_n is the $n \times n$ identity matrix, while $H = (U(1) \times U(n))/Z' \sim U(n)$. Although H and H_0 are both isomorphic to $U(n)$ as groups, H is the $n+1$-fold cover of H_0, with covering map $A \to (\det A)A$.

G itself can be viewed as a bundle of frames, just not of TM. G is the bundle of complex orthonormal frames of $L \otimes_{\mathbb{C}} TM$, where L is the tautological complex line bundle over $\mathbb{CP}^n$. Equivalently, $L \otimes_{\mathbb{C}} TM$ is the orthogonal complement to L in the trivial $\mathbb{C}^{n+1}$ bundle over $\mathbb{CP}^n$.

There are infinitely many nontrivial complex 1-dimensional (or real 2-dimensional) representations of $H = U(n)$, namely all powers of the determinant. The determinant representation itself yields a line bundle of Chern class $+1$. This is the determinant bundle of $L \otimes_{\mathbb{C}} TM$, and is also isomorphic to L^*, the complex dual of L. The k-th power of the determinant yields $(L^*)^k$ if $k > 0$, or L^{-k} if $k < 0$. (For $n = 1$, this replicates our previous analysis of $M = S^2$.)

Note that the determinant representation of H_0 yields the determinant of TM (viewed as a complex vector bundle), which has Chern class $n + 1$, and all complex line bundles constructed from representations of H_0 have first Chern class divisible by $n + 1$. This demonstrates once again that to get all parallel bundles we must consider all representations of H, not just representations of H_0.

After the trivial bundle, the determinant, and powers of the determinant, the smallest unitary representations of $U(n)$ are the fundamental representation and its dual, both of complex dimension n. There are an infinite number of complex representations of this dimension, obtained by multiplying the trivial representation (or its dual) by arbitrary powers of the determinant. The smallest nontrivial real (i.e. not complex) irreducible representation is the adjoint representation, of real dimension $n^2 - 1$.

C. Lie Groups.

Every connected Lie group K which is the product of a compact group and a vector group is a symmetric space. If K is simply connected, we can take $G = K \times K$, and let $H \sim K$ be the diagonal subgroup of G. Parallel bundles over K thus correspond to representations of K itself. The tangent bundle of K corresponds to the adjoint representation, as the tangent space of K at the origin is naturally isomorphic to the Lie algebra of K. It follows from Corollary 5.3 below that if K has no Euclidean factor, then the unit tangent bundle of K admits a metric of constant positive scalar curvature.

Applying this construction to $K = SU(2)$, we recover our previous results about $M = S^3$.

D. Sphere bundles and constant scalar curvature.

Let E denote a rank k vector bundle with parallel connection over $M = G/H$. It follows from the main theorem that if G admits a bi-invariant metric, then E and the unit sphere bundle E^1 of E admit complete metrics of nonnegative sectional curvature: Endow $\mathbb{R}^k$ with any $O(k)$ invariant metric. Then the diagonal action of H on the metric product $G \times \mathbb{R}^k$ is by isometries, and there is a unique metric on the quotient $E = G \times_\rho \mathbb{R}^k$ (respectively $G \times_\rho S^{k-1}$) for which the projection $G \times \mathbb{R}^k \to E$ (resp. $G \times S^{k-1} \to E^1$) becomes a Riemannian submersion, see [C]. Moreover, if H acts transitively on the unit sphere in $\mathbb{R}^k$, then E^1 is a homogeneous space under the isometric action $g[(g', u)] := [(gg', u)]$, $g \in G$, and in particular, E^1 has constant scalar curvature. It is noteworthy that there are vector bundles over the sphere that admit nonnegatively curved metrics but no parallel connections: One of these, in fact, is a rank 4 bundle over S^4 whose unit sphere bundle is a

Milnor exotic 7-sphere, see [G-M] and [R].

If H does not act transitively on S^{k-1}, then the resulting metric on the unit sphere bundle is not, in general, homogeneous. Nevertheless, when the action of H is irreducible, then there exists a metric of *constant scalar curvature* on E^1: Decompose $TE = \mathcal{H} \oplus \mathcal{V}$ as a sum of subbundles $\mathcal{H}$ (the horizontal distribution defining the connection) and $\mathcal{V}$ (the vertical distribution tangent to the fibers of $\pi : E \to M$) over E. The inner product on $\mathcal{H}$ is defined by pulling back the one on TM via $\pi_* : \mathcal{H} \to TM$. The one on $\mathcal{V}$ is induced by the $O(k)$ invariant metric on $\mathbb{R}^k$. Finally, declare both distributions to be mutually orthogonal. This defines a *connection metric* on E, and the projection $\pi : E \to M$ is a Riemannian submersion with totally geodesic fibers. It can be shown [S-W] that if the sectional curvature K_M of M is strictly positive (i.e., if M is a compact rank one symmetric space), then $K_E \geq 0$, provided the fiber metric is sufficiently curved.

Proposition 5.2. *The connection metric on E induces a constant scalar curvature metric on the unit sphere bundle E^1.*

Proof. Let s_E, s_M, and s_F denote the scalar curvatures of E, M, and the fiber F respectively. Since $\pi : E \to M$ has totally geodesic fibers,

$$s_E = s_m \circ \pi + s_F - |A|^2, \qquad |A|^2 := \sum_{i,j} |A_{x_i} x_j|^2, \tag{5.4}$$

cf. [B, 9.37]. Here A denotes the O'Neill tensor of π, $A_X Y = \frac{1}{2}[X,Y]^v$ for horizontal X and Y, and x_i is (the horizontal lift of) an orthonormal basis of TM. On the other hand, for unit $u \in E$ and $r > 0$,

$$|A|^2(ru) = \frac{1}{4} G^2(r) \sum_{i,j} |R^E(x_i, x_j)u|^2, \tag{5.5}$$

where $dr^2 + G^2(r)d\sigma^2$ is the fiber metric in polar coordinates, see [S-W]. It therefore suffices to check that the map $C : E^1 \to \mathbb{R}^+$ given by $C(u) = \sum |R(x_i, x_j)u|^2$ is constant. Observe that $C(u) = \langle \tilde{C}u, u \rangle$, where $\tilde{C} : E \to E$ is defined as

$$\tilde{C} = -\sum_{i,j} R^E(x_i, x_j) \circ R^E(x_i, x_j). \tag{5.6}$$

But $\tilde{C}$ is parallel since R^E is, and therefore commutes with the action of the holonomy group of E. The latter is by assumption irreducible, so that by Schur's lemma, $\tilde{C}$ is a multiple of the identity. This implies that C is constant as claimed.

Corollary 5.3. *Let M be an irreducible symmetric space with nonnegative sectional curvature. Then the unit tangent bundle of M admits a metric of nonnegative sectional curvature and constant scalar curvature.*

Proof. The first part of the statement is essentially due to Cheeger [C]. The claim about the scalar curvature is an immediate consequence of Proposition 5.2 together with the irreducibility assumption.

Acknowledgements. We thank Sharad Agnihotri and Jan Segert for helpful discussions. The work of L.S. is partially supported by NSF grant DMS-9626698.

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