WHEN IS A RIEMANNIAN SUBMERSION HOMOGENEOUS?

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Abstract. We study the structure of the most common type of Riemannian submersions, namely those whose fibers are given by the orbits of an isometric group action on a Riemannian manifold. Special emphasis is given to the case where the ambient space has nonnegative curvature.

1. Introduction

Ehresmann showed that a submersion $\pi : M \to B$ with compact fibers is always a locally trivial fibration; i.e, any $b \in B$ has an open neighbourhood U in B such that $\pi^{-1}(U)$ is homeomorphic to a product $U \times F$ (see [3], [4]). It is important to note that although in the original statements of these theorems (or of their extensions in [8]) the reference is to bundles, this word is not used with its usual meaning of *fiber bundle*, namely a space admiting trivializations with a Lie group as structure group (see [9] for definitions). In this note we improve Ehresmann's result to homogeneous submersions; i.e, to Riemannian submersions with "fibers" given by orbits of an isometric group action on the space: specifically, such submersions are bundles with a homogeneous space as fiber that are associated to a principal G-bundle, where G is the holonomy group of the submersion. Furthermore, the latter has a natural principal G-connection determined by the horizontal spaces of the original submersion.The above scenario is next examined from a different perspective: since almost every Riemannian submersion is homogeneous, we try to determine criteria ensuring that a given Riemannian submersion is not of that type. This is done in terms of the substantiality of the O'Neill A-tensor of the submersion. Finally, we use Wilking's dual foliation to show that a homogeneous submersion on a space of quasipositive curvature always has a Lie group as fiber.

2. The holonomy of a homogeneous metric fibration

We will use what is by now standard terminology and notation for Riemannian (also called metric) foliations. Such a foliation $\mathcal{F}$ on a Riemannian manifold M induces an orthogonal splitting $TM = \mathcal{V} \oplus \mathcal{H}$ of the tangent bundle of M, with the *vertical* distribution $\mathcal{V}$ tangent to $\mathcal{F}$. A curve in M is said to be *horizontal* if it is everywhere tangent to the horizontal distribution $\mathcal{H}$. When the leaves of $\mathcal{F}$ are given by the fibers of a (necessarily Riemannian) submersion $\pi : M \to B$, we also refer to π as a *metric fibration*.

Let $G \times M \to M$ denote an isometric action by principal orbits of a compact Lie group G on a complete Riemannian manifold M. The orbit space $B := M/G$ is also a manifold,

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and inherits the quotient metric from B for which $\pi : M \to B$ becomes a Riemannian submersion. π is then said to be a *homogeneous* submersion or fibration.

2.1. **The holonomy group of π is a Lie group:** For $b \in B$, we define *the holonomy group* $\mathrm{Hol}(b)$ of π at b to be the group consisting of holonomy diffeomorphisms of $\pi^{-1}(b)$ induced by loops in B based at b (recall that a loop c at b induces a *holonomy diffeomorphism* of $\pi^{-1}(b)$ that maps $p \in \pi^{-1}(b)$ to the endpoint of the horizontal lift of c starting at p). For a general Riemannian submersion, $\mathrm{Hol}(b)$ need not to be a Lie group, as observed for instance in [1], remark 9.57. Nonetheless, we will see that this can not happen for a homogeneous fibration:

Lemma 2.1. $\mathrm{Hol}(b)$ *is a Lie group.*

Proof: Identify $\pi^{-1}(b)$ with G/H where H is the isotropy group of some $p \in \pi^{-1}(b)$. Denote by $h : G/H \to G/H$ the holonomy diffeomorphism associated to some loop $\bar{\sigma}$ based at b. If σ is the lift to M starting at some $p \in \pi^{-1}(b)$, then $g \cdot \sigma$ is the lift of $\bar{\sigma}$ at the point $g \cdot p$. Thus $h(g \cdot p) = g \cdot h(p)$, and h is a G-equivariant diffeomorphism of G/H. The collection of such diffeomorphisms is precisely $N(H)/H$, where $N(H)$ is the normalizer of H in G, which is known to be a Lie group (see for instance [2]). The connected component of the identity in $\mathrm{Hol}(b)$ lies in the connected component of the identity of $N(H)/H$ as a connected algebraic subgroup, and is therefore a Lie subgroup ([14] or [5]). The collection of connected components can be identified with $\pi_1(B)$, which is countable, thereby establishing the claim. □

Remark: The correspondence between $\mathrm{Hol}(b)$ and $N(H)/H$ is as follows: if $h(H) = aH$, then $a \in N(H)/H$. This forces the action of $\mathrm{Hol}(b)$ in G/H to be *on the right*: if $gH \in G/H$ and $a \in N(H)/H$ is the element corresponding to the holonomy diffeomorphism h, then $h(gH) = gh(H) = gaH = (gH)a$.

2.2. **Fiber bundle structure of M over B.** Fix some $b_0 \in B$, and denote $\pi^{-1}(b_0)$ by F_0.

Theorem 2.2. *$\pi : M \to B$ is a fiber bundle with base M, fiber F_0, and structure group* $\mathrm{Hol}(b_0)$.

Proof: Choose some open covering of B by convex balls $B(b_\alpha, r_\alpha)$ of radius r_α centered at b_α, and a corresponding set of minimal geodesics $c_\alpha : [0, 1] \to B$ with $c_\alpha(0) = b_0$, $c_\alpha(1) = b_\alpha$. For each α define $\phi_\alpha : B_\alpha \times F_0 \to \pi^{-1}(B_\alpha)$ by $\phi_\alpha(b, f) = h_b \circ h_\alpha(f)$, where h_α is the holonomy map between F_0 and F_{b_α} obtained by lifting c_α, and h_b is the corresponding map for the unique geodesic c_b in B_α joining b_α and b. It is clear now from their definition that the transition maps correspond to elements in $\mathrm{Hol}(b_0)$ obtained by lifting a composition of c_α's and c_b's. The smoothness of such maps in b is immediate. □

It is worth pointing out that a similar statement for an arbitrary Riemannian submersion was obtained in [7] *when the submersion has totally geodesic fibers.* This restriction is lifted here, although we have to compensate by assuming homogeneity of the submersion (since, as noted earlier, an arbitrary Riemannian submersion need not be a fiber bundle).

2.3. **The holonomy group of a homogeneous submersion.** Since $\pi : M \to B$ is a fiber bundle all of whose transition maps live in $\mathrm{Hol}(b_0)$, we may consider its associated principal bundle $\pi_P : P \to B$. We claim that P can be described as the set $\mathcal{S}$ of all possible holonomy diffeomorphisms $h : F_0 \to F_b$ for $b \in B$, with the bundle projection sending each map

$h : F_0 \to F_b$ to b. To see this, consider the action of $\mathrm{Hol}(b_0)$ on $\mathcal{S}$ given by precomposition. This is clearly a free action preserving the fibers of the projection. Also in the notation of Theorem 2.2, the maps

$$\psi_\alpha : \pi_P^{-1}(B_\alpha) \longrightarrow \mathrm{Hol}(b_0);$$
$$h \longmapsto h_{-c_\alpha} \circ h_{-c_b} \circ h,$$

are $\mathrm{Hol}(b_0)$ invariant (here, $-c$ denotes the curve $t \mapsto c(1-t)$); in fact, if $h_0 \in \mathrm{Hol}(b_0)$, then

$$\psi_\alpha(h_0 h) = h_{-c_\alpha} \circ h_{-c_b} \circ (h \circ h_0) = \psi_\alpha(h) \circ h_0 = h_0 \psi_\alpha(h).$$

This allow us to construct trivializations of $\mathcal{S}$ with the same transition functions as those of the bundle $\pi : M \to B$, and $\mathcal{S} \to B$ must therefore be the associated $\mathrm{Hol}(b_0)$-principal bundle, which from now on will be denoted $P \to B$.

As a consequence, we get a $\mathrm{Hol}(b_0)$ action on $F_0 \times P$ whose quotient $F_0 \times_{\mathrm{Hol}(b_0)} P$ is equivariantly diffeomorphic to M. In the above description of P, the diffeomorphism $\Phi : F_0 \times_{\mathrm{Hol}(b_0)} P \to M$ is given by $\Phi([m,h]) = h(m)$, where $[m,h]$ is the class of (m,h) in $F_0 \times_{\mathrm{Hol}(b_0)} P$.

Recall that a *G-connection* on a principal G-bundle $\pi_P : P \to B$ is a distribution $\widetilde{\mathcal{H}}$ on P that is complementary to the kernel of π_{P*} and invariant under the action of G; i.e., $g_* \widetilde{\mathcal{H}}_p = \widetilde{\mathcal{H}}_{gp}$, for $g \in G$ and $p \in P$. The *holonomy group of the connection* at $b \in B$ is, as usual, the group of diffeomorphisms of the fiber over b obtained by lifting loops at b to curves everywhere tangent to $\widetilde{\mathcal{H}}$. We claim that the horizontal distribution of the submersion $\pi : M \to B$ induces in a natural way a connection on $P \to B$, the holonomy group of which coincides with the holonomy group of π.

To this end, consider a point $p = h_c \in P$ and a curve $c_0 : [t_0, t_1] \to B$ with $c_0(t_0) = \pi_P(h_c)$. By the definition of π_P, the curve $c_0^p : [t_0, t_1] \to P$ defined as $c_0^p(t) = h_{c_0|_{[t_0,t]}} \circ h_c$ is a lift of c_0 to P with $c_0^p(t_0) = h_c$. In order to see how the horizontal spaces for the submersion π induce horizontal spaces for a connection in P, we proceed as follows:

(1) Fix some $m \in F_0$; after following the inclusion $P \to F_0 \times P$, $h_c \to (m, h_c)$, by the projection $\rho : F_0 \times P \to F_0 \times_{\mathrm{Hol}(b_0)} P$, and then by the diffeomorphism $\Phi : F_0 \times_{\mathrm{Hol}(b_0)} P \to M$, we obtain a map $\psi : P \to M$ sending the curve $t \mapsto c_0^p(t)$ to $t \mapsto h_{c_0|_{[t_0,t]}} \circ h_c(m)$; the tangent vectors to the latter curves fill the horizontal space of π at $h_c(m)$. Moreover, the restriction of $\Phi_{*[m,h_c]}$ to $\rho_*(0 \times T_{h_c} P)$ must be onto the horizontal space $\mathcal{H}$ at M. As a consequence we can find a unique distribution $\widetilde{\mathcal{H}}$ in P that is ψ-related to $\mathcal{H}$; i.e., such that $\psi_* \widetilde{\mathcal{H}} = \mathcal{H} \circ \psi$.

(2) $\widetilde{\mathcal{H}}$ is $\mathrm{Hol}(b_0)$-invariant: since $[m, c_0^p]$ is $\mathcal{H}$-horizontal in M, $[h(m), c_0^p] = [m, hc_0^p]$ is also horizontal for $h \in \mathrm{Hol}(b_0)$. Therefore $h_* \widetilde{\mathcal{H}} = \widetilde{\mathcal{H}} \circ h$.

Finally, we still have to show that $\mathrm{Hol}(b_0)$ is the holonomy group of this connection on P; but this is clear once we compute this group at b_0: the fiber of P at b_0 is precisely $\mathrm{Hol}(b_0)$, and the horizontal lifts of loops c_0 in B to P are of the form c_0^p, which map the point h_c to $h_{c_0} \circ h_c$, *the right action* of $\mathrm{Hol}(b_0)$ on itself. We have proved:

Theorem 2.3. *Consider the Riemannian submersion from Theorem 2.2, and the associated principal* $\mathrm{Hol}(b_0)$*-bundle* $\pi_P : P \to B$*. Given* $m \in F_0$*, define* $\psi : P \to M$ *by* $\psi(h_c) = h_c(m)$*, for* $h_c \in P$*. Then there is a unique connection* $\widetilde{\mathcal{H}}$ *on the principal bundle* π_P *that is* ψ*-related to the horizontal distribution* $\mathcal{H}$ *of the submersion* π*,* $\psi_* \widetilde{\mathcal{H}} = \mathcal{H} \circ \psi$*. The holonomy group of* *this connection coincides with the holonomy group of the submersion acting on itself by right multiplication.*

3. Homogeneous versus substantial

When dealing with foliations rather than fibrations, some extra care is needed. Let $\mathcal{F}$ be a Riemannian foliation on a manifold M. $\mathcal{F}$ is said to be *homogeneous* if there is an isometric group action on M that preserves the foliation, and such that the isotropy group of each leaf is transitive on the leaf. If this only holds in a neighborhood of each point of M, then $\mathcal{F}$ is said to be *locally homogeneous*, cf. also [6]. In particular, leaves need not share the same topology. One such example is that of the orbit foliation of the $\mathbb{R}$-action on S^3 given by $(t,(z_1,z_2)) \mapsto (e^{it}z_1, e^{i\alpha t}z_2)$ for $z_1,\ z_2 \in \mathbb{C}$, $|z_1|^2+|z_2|^2 = 1$. When α is irrational, then exactly two leaves are circles, whereas the others are immersed copies of $\mathbb{R}$. We will say that $\mathcal{F}$ is *substantial at* $p \in M$ if the A-tensor is onto the vertical space at p; i.e, if for any $v \in \mathcal{V}_p$, there exist $x,\ y \in \mathcal{H}_p$ such that $A_x y = v$. When this holds at every point of M, we say $\mathcal{F}$ is *substantial*. Notice that the version of substantiality that we are using here is weaker than that used in [6].

Theorem 3.1. *Let $\mathcal{F}$ be a homogeneous foliation on M. If $\mathcal{F}$ is substantial, then the action is free. In particular, if $\mathcal{F}$ is a homogeneous fibration that is substantial at one point, then it is a principal bundle fibration.*

Proof: Consider $p \in M$. We claim the isotropy group G_p of the action of G at p is trivial, so that the action is free. The derivative of any $g \in G_p$ must fix the horizontal distribution at p, for if $g_*x = y$, then for any t, $g \circ \exp(tx) = \exp(tg_*x) = \exp(ty)$; i.e., $\exp(tx)$ and $\exp(ty)$ belong to the same fiber, and thus project to intersecting geodesics downstairs. This implies that $x = y$.

Next, consider a vertical vector $v \in M_p$. Since the foliation is substantial, v can be written as $A_x y$ for some horizontal x, y. Extend the latter to basic fields X, Y. Now, any basic field is g-related to itself, and therefore so is the bracket $[X,Y]$. Since this is also true for the basic horizontal component of $[X,Y]$, it holds for the vertical component. When evaluated at p this component is just $2A_x y$. Thus, $g_*v = g_*A_x y = A_x y = v$, and the derivative of g is the identity. Since g is an isometry, it is the identity map. This establishes the first statement. For the fibration case, observe that by the above argument, if $\mathcal{F}$ is substantial at p, then the isotropy group at p is trivial. Since orbits are equivariantly diffeomorphic, all isotropy groups are trivial. Thus, the action is free, and the second statement now follows from Theorem 2.2. Notice that in this case, the bundle described in Theorem 2.3 is usually a reduction of the G-principal bundle $M \to B := M/G$, because $\mathrm{Hol}(b_0)$ is identified with a subgroup of G acting on G by right multiplication. □

Theorem 3.1 implies that a substantial metric fibration must have a Lie group as fiber, if it is to be homogeneous. We can give now an example of a nonhomogeneous metric fibration:

Corollary 3.2. *The Hopf fibration $S^{15} \to S^8$ is not homogeneous.*

Proof: The fiber is S^7, which is not a Lie group. Furthermore, the fibration has totally geodesic fibers and the total space has positive sectional curvature, implying that the fibration is fat in the sense of Weinstein [10], and therefore substantial in a strong sense: Indeed, the fact that a plane spanned by a horizontal x and a vertical v has positive curvature implies that $A_x^* v \neq 0$, where $A_x^* : \mathcal{V} \to \mathcal{H}$ denotes the pointwise adjoint of A_x. Thus, A_x is onto for any nontrivial horizontal x. □

The other Hopf fibrations (over complex and quaternionic projective spaces) are of course homogeneous. Together with the one above, they are known to be the only metric fibrations of Euclidean spheres [6], [11].

4. Obstructions due to the dual foliation.

Wilking has shown that a metric foliation $\mathcal{F}$ on M induces another so-called dual (possibly singular) foliation that contains the horizontal distribution of $\mathcal{F}$, and used it to prove that when M has positive curvature, any two points in M can be connected by a piecewise smooth horizontal geodesic [12]. In fact, this conclusion holds also under the weaker condition of nonnegative sectional curvature and positive curvature at least at one point. This latter situation is often refered to as *quasipositive curvature*, see for instance [13].

In this section we explore some consequences of Wilking's connectedness theorem on homogeneity of metric foliations:

Lemma 4.1. *Let $\mathcal{F}$ be a homogeneous metric foliation on a manifold M. If $\gamma : [0, a] \to M$ is a horizontal geodesic, then the isotropy groups of the action coincide for every point of γ.*

Proof: This is contained in [2], but we sketch a (more geometric) proof here for completeness. Let $p = \gamma(0)$ and $u = \dot\gamma(0)$. If $h \in G_p$, then, as observed in the proof of Theorem 3.1, its derivative h_* fixes the whole horizontal space at p, and thus $(h \circ \gamma)(t) = h(\exp_p tu) = \exp_p(t h_* u) = \gamma(t)$. It follows that G_p is contained in the holonomy group of any point $\gamma(t)$. By symmetry, it must equal that group. □

Theorem 4.2. *Let G act with principal orbits by isometries on a quasipositively curved manifold M. Then the isotropy group of any point is a normal subgroup of G, and the fiber of $M \to M/G$ is a Lie group.*

Proof: Let $p \in M$, and denote by H its isotropy group. By Wilking's dual foliation theorem and the previous lemma, the isotropy group of any other point must coincide with H. But it is easy to see that if $g \in G$, then the isotropy group of $q = gp$ is gHg^{-1}, so that H is normal in G, and the fiber G/H is a Lie group. □

Notice that this provides another proof of Corollary 3.2.

Corollary 4.3. *Under the above hypothesis, if G is a simple group, then the action of G on M is either free or transitive.*

Proof: Since G contains no nontrivial normal subgroups, it follows that the isotropy group of any point must be either the identity or the full group G. □

We saw in the proof of Theorem 4.2 that under the given hypotheses, all points share the same isotropy group. Thus, H is in the kernel of the representation, and we deduce the following:

Corollary 4.4. *Let G act with principal orbits by isometries on a quasipositively curved manifold M. If the action is effective, then it is necessarily free.*

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