Isometric Immersions without Positive Ricci Curvature

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ABSTRACT. In this note we study isometric immersions of Riemannian manifolds with positive Ricci curvature into an Euclidean space.

1. Introduction

When studying isometric immersions, one often realizes that restrictions on the curvature result in the need for extra space. For instance, a classical result of Hilbert affirms that in $\mathbb{R}^3$ there are no complete surfaces whose induced metrics have constant negative curvature [**5**]. In the same spirit, Chern and Kuiper proved that if an n-dimensional compact Riemannian manifold M with nonpositive sectional curvature immerses isometrically in some $\mathbb{R}^N$, then $N \geq 2n$ [**1**], [**2**].

When turning to the positive sectional curvature case, the main observation was due to A. Weinstein [**14**], who showed that for immersions of M^n in $\mathbb{R}^{n+2}$, the curvature operator $R : \Lambda^2 M \to \Lambda^2 M$ had to be positive. As a consequence, if M^n is simply connected, then it has to be homeomorphic to a sphere. Related research appeared in [**8**] and [**11**].

Our aim in this note is to collect a few observations about the positive Ricci curvature case, by taking into account the type of the normal bundle to the immersion as a possible obstruction. Our main tool is Lemma 2.1. We then apply it to several situations, and among them, we get

THEOREM 1.1. *Let $M^4 = \mathbb{CP}^2 \# \ldots \# \mathbb{CP}^2$ a connected sum of complex projective spaces ($\mathbb{CP}^2$ itself is allowed). Then no immersion $f : M^4 \to \mathbb{R}^7$ induces a metric with positive Ricci curvature.*

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2. Positive Ricci curvature and normal bundles

Let (M, g) be a Riemannian manifold of dimension n and $f : M \to \mathbb{R}^m$ an isometric immersion. Denote by $\nu(f)$ the normal vector bundle of the immersion;

i.e,

$$\nu(f) = \{\, (x,v) \,:\, x \in M, v \in T_x\mathbb{R}^n, v \perp f_*(T_xM) \,\}.$$

There is an isomorphism of vector bundles $TM \oplus \nu(f) \simeq \mathbb{R}^n$, where, without risk of confusion, $\mathbb{R}^n$ denotes the product n–bundle over M.

LEMMA 2.1. *If (M, g) has positive Ricci curvature, then the mean curvature vector H is a non-vanishing section of $\nu(f)$.*

Proof: Denote by $B : TM \times TM \to \nu(f)$ the second fundamental form of the immersion, and $x \in T_pM$ a unit vector field that we complete to an orthonormal basis $\{x = e_1, e_2, \ldots, e_n\}$. Gauss' formula implies that

$$Ric(x,x) = \sum_{i=2}^{n} K(x, e_i) = \sum_{i=2}^{n} \langle B(x,x), B(e_i, e_i) \rangle - \|B(x,e_i)\|^2.$$

Thus

$$\langle B(x,x), H \rangle = Ric(x,x) + \sum_{i=1}^{n} \|B(x,e_i)\|^2 > 0.$$

$\square$

Although we have phrased the Theorem in terms of positive Ricci curvature, the proof shows that it suffices to have a vector x with $Ric(x) > 0$ at each point.

3. Immersions of $\mathbb{CP}^2 \# \ldots \# \mathbb{CP}^2$ in $\mathbb{R}^7$

Whitney's immersion theorem asserts that any four-dimensional manifold can be immersed in $\mathbb{R}^7$. We use the lemma to give the application mentioned in the introduction:

THEOREM 3.1. *$M^4 = \mathbb{CP}^2 \# \ldots \# \mathbb{CP}^2$ cannot be immersed into $\mathbb{R}^7$ with an induced metric of positive Ricci curvature.*

Proof: If not, then by Lemma 2.1 we have a plane bundle $E \to M$ such that

$$M \times \mathbb{R}^7 \simeq TM \oplus E \oplus \mathbb{R}.$$

Since M is simply connected, E is oriented, and a simple topological argument shows that $H^2(M, \mathbb{Z})$ is torsion–free ([**4**], proposition E.1). Thus, the product formula for Pontryagin classes (see [**9**], theorem 15.3) gives:

$$(1 + p_1(M))(1 + p_1(E)) = 0.$$

Hence $p_1(E) = -p_1(M)$. By Hirzebruch's theorem in dimension 4 ([**9**] Theorem 19.4), $\langle p_1(M), [M] \rangle = 3\sigma(M)$. Since by hypothesis, this is positive, we have

$$\langle p_1(E), [M] \rangle < 0.$$

On the other hand, by Corollary 15.8 in [**9**], $p_1(E) = e(E)^2$. Thus we can use the intersection form Q of M to give an alternate computation of $p_1(E)$ as

$$\langle p_1(E), [M] \rangle = \langle e(E)^2, [M] \rangle = Q(e(E), e(E)).$$

However, M was chosen so that Q was positive definite (see [**3**], for instance), and we obtain a contradiction. $\square$

The theorem is optimal, since the Veronese embedding

$$\phi : \mathbb{CP}^2 \longrightarrow \mathrm{Herm}_0(\mathbb{C}, 3)$$

where $\mathrm{Herm}_0(3, \mathbb{C})$ are the 3×3 trace–free hermitian matrices, defined by

$$\phi[z_0, z_1, z_2] = I - 2(\overline{z_0}, \overline{z_1}, \overline{z_2})^t (z_0, z_1, z_2)$$

induces the Fubini–Study metric in $\mathbb{CP}^2$, which has positive sectional curvature. In fact, the natural action of $SU(3)$ on $\mathbb{CP}^2$ induces an isometric action on its image by ϕ with $U(2)$ as isotropy, and hence the induced metric agrees with the quotient metric $SU(3)/U(2)$. Observe that the dimension of $\mathrm{Herm}_0(\mathbb{C}, 3)$ is eight.

Remark: There are several extensions of the above proof. For instance, the same argument shows that the complex projective plane $\mathbb{CP}^n$ cannot be immersed with positive curvature in codimension 3. But also any proof of the impossibility of immersing a given manifold M^n into R^N that uses the impossibility of splitting the trivial rank N bundle as a sum $TM \oplus E$ can be adapted to our situation to show that M^n does not immerse in R^{N+1} with positive curvature. As a concrete example, RP^n can be immersed in R^{2n-1} by Whitney's theorem, but when n is a power of two, the induced metric cannot have positive curvature everywhere. Otherwise, the normal bundle would split a line and we would get a contradiction to the product formula for Stiefel-Whitney classes as in [**9**].

4. Immersions of the sphere with positive Ricci curvature

Due to the work of Smale in [**12**], we have a good understanding of the immersions of n-spheres in the euclidean space of dimension $n + k$. Up to isotopy, they are in one-to-one correspondence with elements of $\pi_n(V_{n+k,n})$, where $V_{n+k,n}$ is the Stiefel manifold of n-orthonormal frames in $\mathbb{R}^{n+k}$.

In a development of [**12**], Kervaire gave a characterization of the bundles over S^n that can appear as a normal bundle of an immersion into $\mathbb{R}^{n+k}$. Namely, the isomorphism classes of rank k vector bundles over S^n correspond to elements in $\pi_{n-1}(SO(k))$ through their clutching maps [**13**]. On the other hand, the usual fibration of the Stiefel manifold $SO(k) \to SO(n+k) \to V_{n+k,k}$ induces a homotopy sequence

$$\cdots \to \pi_n(SO(n + k)) \to \pi_n(V_{n+k,k}) \to \pi_{n-1}(SO(k)) \to \ldots$$

where $\partial : \pi_n(V_{n+k,k}) \to \pi_{n-1}(SO(k))$ is the boundary homomorphism. Kervaire proved that there is a bijection between the set of clutching maps for normal bundles of immersions in $\mathbb{R}^{n+k}$ and the image of ∂. He denoted this subgroup by $J_{n,k}$ and proceeded to compute it for some values of k and n.

The results in this section follow from combining some of these facts with the information given by Lemma 2.1.

THEOREM 4.1. *Let n and k positive integers such that $J_{n,k} = 0$. Then any immersion $f : S^n \to \mathbb{R}^{n+k+1}$ with positive Ricci curvature can be isotoped to the map $S^n \xrightarrow{i} \mathbb{R}^{n+1} \to \mathbb{R}^{n+k+1}$, where i is the standard inclusion.*

Proof: The normal bundle $\nu(f)$ to the immersion admits a section without zeros due to lemma 2.1. Thus there is a splitting $\nu(f) = \mathbb{R} \oplus E$, where E has rank k. Theorem 6.1 in [**6**] asserts that f can now be isotoped to an immersion $g : S^n \to \mathbb{R}^{n+k}$, and the bundle E is isomorphic to $\nu(g)$. The hypothesis on $J_{n,k}$ $(= 0)$ implies that E is the trivial bundle, and correspondingly $\nu(f)$ is just $S^n \times \mathbb{R}^{k+1}$. But corollary 6.2 from [**6**] affirms that in this case, f can be isotoped into R^{n+k+1}, and the theorem is proved. $\qquad\square$

COROLLARY 4.2. *Any immersion of S^{8s+5} into $\mathbb{R}^{16s+6}$ or $\mathbb{R}^{16s+7}$ with an induced metric of positive Ricci curvature can be isotoped into the standard immersion of S^{8s+5} in $\mathbb{R}^{8s+6}$.*

Proof: In [**7**], it was shown that $\pi_{8s+4}(SO(8s)) = \pi_{8s+4}(SO(8s+1)) = 0$. Thus $J_{8s+5,8s} = J_{8s+5,8s+1} = 0$, and we are in the hypothesis of theorem 4.1. $\qquad\square$

It is somewhat unsatisfactory that the theorem does not give any information about the induced metrics at each stage of the isotopy between f and the inclusion $S^n \to \mathbb{R}^{n+1}$. The induced metrics at times $t = 0$ and $t = 1$ of the isotopy have both positive Ricci curvature, and we would like to know whether the deformation could be done entirely by such metrics. However, in the codimension two case, Moore showed that there are not obstructions to such deformations in the class of *positive sectional curvature metrics* [**10**].

References

[1] S.S. Chern and N. Kuiper, *Some theorems on the isometric imbedding of compact Riemannian manifolds in Euclidean space*, Annals of Mathematics, 56 (1952), pp. 422-430.

[2] M. Dajczer, *Submanifolds and Isometric Immersions*, Mathematics Lecture Series, 13 , Publish or Perish Inc. (1990).

[3] S. Donaldson and P. Kronheimer, *The geometry of four-manifolds*, Oxford Science Publications (1990).

[4] D. Freed and K. Uhlenbeck, *Instantons and Four-Manifolds*, MSRI publications 1 (1984).

[5] D. Hilbert, *Über Flächen von konstanter Gausscher Krümmung*, Transactions of the American Mathematical Society, 2 (1901), pp. 87-99.

[6] M. Kervaire, *Sur le fibré normal à une sphère immergée dans un espace euclidien*, Commentarii Mathematici Helvetici, 33 (1959), pp. 121-131.

[7] M. Kervaire, *Some nonstable homotopy groups of Lie groups*, Illinois Journal of Mathematics, 4 (1960), pp. 161-169.

[8] N. Kleinjohann and R. Walter, *Nonnegativity of the Curvature Operator and Isotropy for Isometric Immersions*, Mathematische Zeitschrift, 181 (1982), pp. 129-142.

[9] J. Milnor and J. Stasheff, *Characteristic Classes*, Annals of Mathematical Studies, Princeton University Press, 76 (1974).

[10] D. Moore, *Isometric homotopy in codimension two*, Transactions of the American Mathematical Society, 292 (1985), pp. 653-663.

[11] K. Sarkaria and S. Zoltek, *Positively curved bilinear forms*, Proceedings of the AMS, 97 (1986), pp. 577-584.

[12] S. Smale, *The classification of immersions of spheres in euclidean spaces*, Annals of Mathematics, 69 (1959), pp. 327-344.

[13] N. Steenrod, *The topology of fiber bundles*, Princeton University Press (1951).

[14] A. Weinstein, *Positively curved n-manifolds in* $\mathbb{R}^{n+2}$, Journal of Differential Geometry, 4 (1970), pp. 1-4.

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