The free boundary problem for the heat equation with fixed gradient condition *

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1 Introduction

The Stefan problem is a very well-known example of free boundary problem (FB problem) for the heat equation. In its one-phase formulation it consists of the determination of *a domain* in space-time, Ω, a subset of $Q_T = \mathbf{R}^N \times (0,T)$, and *a function* $u(x,t)$ defined and positive in such domain, which represents the temperature of the phase under consideration, and satisfies a parabolic PDE, typically the heat equation. We have thus

$$u_t = \Delta u, \quad u > 0 \quad \text{in } \Omega. \tag{1}$$

In FB problems with second-order equations two conditions are given on the moving interface, Γ, which is the a priori unknown lateral boundary of Ω, $\Gamma = \partial\Omega \cap Q_T$. In the standard Stefan problem (SP) these conditions are (i) the condition of temperature continuity

$$u = 0, \tag{2}$$

and (ii) the kinetic condition

$$L\mathbf{v} = -\frac{\partial u}{\partial n}\mathbf{n}, \tag{3}$$

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where $\mathbf{v}(x,t)$ denotes the normal velocity of motion of the free boundary at a point (x,t) and $\partial u/\partial n$ denotes the gradient of u in the direction of the outward spatial normal $\mathbf{n}$ to Γ, taken as a limit value as $y \to x$, $u(y,t) > 0$ (i.e., from inside Ω). The constant L is the latent heat. In order to complete the conditions we add initial data

$$u(x,0) = u_0(x) > 0 \qquad \text{for } x \in \Omega_0, \tag{4}$$

where Ω_0 is the initial domain. We have posed the problem in the infinite ambient space $\mathbf{R}^N$. In case a bounded space is taken, appropriate boundary conditions should be given on the fixed boundary. Though clearly formulated in the XIX century, a rigorous theory for this problem was obtained only in the last 4 decades. There exists now an extensive literature of the problem, its numerous variants and applications, cf. e.g. [Ru], [M1].

In this article I will deal with a different free boundary problem for the heat equation that has recently attracted the interest of researchers. The problem consists of finding a function $u(x,t) > 0$ which solves the heat equation (1) in an a priori unknown domain $\Omega \subset Q_T = \mathbf{R}^N \times (0,T)$ with lateral boundary Γ. On Γ we impose the conditions (2) and

$$\frac{\partial u}{\partial n} = -1, \tag{5}$$

This condition replaces (3) and makes the problem quite different from the Stefan problem and its numerous variants studied in the literature. The constant in the second member of (5) is put to 1 as a normalization and can be replaced by any positive number by a simple rescaling. In more general versions of the problem it is replaced by a fixed positive function of x and t. Initial conditions like (4) are also needed. Here we will refer to this problem as Problem (FGP).

Assuming that we have smooth initial data in Ω_0 which are continuous up to the smooth boundary and take the value 0 on $\Gamma_0 = \partial\Omega_0$, we understand by *classical solution* of Problem (FGP) a smooth surface Γ which starts from $\Gamma_0 = \partial\Omega_0$ and a smooth function u defined in a domain Ω with lateral boundary Γ and satisfying (1), (2), (4), (5). For brevity when the context is clear we will simply say that u and not (Ω, u) is the solution. Classical solutions to problem (FGP) in one dimension are relatively easy to construct until the possible occurrence of certain singularities which can be described. The problem is much more difficult in several space dimensions so that a concept of weak solution is needed and has been introduced in [CV].

The article presents a report of progress obtained to this date and known to the author. It also discusses the main applications which use this model under suitable physical assumptions. These applied problems were formulated decades ago, but the time was not ripe then for a complete and rigorous analysis. I will present some highlights of the mathematical development, discussing in particular the existence and properties of classical, weak and limit solutions and the methods of constructing solutions. The text touches also the questions of uniqueness, asymptotic behaviour, relation with other equations and alternative mathematical formulations.

2 The combustion model

The above problem arises in a quite natural way in combustion theory to describe the propagation of curved premixed equi-diffusional deflagration flames in the limit of high activation energy. This is an asymptotic method which simplifies the complicated system of nonlinear equations describing the process of combustion on the basis of physically sound approximations, making it thus amenable to further qualitative analysis. As in all simplified models it is of great importance to keep in mind the approximations involved when analyzing the meaning of the mathematical results. Therefore, it will not be useless to review the main lines of the well-known derivation of the thermo-diffusive model for flame propagation together with the high activation energy asymptotics. The main assumption in the present modelization is that of taking to the limit the activation energy of the chemical reaction.

As explained in the classical textbooks, the problem of flame propagation can be written in its generality as a system of PDE's (conservation laws) for the variables ρ, density of the mixture, $\mathbf{v}$, velocity, p pressure, T temperature, and Y_α, the mass fractions of the different combustible components and products, connected also by suitable constitutive relations. Obviously, the chemistry is a crucial part of the problem and usually many, even hundreds of reactions are involved, which complicates enormously the analysis. But in some basic respects such a complication is not necessary and as a first approximation we will consider the simple case of one fuel and one oxidizer giving one product according to the overall irreversible chemical reaction

$$F + \nu_O O_2 \to (1+\nu_O)\, P + (q), \tag{6}$$

where a mass ν_O of oxygen is consumed per unit mass of fuel to yield a mass $1+\nu_O$ of products plus a thermal energy q. The system is written as follows. First, we have the hydrodynamic equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{7}$$

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) = -\nabla p + \nabla \cdot \tau', \tag{8}$$

where we have neglected the effect of gravity and τ', the viscous stress tensor is given by the Navier-Stokes law. We then have the energy balance equation for the common temperature of the premixed flame

$$\frac{\partial}{\partial t}(\rho c_p T) + \nabla \cdot (\rho c_p \mathbf{v} T) = \nabla \cdot (k \nabla T) + q W_F. \tag{9}$$

According to the *isobaric approximation*, usually accepted in the description of slow deflagrations, pressure variations can be neglected but for the momentum equation (8). In particular, the corresponding work terms have been neglected in the second member of (9). Besides, one usually writes $k = \rho c_p D_T$, where D_T is the thermal diffusivity. Moreover, we have the component equations

$$\frac{\partial}{\partial t}(\rho Y_\alpha) + \nabla \cdot (\rho \mathbf{v} Y_\alpha) = \nabla \cdot (\rho D_\alpha \nabla Y_\alpha) - W_\alpha. \tag{10}$$

To this we have to add the law of state

$$p = \rho R T / M \tag{11}$$

and the chemistry is governed by the Arrhenius law

$$W_F = \rho B Y_F^m Y_0^n e^{-E/RT}, \qquad W_O = \nu_O W_F, \tag{12}$$

where m and n are the orders of reaction. We will see later that the very precise form of the Arrhenius reaction term will not be important for our study, only its asymptotic properties will matter. In formulas (9)-(12) c_p, q, D_T, R, M, B, m, n and E will be considered as functions of T, Y and ρ, but such dependence is not crucial and is neglected in this modelization. The last constant, called the *activation energy* of the reaction, will play a fundamental role in what follows. Actually, the difference in activation energies selects the reactions that really matter in the flame description in a many-step reaction chain, by eliminating those with too small or too large relative energies because their processes are too fast or too slow. Another important constant is the relation between the thermal diffusivity and the diffusivity of the species α

$$Le_\alpha = \frac{D_T}{D_\alpha}. \tag{13}$$

It is the called the *Lewis number* and is also a crucial parameter in the combustion process. For more details on the formulation cf. [BL], [W], [Z4]. In this generality the problem is mathematically too difficult, but cf. [La].

A way to perform a further mathematical investigation and derive the basic qualitative properties of the solutions consists in passing to a limit situation. In 1938 Zeldovich and Frank-Kamenetski, [ZF], proposed to make a limit analysis for flame propagation for very large E, taking into account the high sensitivity of the Arrhenius factor of the chemical reaction with respect to the temperature when the activation energy is very large. These methods have been very popular in engineering in the study of both *premixed* and *diffusion flames*, and are explained in [Ba], [Z4], [BL], [W] or [Li]. Only recently a mathematically rigorous mathematical investigation has been addressed on those issues.

We want to describe a premixed deflagration flame in which a homogeneous mixture of fuel and oxygen is attained. We are interested in the situation of a one-species reaction and we will now assume moreover that the amount of oxygen is considered so large, $Y_O \approx 1$, that only the amount of fuel matters, hence only the fraction $Y_F = Y$ is of importance (deficient species). We can then write a system for the temperature and the fuel mass fraction in the form

$$\rho(T_t + \mathbf{v} \cdot \nabla T) - \nabla \cdot (\rho D_T \nabla T) = A\rho Y^m e^{-E/RT} \tag{14}$$

$$\rho(Y_t + \mathbf{v} \cdot \nabla Y) - \nabla \cdot (\rho D_F \nabla Y) = -B\rho Y^m e^{-E/RT}. \tag{15}$$

Here $A = qB/c_p$. The analysis of system (14)-(15) under appropriate initial and boundary conditions is one of the main subjects of current investigation in the mathematical theory of combustion, both for Lewis number $Le = D_T/D_F$ unity or different from one. It is apparent that the hypothesis of equi-diffusion, i.e. $Le = 1$, will simplify considerably the analysis, in fact much more progress has been achieved under this assumption. In the present generality the hydrodynamics appears through the variables $\mathbf{v}$ and ρ. Under the assumption of almost constant pressure the law of state (11) allows to explicit ρ as a function of T alone and the hydrodynamics equations can be solved independently giving $\mathbf{v}$ as a function of x and t.

Let us proceed now under the *assumption of equidiffusion*. We introduce the enthalpy function

$$H = T + \frac{q}{c_p}Y, \tag{16}$$

which for $Le = 1$ satisfies the equation

$$\rho(H_t + \mathbf{v}\cdot\nabla H) - \nabla\cdot(\rho D\nabla H) = 0, \tag{17}$$

where $D = D_T = D_F$. Assuming that we work in the whole space, $x \in \mathbf{R}^N$, and assuming that the initial enthalpy is constant, $H(x,0) = H_0$, we obtain the *Lewis-Elbe* law

$$H(x,t) = H_0 \tag{18}$$

for all x and all t. This is a further simplification from the real problem. It can be justified for non-homogenoeus enthalpy when its relaxation time is much smaller than the characteristic times of the main process. In any case, it allows us to explicit Y in terms of T,

$$Y = \frac{c_p}{q}(H_0 - T). \tag{19}$$

and substitute in (14) to get an equation for T:

$$\rho(T_t + \mathbf{v}\cdot\nabla T) - \nabla\cdot(\rho D_T\nabla T) = \rho f(T), \quad f(T) = d(H_0 - T)^m e^{-E/RT}. \tag{20}$$

We have accomplished the first stage of our modelization, obtaining a single equation for T, though it still contains the variables ρ and $\mathbf{v}$. Let us recall that we are assuming that $0 \le T \le H_0$. The reaction function in the second member of (20) is defined and positive for $0 < T < H_0$ with $f(T) = 0$ for

$T = H_0$ and $T = 0$. Moreover, $f(T)$ has a single maximum for some value $T_* < H_0$.

Our next step will be to investigate the limit situation when the activation energy $E \to \infty$, i.e. when a certain distinguished limit is taken in the reaction function $f(T)$ of (20). In doing this it is important to investigate first the limit for travelling wave solutions which leads to problem (FGP).

3 Travelling wave analysis

A plane travelling-wave solution is most easily realized in a tube setup, where we assume an infinitely long tube filled with fuel and oxygen under the above assumptions, with initial hot temperature, H_0, on one end, say at $x = \infty$, and a fresh cold mixture at $T = T_0 < H_0$ on the other end, $x = \infty$. We then look for plane travelling-wave (TW) solutions, of the form

$$T(x,t) = T(\xi), \quad \xi = x_1 + ct, \tag{21}$$

where c is the speed of the wave. The flame will travel from $x = \infty$ to $x = -\infty$ with an increasing temperature profile from the cold end to the hot end. Accordingly, the fuel concentration, given by (19), has a decreasing profile along the ξ-axis, going from $Y = (c_p/q)(H_0 - T_0)$ at $x = -\infty$ to $Y = 0$ at infinity. This represents the fact that at the hot end the flame has depleted the fuel (*burnt zone*) and reaction stops. It is also easy to see from the form of the reaction function that $f(T)$ will have a maximum at an intermediate zone, where most of the reaction takes place (*reaction zone*). The zone to the left is called the *fresh zone*.

Following [ZF], see also [Ba], equations (20), (7) are written for a TW as

$$\rho(v+c)T' = (\rho D T')' + \rho f(T), \tag{22}$$

$$(\rho(v+c))' = 0, \tag{23}$$

where primes denote differentiation with respect to ξ (or x_1 if you like). The last equation implies $\rho(v+c)$ =const $= \rho_0 c$, with ρ_0 the density at $x = \infty$. Then

$$c\rho_0 T' = D(\rho(T)T')' + \rho(T) f(T). \tag{24}$$

When we try to calculate the form of the TW it is convenient to introduce the functions

$$U(\xi) = \frac{D}{\rho_0}\int^{\xi} \rho(T(s))T'(s)ds, \quad \Phi(\xi) = U'(\xi). \tag{25}$$

so that equation (24) is analysed as the system

$$D\frac{\rho}{\rho_0}T' = \Phi, \tag{26}$$

$$\Phi' = cT' - \frac{\rho}{\rho_0}f(T), \tag{27}$$

whose orbits in the phase plane (T, Φ) are given by the equation

$$\frac{d\Phi}{dT} = c - \frac{D\rho}{\rho_0^2}\frac{f(T)}{\Phi}. \tag{28}$$

We want to find a connection from $(T = T_0, V = 0)$ to $(T = H_0, V = 0)$ which lies in the $\Phi \geq 0$ half-plane. Given the form of the second-member no such connection exists if $T_0 > 0$! This is the so-called *cold boundary difficulty*, and is due to the form of the reaction term and the simplifications of the problem, that does not admit a complete TW as solution (though such solutions are actually experimentally observed). Since the TW is important as asymptotics in the reaction zone, a simple correction has been found to allow for the existence of a TW. It consists in *cutting the tail* of function f, which becomes a function with compact support in an interval $T_1 = H_0 - h \leq T \leq H_0$. T_1 is the ignition temperature and h measures the temperature-range in which we take into account the reaction effect, which is switched off for values of T less than T_1. The reader will easily show that, after this correction, equation (28) admits a TW connecting $(H_0, 0)$ with a point of the horizontal axis. This point depends continuously from c. Inversely, we can calculate for given reaction function f and temperature jump $H_0 - T_0$ the speed of the corresponding TW, c.

Zeldovich and Frank-Kamenetskii [ZF] understood that a simple and the same time representative analysis of the TW behaviour can be performed in the limit where we assume that the reaction function f concentrates in a narrow zone near $T = H_0$, i.e., when h is very small. In that case, putting $u = H_0 - T$, we can separate the analysis into 2 zones, the left-hand side where $u > h$ and reaction does not occur (fresh zone) and the right-hand side where $\beta(u) = f(T)$ acts but u is very small, $0 < u < h$. A most important quantity in this analysis will be the total reaction energy

$$M = \int \beta(u)du. \tag{29}$$

Morever, we disregard in first approximation the density variations, $\rho \approx \rho_0$ (*constant density approximation*). The reader will observe that this assumption is not an essential qualitative restriction in the analysis that follows.

In the first zone which we assume to be the interval $\{-\infty < \xi < 0\}$ we have to solve the problem

$$Du'' = cu', \qquad u(0) = h, u(-\infty) = A, \tag{30}$$

(where $A = H_0 - T_0$), which gives as solution

$$u(\xi) = A(1 - e^{\lambda\xi}) + he^{\lambda\xi}. \tag{31}$$

with $\lambda = c/D$. On the other hand, for $\xi > 0$ we have to solve $Du'' - cu' = \beta(u)$. Multiplying by u' and integrating from ξ to ∞ we have

$$\frac{D}{2}(u'(\xi))^2 + c\int_\xi^\infty (u')^2 d\xi = \int_0^h \beta(u)du. \tag{32}$$

Now, the last quantity is just the total β-integral, M. Hence, we obtain the estimate of the slope at the zone transition, $\xi = 0$:

$$Du'(0)^2 = 2M - 2c\int_\xi^\infty (u')^2 d\xi. \tag{33}$$

Now, for a very concentrated function β with constant energy M the last integral can be disregarded to arrive at the asymptotic formula for the slope.

$$|u'(0)| \approx \frac{\sqrt{2M}}{D}. \tag{34}$$

Together with the C^1 agreement of both zones and formula (31) we arrive at the limit value for the wave speed

$$c = \frac{\sqrt{2M}}{AD}. \tag{35}$$

In fact, this limit process can be taken by considering a sequence of reaction functions β_ε with same total energy M but such that

$$\beta_\varepsilon(s) \to M\delta(s), \quad \text{as } \varepsilon \to \infty. \tag{36}$$

In the limit $\varepsilon \to 0$ in the previous TW analysis we observe that the zone $\xi > 0$ attains a constant value $u = 0$ (i.e., $T = H_0$, it is a burned zone), and the effect of the reaction concentrates on the point $\xi = 0$ in the form of a *fixed gradient jump* given by formula (34) and which is *not* related to the wave speed. In (x, t) variables the reaction zone is the line $x = -ct$ and (34) is just the gradient jump condition (5) we were looking for.

Additional Comments. 1) It is to be noted that the division into 3 zones thus obtained and in particular the analysis of the thin reaction zone does not depend on very specific properties of the reaction term, only the concentrated character and the value of the integral M really matter. Besides, it is not a particular feature of plane travelling waves, but it is a valid paradigm for more general curved configurations. This is explained in [Ba], [Z4], by means of an asymptotic analysis with slow and fast times. For the analysis of the internal layer cf. [Fi].

2) This problem is closely related to the famous KPP problem, [KPP], [Fs], studied at the same time (1937), and originated in population dynamics. There is however an essential difference in the assumptions, since the reaction term in the KPP problem is not concentrated and morevoer $\beta'(u)$ has a maximum for $u = 0$. The conclusion is that the *minimal* wave speed is given by a different formula (with $D = A = 1$)

$$c = \sqrt{2\beta'(0)}. \tag{37}$$

On the other hand, and this important for our presentation, there is no room for a jump condition like (34).

3) The above one-dimensional limit process was rigorously studied in [BNS] (1985), where also the limit for the system of equations for T and Y without the equidiffusion assumption is analysed. Work on many-dimensional TW's has been continued by Berestycki, Larrouturou and their collaborators, cf. [BrL], see also [Ve] and [Vo] and their references. Typically, a curved flame $T(x, y)$ is considered in $S = \{(x, y) \in \mathbf{R}^2, 0 < y < L\}$, solution of

$$\Delta T - \alpha(y)T_x + g(T) = 0. \tag{38}$$

For more information on the cold boundary difficulty we refer to [BLR]. The analysis of the (T, Y) system with variable densities and diffusivities is done in [Ro], where its connection is shown with the equations of nonlinear diffusion. The lack of stability of the plane travelling waves for the (T, Y) system with Lewis number less than one is an important research problem studied by Sivashinski that falls out of the scope of this lecture, cf. [Si].

4 The existence of solutions with general data

We have seen the asymptotic limit derived for TWs in the combustion model by letting $\varepsilon \to 0$, i.e., $E \to \infty$. This is the origin of the name *high activation energy asymptotics*. The transition line is called the flame front, which in the limit becomes an inifintely thin zone, i.e. a surface, which for a plane TW solution takes the form $x_1 + ct$ =constant. Such TWs are particular cases of the free boundary problem (FGP), i.e., (1), (2), (4), (5), where u is rescaled so that the slope $\sqrt{2M}$ becomes 1 and D is also put to 1.

Our next objective is to show that the FB problem (FGP) is attained in the limit of high activation energy in one or several space dimensions for solutions with general initial data, which not necessarily have the TW structure. This project was undertaken by Caffarelli and Vazquez, [CV], and solutions of Problem (FGP) were obtained in the limit of simplified problems (P_ε) of the form

$$u_t = \Delta u_\varepsilon - \beta_\varepsilon(u_\varepsilon) \quad \text{in } Q_T, \tag{39}$$

with initial conditions

$$u_\varepsilon(x,0) = u_{0\varepsilon}(x)\,, \tag{40}$$

where the $u_{0\varepsilon}$ are C^∞-smooth and nonnegative approximations of u_0, and then taking $\varepsilon \to 0$. This generality is assumed for mathematical convenience; actually, we may take $u_{0\varepsilon} = u_0$ fixed with ε. Equation (39) is just equation (20) with $u = k(H_0 - T)$ as in (19), after assuming the constant density approximation and neglecting convection terms in comparison with the main process of reaction and diffusion (thermo-diffusive model).

Regarding the reaction term we assume that the functions $\beta_\varepsilon : \mathbf{R} \to \mathbf{R}$ are C^∞-smooth, nonnegative and bounded, with $\beta_\varepsilon(s) = 0$ for $s \le 0$ and support in a small neighbourhood of $s = 0$ of size $O(\varepsilon)$. We will keep a constant heat production M, normalized to $1/2$. A simple way of obtaining that is the following. We define the family β_ε in terms of a single function β by

$$\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})\,. \tag{41}$$

This is very convenient in order to use scaling arguments. We will assume that the function $\beta : \mathbf{R} \to \mathbf{R}$ satisfies the following assumptions:

(i) β is positive in the interval $I = \{0 < s < 1\}$ and 0 otherwise,
(ii) it is a C^∞ function in $[0, \infty)$,
(iii) it is increasing for $0 \le s < 1/2$, decreasing for $1/2 < s \le 1$,
(iv) the integral of β, $\int \beta(s)\,ds$, equals $1/2$.

With this choice, and if the TW analysis has general validity, formula (34) will lead precisely to the jump condition (5). Observe that the term $\beta_\varepsilon(u)$ acts as an absorption term in equation (39). Since $T = H_0 - (u/k)$, it is in fact a reaction term for the temperature, representing the effect of the exothermic chemical reaction.

Let us recall that we want problem (P_ε) to approximate the FB problem (FGP). In doing that a certain liberty exists in choosing the initial data $u_{0\varepsilon}$ and the absorption functions β_ε. But there are definite constraints. One of the difficulties we face consists in finding a solution with a free boundary Γ which starts from the initial boundary Γ_0. Now, if for example we take compactly supported initial data u_0, functions β_ε with support in the interval $[0, \varepsilon]$ and approximations $u_{0\varepsilon}$ to the data such that $u_{0\varepsilon} \ge \varepsilon$, it clearly follows that the absorption term has no effect and we obtain just positive solutions of the heat equation in Q with no free boundary. Such a difficulty is already studied in the stationary case by Berestycki, Caffarelli and Nirenberg [BCN]. In order to represent in the limit the FB problem $\beta_\varepsilon(s)$ has to be roughly speaking concentrated in a right neighbourhood of $s = 0$ and its mass $M = \int \beta_\varepsilon(s)\,ds$ has to be directly related to the value $u_\nu = -1$ that we seek to obtain on the free boundary in the limit.

5 Limit solutions and weak solutions

Problem (P_ε) admits a unique classical solution $u_\varepsilon \in C^\infty(Q)$, $Q = \mathbf{R}^N \times (0,\infty)$, which is positive everywhere in Q. The Maximum Principle holds. We want (P_ε) to approximate (FGP) as $\varepsilon \to 0$. In the limit $\varepsilon \to 0$ we want to obtain a

$$u(x,t) = \lim_{\varepsilon\to 0} u_\varepsilon(x,t). \tag{42}$$

This is called a *limit solution*. The question is now *whether u solves the free boundary problem (FGP) and if it does, in which sense?* Uniqueness is the next basic concern.

We want to obtain a classical solution of (FGP) if possible, but the existence of classical solutions will not be guaranteed for general initial data. A natural *weak formulation* for problem (FGP), as introduced by [CV], asks for a domain $\Omega \in \mathbf{R}^N \times (0,T)$ with Lipschitz continuous lateral boundary Γ, and a function $u \in C(\Omega \cup \Gamma)$ such that:

(i) for every test function $\phi \in C_0^\infty(\mathbf{R}^N \times [0,T))$

$$\int\int_\Omega u\,(\phi_t + \Delta\phi)\,dx\,dt + \int_{\Omega_0} u_0\,\phi\,dx = \int_\Gamma \phi\,d\Sigma\,\cos\alpha\,, \tag{43}$$

(ii) u vanishes on Γ, and

(iii) the free boundary Γ starts from $\Gamma_0 = \partial\Omega_0$, i.e the section Γ_t at time t converges to Γ_0 as $t \to 0$ in some sense.

In (43) $d\Sigma$ is the area element on Γ and α is the angle formed by the exterior normal $\nu(x,t)$ at a point $(x,t) \in \Gamma$ and the hyperplane $t =$ constant, so that $dS = d\Sigma\,\cos\alpha$ is the space projection of the element $d\Sigma$. The reader will have no difficulty in checking that a classical solution is a weak solution in the sense just defined.

Corresponding definitions apply to the problem posed in a proper subspace D of $\mathbf{R}^N$, bounded or unbounded. Then, Ω is sought as a subdomain of $D \times (0,T)$ and boundary conditions have to be given on the fixed lateral boundary, either of Dirichlet or Neumann type. A typical configuration is a cylindrical domain of the form $D = \omega \times \mathbf{R}$ with homogeneous Neumann (no flux) conditions on $S = \partial\omega \times \mathbf{R}$. The set of test functions in formula (43) has to be changed accordingly.

The main result of [CV] consists in proving that limit solutions exist and under certain conditions they are weak solutions of (FGP).

Theorem 1 *(i) Let u_0 be a Lipschitz continuous, bounded and nonnegative function in $\mathbf{R}^N$. Then the appoximation of previous section produces in the limit (42) a continuous function $u(x,t)$, solution of the heat equation in its positivity set, Ω. The function u is Lipschitz continuous in x and $C^{1/2}$-Hölder continuous in t. It is C^∞ smooth in x and t in Ω.*

(ii) Under the assumption that Δu_0 be strictly negative in the closed set $\overline{\Omega_0}$, and $|\nabla u_0|$ be less than 1 at the boundary of Ω_0, then the limit solution is a weak solution with free boundary Γ given by a Lipschitz continuous function $t = \tau(x)$. At all regular points of the free boundary (i.e., almost everywhere in x for almost every t) we have the boundary condition $\partial_n u = -1$.

The proof of the first part is based on a priori estimates of three kinds: a) integral estimates, b) a Bernstein estimate for ∇u, and c) a $C^{1/2}$-estimate for u_t. The passage to the limit produces a smooth solution of the heat equation in the positivity set Ω and the second member converges to a singular Borel measure supported on the lateral boundary of Ω. Identifying this measure as equivalent to the gradient jump condition (5) is the most delicate part and the proof is only performed in [CV] under the above-listed additional conditions, which are selected to imply $u_t > 0$ in Ω (contracting flame). Let us remark that the main idea of that restriction is monotonicity, and the proof is equally valid if we ask that Δu be strictly positive in Ω_0 and $\partial_n u_0$ be larger that 1 (so that $u_t > 0$ in Ω).

Subsequently, Caffarelli [C] has shown that the Lipschitz bound on spatial derivatives ∇u can be derived for general initial data as a consequence of a powerful *monotonicity formula*.

6 Typical examples of classical solutions

• The plane TWs studied in Section 3 are examples of classical solutions of Problem (FGP). Let us recall their form. Assume without loss of generality that $N = 1$ and they are monotone decreasing. Then they are given by

$$u(x,t) = \frac{1}{c}(1 - e^{c(x+ct)})_+ \,, \tag{44}$$

when they travel towards the negative x-axis (the free boundary recedes towards the support; in combustion terms, the flame advances). On the contrary, for progressing free boundary

$$u(x,t) = \frac{1}{c}(e^{c(ct-x)} - 1)_+ \,. \tag{45}$$

For $c = 0$ we get the stationay profile $u(x,t) = (-x)_+$. All these solutions have a flat free boundary. As usual the symbol $(\cdot)_+$ means positive part.

• An example with curved free boundary is constructed in [CV] in the form of a *self-similar solutions* of the form

$$u(x,t) = (t_1 - t)^{1/2} f(|x|/(t_1 - t)^{1/2}). \tag{46}$$

For every $t_1 > 0$ precisely one such solution with compact support is constructed, it solves (FGP) in the classical sense and vanishes identically at time T. Such a solution is the unique limit of the solutions of the approximate problems (P_ε). Moreover, a stability result is established. The free boundary has the form

$$\Gamma = \{(x,t) : 0 \le t \le t_1,\ |x| = R\sqrt{t_1 - t}\}. \tag{47}$$

The self-similar solutions extended by zero outside of the support are globally Lipschitz functions for $0 \le t < t_1$ with maximum spatial gradient 1 taken at the free boundary. At the extinction point $(x = 0, t = t_1)$ the regularity decreases however, it is only $C^{1/2}$. Besides, it is important to remark that near extinction the free boundary has vanishing curvature radii, which are then comparable to the thickness of the transition zone even for large E, so that the model loses its validity as an asymptotic limit. It conserves however its validity as *intermediate asymptotics*, as explained by [Ba]. It is proved in [CV] that the self-similar solutions (46) are indeed the unique limit solutions of the corresponding approximate problems, and that the Maximum Principle applies to them with respect to other limit solutions. The same applies to the plane TW's. Using the self-similar solutions as comparison terms it is proved in [CV] that

Theorem 2 *If the initial data is compactly supported the limit solution vanishes identically in finite time.*

• Further examples of classical solutions are the *stationary solutions with a hole*. Assume that $N>1$. We have to find a number $R>0$ and a function $u(x)$ defined for $x>R$ such that

$$\Delta u=0 \quad \text{for} \quad |x|>R, \quad u(R)=0, \quad \partial_n u(R)=-1$$

In two space dimensions, $N=2$, such a solution takes the form

$$u(r)=R\log(r/R), \quad r=|x|\,.$$

Suppose now that we fix at $r=1$ the value of the solution, say $u(1)=A$. Then, R is determined by the equation

$$A=-R\log R\,.$$

This equation does not admit a solution if $A>A_*=1/e$, it has a unique solution $R=1/e$ if $A=A_*$ and it has two solutions if $0<A<A_*$, corresponding to a big hole and a small hole. Moreover, the radius of the small hole decreases to 0 as $A\to 0$ while the radius of the large hole goes to 1. The small holes show that we can have stationary solutions as close to 0 as we like in a certain region having holes of small radius.

A similar situation happens in $N\geq 3$. Now the solution with hole of radius $R>0$ takes the form

$$u(r)=\frac{R^{N-1}}{N-2}\Big\{\frac{1}{R^{N-2}}-\frac{1}{r^{N-2}}\Big\}\,.$$

Observe that the solution is now bounded in its domain $\{|x|>R\}$. This is the famous Zeldovich flame (with $N=3$, cf. [Z4]).

7 Kinetic equation on the free boundary

We now analyze the equation describing the movement of the FB. Let us consider a classical solution with smooth free boundary Γ. The kinetic equation is derived by differentiating along a normal spatial direction the equation $u(x,t)=0$ which holds at the free boundary. If the free boundary moves along such direction with velocity $\mathbf{v}$ we get $\nabla u \cdot \mathbf{v} + u_t = 0$, from which, thanks to the boundary condition (5): $\partial_n u = -1$, it follows that

$$\mathbf{v} = \mathbf{n}\, u_t = \mathbf{n}\, \Delta u\,, \tag{48}$$

where $\partial_t u$ and Δu are understood as limits approaching the boundary from the region $\{u > 0\}$). We now observe that for radially symmetric solutions it means (using again the condition $\partial_r u = -1$ on the free boundary)

$$\mathbf{v} = \mathbf{n} \left(\partial_r^2 u\Big|_\Gamma - \frac{N-1}{r(t)} \right). \tag{49}$$

This is in particular true for the self-similar solutions (46). With a bit more of work we prove that for general configurations the formula reads

$$\mathbf{v} = \mathbf{n} \left(\partial_n^2 u\Big|_\Gamma - K(x,t) \right), \tag{50}$$

where $K(x,t)$ is the Gauss curvature at a point of the free boundary as a surface in $\mathbf{R}^N$ for constant t. Formula (50) shows an unexpected relation of this equation with the now famous model of **motion by curvature**, cf. [CGG], [ES], [GH], [Ang].

8 Filtration in compressible porous media

The same type of free-boundary problem in one space dimension was proposed by Florin [Fl] in 1951 in the study of groundwater filtration in compressible media taking into account the effects of connate water and the *modified Darcy law with initial pressure gradient*. The first hypothesis is explained in the modeling by taking into consideration 3 constitutive elements: the free water, the solid matrix and the bound or connate water. Let x represent vertical distance measured downwards from the ground surface, let m, n and s be the respective concentrations and let u, v, w be the respective average velocities. We have $v = w$ since the connate water, located a thin layer next to the solid matrix, moves rigidly attached to it by molecular forces. The second hypothesis says that under those assumptions the standard Darcy law which relates linearly the relative velocity $u - v$ to the pressure gradient $\partial H/\partial x$ (more accurately H is the total pressure head) has to be replaced by the nonlinear Darcy law

$$u - v = \frac{k}{m}\left(\frac{\partial H}{\partial x} - J_0\right), \tag{51}$$

valid whenever $\partial H/\partial x \geq J_0$, and $u = v$ otherwise. $J_0 > 0$ is the initial pressure gradient, also called the *threshold gradient*, below which no flow occurs. Under standard assumptions of filtration theory for which we refer to [Fl] one gets the equation for H in the filtration zone

$$\frac{\partial H}{\partial t} = C\frac{\partial^2 H}{\partial x^2}, \tag{52}$$

for some $C > 0$. This zone takes the form $\Omega = \{0 < x < r(t)\}$, where $x = r(t)$ is the a priori unknown moving interface between the filtration zone and the immobile zone. On this interface we put the conditions

$$H = H_0 > 0, \quad \frac{\partial H}{\partial x} = J_0. \tag{53}$$

We add Dirichlet data

$$H = 0 \quad \text{on } x = 0, t \geq 0. \tag{54}$$

We also add the condition that the interface starts at $t = 0$ at the surface $x = 0$, which eliminates the need for initial conditions on H, but these additional assumptions are not essential for the mathematical treatment. In this way we obtain problem (FGP) after setting $u = H_0 - H$.

Bear [Be, Chapter 5] discusses the above nonlinear Darcy law that he writes in the form

$$\mathbf{q} = K\mathbf{J}(J - J_0)/J \quad \text{for } J = |\mathbf{J}| \geq J_0, \tag{55}$$

with $\mathbf{q} = 0$ for $J < J_0$. Here $\mathbf{q}$ is discharge and $\mathbf{J}$ hydraulic gradient. The need for the correction J_0 appears in different contexts, in particular in connection with fine-grained cohesive soils. Problems with nonlinear laws like (51), (55) are found in viscoplastic filtration, cf. [BER]. The corresponding movement equations in several space dimensions lead to systems of a more complex form than problem (FGP).

Existence and uniqueness of classical solutions for the one-dimensional problem proposed by Florin was proved by Ventsel' [Vn] in 1960. The free boundary and lateral data are allowed to be variable under certain precise conditions. The proof is based on discretization in time which leads to a FB problem of elliptic type, plus an integral representation for the solutions of this problem.

9 A model for the deflagration-detonation transition

In 1983 Lundford and Stewart [SL] proposed a model for the study of fast deflagration waves in order to understand the transition from deflagration to detonation (DDT), which is closely related to our investigation, see also [S]. They still deal with an ideal premixed gas undergoing a one-step Arrhenius reaction in the limit of high activation energy. But now the modelization leads to a one-dimensional two-phase problem. The normalized temperature satisfies Burger's equation

$$T_t = T_{xx} + TT_x, \tag{56}$$

on both sides of a line $x = \zeta(t)$ where the reaction is concentrated, which is a priori unknown but for the initial location ζ_0, and is determined from the two following data: the temperature is prescribed

$$T(\zeta(t), t) = T_s, \tag{57}$$

as well the gradient jump

$$T_x(\zeta(t)+), t) - T_x(\zeta(t)-), t) = -1. \tag{58}$$

We add suitable intial data and contiditions as $x \to \pm\infty$. When $T(x, 0) = T_s$ for $x \geq \zeta_0$ we get the one-phase problem (FGP) after putting $u = T_s - T$ and *disregarding the convective term* TT_x. The existence and stability of travelling waves were studied by Brauner, Lunardi and Schmidt-Lainé in a series of papers, see [BLS] and its references. The question of well-posedness of the (two-phase) Cauchy Problem is solved by Bertsch, Hilhorst and Schmidt-Lainé [BHS]. Their method is based on the consideration that the problem admits an implicit formulation in the form

$$T_t = T_{xx} + TT_x + (H(T - T_s))_x, \tag{59}$$

where H is the Heaviside function. In this way the free boundary disappears and we enter the theory of quasilinear parabolic equations. It is to be noted that such an approach does not work in several space dimensional. A multi-dimensional analysis of the stability of travelling waves is done in [BLS2] in the framework of fully nonlinear parabolic equations.

10 The elliptic-parabolic model for partially saturated porous media

The equation

$$\partial_t c(u) = \Delta u \tag{60}$$

has been proposed to describe fluid flow in a partially saturated porous medium, cf. [Be]. Here u is the hydrostatic potential due to capillary suction, c is the moisture content and the dependence $c = c(u)$ takes the form of a monotone function such that $c(-\infty) = 0$, $c(\infty) = 1$ and morevoer, $c(u)$ is strictly increasing for $u < 0$ and constant for $u > 0$. Hence, the points (x, t) at which $u \geq 0$ correspond to the saturated zone where $c = 1$. In this zone $\partial_t c = 0$ and the equation becomes elliptic, $\Delta u = 0$. On the contrary, for $u < 0$ the flow is unsaturated and obeys the nonlinear parabolic equation

$$\partial_t u = \frac{1}{c'(u)} \Delta u. \tag{61}$$

This problem has been studied in one dimension by van Duyn and Peletier [D], [DP] and by Hulshof in a series of papers, cf. [H] and its references. Suppose for definiteness that the problem is posed for $0 < x < 1$ and $0 < t < T$ with boundary conditions

$$u_x(0, t) = 0, \quad u_x(1, t) = f(t) > 0, \tag{62}$$

and monotine initial conditions $c_0(x) \in [0,1]$. It is then shown that there appears a continuous curve $x = \zeta(t)$ (the free boundary) separating the saturated and unsaturated zones and on the FB we obtain the conditions

$$u = 0, \quad u_x = f(t). \tag{63}$$

They follow from observing that for $x \geq \zeta(t)$ we have $c = 1$, hence $u_{xx} = 0$, so that u_x is constant, namely the prescribed Neumann data. We have thus obtained a problem very close to (FGP) with the correct free boundary conditions. The only difference is that for (FGP) the function c has to be taken as

$$c(u) = 1 - u \quad \text{for } u \leq 0. \tag{64}$$

This causes only minor mathematical changes. A complete theory of existence, uniqueness, comparison and regularity is developped in the framework of the elliptic-parabolic theory for equation (60), [H]. See [HH1] for convergence to travelling waves.

However, the problem in several space dimensions has in principle nothing to do with our free boundary problem (FGP), though we will show in Section 13 that it is possible to use a method of extension of the FB problem to an elliptic-parabolic form for radially symmetric solutions.

11 Relation with the Stefan problem. Undercooled solutions

In one space dimension there is a simple connection of our problem with the standard Stefan Problem (SP) described in Section 1. Namely, suppose that we have a solution of problem (FGP) with ambient space $D = \mathbf{R}_+$, positivity domain $\Omega = \{(x,t) : 0 < x < \zeta(t)\}$ and assume also we take Neumann data on the fixed lateral boundary $x = 0$:

$$u_x(0,t) = g(t). \tag{65}$$

For instance, we can take the restriction to $x > 0$ of solutions defined in the whole line under the assumption of symmetry in the x variable and then the lateral condition is $g(t) = 0$ (think of the self-similar solutions (46)). We take as new function

$$w(x,t) = -1 - u_x(x,t). \tag{66}$$

It safisfies equation (1) with corresponding initial data and Dirichlet data on the fixed lateral boundary:

$$w(0,t) = g(t). \tag{67}$$

Let us check the conditions on the moving boundary $\{x = \zeta(t)\}$. Firstly, the gradient condition (5) implies a Dirichlet condition $w = 0$. The second condition is derived from the kinetic equation of Section 7. Thus,

$$\mathbf{v} = u_{xx} = -w_x, \tag{68}$$

which is precisely the kinetic condition (3). We obtain in this way a solution of the Stefan problem. There is only one caveat to that construction. It is not a priori clear that $w \geq 0$, so that it can happen that we will have a nonstandard from of the problem. The condition $w \geq 0$ will be obtained if $w \geq 0$ on the fixed parabolic boundary, i.e. if $u_x(x,0) \leq -1$ and $g(t) \leq -1$. Now, if $u_x \geq -1$ everywhere in the fixed parabolic boundary we will have a case of **undercooled Stefan problem**, $w < 0$ in Ω. This is precisely what happens for the self-similar solutions (46)! In the more general case where w changes sign inside Ω we cannot interpret the problem in terms of the two variants of the (SP) mentioned.

Inversely, using formula (66) in this type of domain we can produce solutions of problem (FGP) from standard or undercooled solutions of (SP). The only crucial points are: (i) checking that u is zero (i.e, constant, we can always normalize the value). Indeed, we have

$$\frac{d}{dt} u(\zeta(t), t) = 0, \tag{69}$$

as a consequence of the equation and the SP free-boundary conditions. (ii) Checking that u is positive in Ω. This will depend again on the data of w.

A similar analysis applies to solutions with one interface whose positivity domain extends to $x = -\infty$. Let us take a brief look at the more general one-dimensional case with two interfaces where we have a solution of (FGP) defined in a domain $\Omega = \{(x,t) : \sigma(t) < x < \zeta(t)\}$, not necessarily symmetric, formula (66) produces as before a solution of the heat equation with correct conditions for an (SP) problem on the right-hand side interface $x = \zeta(t)$. On the left-hand interface $x = \sigma(t)$ we have

$$w = -2, \quad w_x = \mathbf{v}_l, \tag{70}$$

the velocity of movement fo the left interface. We have a model for two phase transitions at 'temperatures' $w = 0$ and $w = -2$.

There is *no* obvious relation between problems (FGP) and (SP) in several space dimensions, even in the presence of radial symmetry.

12 Existence of classical solutions in N dimensions

In [M2] Meirmanov extended Ventsel's results to local-in-time existence for the following problem in two space dimensions. He considered the equation

$$\partial_t \theta - \sum_{i,j} \partial_i (a_{ij}(x,t,\theta).\partial_j \theta) + a(x,t,\theta,\nabla\theta) = 0, \tag{71}$$

in a two-dimensional domain of the *special form* $G(t) = \{(x,y) : 0 < x < 1, 0 < y < R(x,t)\}$ with periodic conditions in the x variable, a condition $\theta = f(x,t)$ on the bottom boundary $y = 0$, given initial conditions $\theta = \theta_0(x,y)$ in the initial support and conditions $\theta = 0$ and

$$\sum_{i,j} a_{ij}\partial_i\theta.\partial_j\theta = g(x,t) \geq a_0 > 0 \tag{72}$$

on the free boundary $y = R(x,t)$. Under suitable assumptions on the data a classical solution of this free boundary is obtained for a small time $0 < t < T_*$. But when the equation is the heat equation, the bottom data are constant > 0 and g and θ_0 satisfy very precise assumptions the solution is *unique and global in time*. The method follows his well-known proof of classical solutions for the Stefan problem.

Andreucci and Gianni [AG] prove local in time existence and uniqueness for a two-phase problem that generalizes the above formulation. Namely, they propose to find a decomposition of the ambient domain G into two domains G_+ and G_- varying with time and separated by a smooth surface S. In each domain an equation of the above type is to be solved. Initial data are to be given as well as boundary conditions on the fixed (external) boundary. On the free boundary we have the conditions: $u = 0$ and the gradient jump condition

$$[|\partial_n u|] = \partial_n^+ u - \partial_n^- u = 1. \tag{73}$$

Galaktionov, Hulshof, Vazquez, [GHV] solve the problem of existence and uniqueness of radially symmetric solutions supported in a ball. Classical solutions are produced that exist until they vanish identically at the origin of coordinates. The proof proceeds in two steps. In the first the *elliptic-parabolic theory* is adapted to several dimensions in the form of a non-standard boundary value problem. In order to keep the notation of the references we will work in this development with nonpositive solutions, just changing u into $-u$, which is anyway closer to the proposed physical model where temperatures in the fresh zone are below the critical temperature. In doing this we have to replace the radial free-boundary condition by

$$u(\zeta(t), t) = 0, \quad u_r(\zeta(t), t) = 1, \tag{74}$$

where $r = \zeta(t)$ denotes the interface. The crucial observation is that radial (negative) solutions can be naturally extended as solutions to an *elliptic-parabolic* problem on a large fixed ball containing the supporting balls for all $0 \le t \le \tau$. This leads to a problem of mixed type in a fixed domain, which is easier to solve. This is done as follows. We take $N \ge 3$, the adaptations to perform in case $N = 2$ are clear. Consider the equation

$$(c(u))_t = \Delta u, \tag{75}$$

where $c(s) = \min\{0, s\}$, and suppose that u is a solution. Then u solves the heat equation if $u < 0$, while for $u > 0$ it is harmonic in x. We can extend u to the region $r > \zeta(t)$ by setting $\Delta u = 0$ there, which by (1.1) implies that

$$(r^{N-1} u_r(r, t))_r = 0 \quad \Rightarrow \quad u_r(r, t) = (\frac{\zeta(t)}{r})^{N-1},$$

hence

$$u(r,t) = \int_{\zeta(t)}^{r} \left(\frac{\zeta(t)}{\rho}\right)^{N-1} d\rho = \frac{\zeta(t)}{N-2}\left[1 - \left(\frac{\zeta(t)}{r}\right)^{N-2}\right].$$

This turns u into a radial solution of (75) which has a jump in u_t and Δu across the free boundary. Fixing a ball with radius R containing the support of the original solution for all $t \in [0,T]$, we obtain nonlocal boundary conditions on ∂B_R of the form

$$u_r(R,t) = (\zeta(t)/R)^{N-1} \quad \text{(Neumann)}, \tag{76}$$

and

$$u(R,t) = \frac{\zeta(t)}{N-2}\left[1 - \left(\frac{\zeta(t)}{R}\right)^{N-2}\right] \equiv F(\zeta(t),R) \quad \text{(Dirichlet)}. \tag{77}$$

Here $r = \zeta(t)$ is now the a priori unknown level set of $u = 0$. We can eliminate $\zeta(t)$ from these two conditions to obtain

$$u(R,t) = \frac{R}{N-2}(u_r(R,t)^{\frac{1}{N-1}} - u_r(R,t)) = \mathbf{G}(u_r(R,t)), \tag{78}$$

which however is not completely straightforward to work with, in view of the fact that $\mathbf{G}$ is a nonmonotone function of u_r. Both functions F and $\mathbf{G}$ depend on N. An iterative scheme based on the solution of the filtration problems (76) and (77) allows to find a solution of the mixed problem (78). We note that as a function of ζ, function $F(\zeta,R)$ is increasing for $0 \le \zeta < R(N-1)^{-1/(N-2)}$ and decreasing for $R(N-1)^{-1/(N-2)} < \zeta \le R$. We have to choose R appropriately to ensure that we will be in the latter situation, for then, bearing in mind that larger solutions have smaller interfaces, we can set up a monotone iteration scheme.

Theorem 3 *Suppose that v_0 is continuous and radially symmetric on B_R, negative on $\{|x| < \zeta_0\}$, zero on $\{|x| \ge \zeta_0\}$, where $\zeta_0 < R$. Assume that ζ_0 lies in the interval $(R(N-1)^{-1/(N-2)}, R)$ and that $r^{N-1}v_0'(r)$ is bounded. Then there exists $T > 0$ and a unique function $u \in L^2(0,T;H^1(B_R))$ which has a continuous interface $r = \zeta(t)$ such that u is a weak solution to the*

Neumann problem with (76) and to the Dirichlet problem with (77). The weak formulation is equivalent to

$$\int_0^T\int_0^{\zeta(t)}(-r^{N-1}\varphi_t c(u)+r^{N-1}u_r\varphi_r)drdt+\int_0^{\zeta(T)} r^{N-1}\varphi(r,T)c(u(r,T))dr$$
$$=\int_0^{\zeta_0} r^{N-1}\varphi(r,0)u_0(r)dr+\int_0^T \varphi(\zeta(t),t)\zeta(t)^{N-1}dt,$$

for all $\varphi\in H^1(Q_T)$. In this sense the pair (u,ζ) is the unique solution. Moreover, the comparison principle holds: if we have two solutions (u_i,ζ_i), $i=1,2$ and their initial data v_{0i} and ζ_{0i} are ordered, $v_{01}\le v_{02}$ and $\zeta_{01}\ge\zeta_{02}$, then the solutions (and interfaces) are ordered in the same way.

In a second step we use von Mises variables to straighten the free boundary and prove that both the solution and its free boundary are smooth.

Theorem 4 *(i) Suppose Ω_0 is a ball B, and u_0 is radially symmetric, continuously differentiable, zero on the boundary, with normal derivative equal to one. Then the combustion problem has a unique continuous solution on some interval $(0,T]$, which is real analytic for $t>0$.*

(ii) Let $(0,T_m)$ be the maximal open time interval on which a unique analytic solution exists. Then

$$\lim_{t\to T_m}\zeta(t)=0,$$

and the solution vanishes identically at $t=T_m$.

Similar results hold for other configurations, like data supported in an annulus or in an exterior domain, cf. [GHV]. In the latter case the hole may evolve in different ways depending on the initial data: it can stay (as in the Zeldovich flame commented above), it can expand to fill the whole space or it can contract and even disappear in finite time. This last case, called focusing, will be described in Section 14.

It is also proved there that under the conditions of Theorem 6.1 the solution coincides with the limit solutions of the approximate problems (P_ε), as constructed in [CV].

13 The problem of nonuniqueness

Consider radial symmetric initial data in the form of a hump with compact support and a bell-shaped form. As we have just seen, this gives rise to a classical solution in a certain time interval $0 < t < \tau$. If the initial gradient is very large then the support of the solution begins its evolution by expanding. This is easily shown by comparison with suitable barriers. A solution with compact supoort must eventually collapse to the origin, $r(t) \to 0$ as $t \to \tau$. Hence, there is a time of maximum expansion, say $t_1 > 0$, $t_1 < \tau$, with maximum free boundary radius r_1. Since the problem is invariant under x translations the same picture holds after shifting the solution a fixed distance in space.

Now consider the solution of problem (FGP) corresponding to initial data formed by two humps, the former one plus another similar one centered at a point $(x_1, 0, ...0)$. It is clear that for every value of x_1 larger than $2r_1$ a classical solution of the problem is constructed by just superimposing the separate evolutions of the two humps, since they have disjoint supports. Taking limits we easily conclude that for $x_1 = 2r_1$ there is a possible solution obained by superposition, which develops a point of irregular free boundary at $t = t_1$ where the two separate supports make contact. However, they separate later to undergo extinction in the form of two separate balls. This is a first weak solution for the problem with $x_1 = 2r_1$.

There is however a different continuation where the two supports merge at $t = t_1$ and do not separate in the future. This is most easily seen in 1D by considering first the case $x_1 < 2r_1$ where the two supports meet at a time t_1' before t_1 with nonzero speed, so that any physical continuation implies a nontrivial superposition immedaitgely after t_1'. The coincidence of the two flame fronts makes for a singularity that has to be resolved if we want to continue the solution. The natural way is to cancel the collapsing (inner) interfaces and to continue the solution as the solution of the (FGP) with connected support the union of the two intervals and free boundary consisting of the two outer interfaces. In this process we lose two of the four former flame fronts, precisely the two inner ones which recede until they meet and the reaction freezes. This means that we will lose twice the production rate (normalized to 1) in the solution, which will evolve with different mass. There is no difficulty in checking that such a solution is a weak solution of

(P) and is classical for $t \neq t_1'$. Finally, by taking the limit $x_1 \to 2r_1$ we obtain a second solution for the problem with $x_1 = 2r_1$ which coincides the former one for $t < t_1$ but is different for $t > t_1$. In fact, the mass rule gives

$$\int u_2(x,t)dx - \int u_1(x,t)dx = 2, \tag{79}$$

so that even their extinction times are different. Clearly, one of them is the maximal and another one the minimal solution. A more general picture of nonuniqueness based on these ideas is still pending. It also unclear which solutions should be preferred in the different applications for which this problem is only an approximation.

Connected to this problem is the problem of *pulse splitting*. Suppose that in the situation of the previous example with two humps and $x_1 > 2r_1$ we add a very thin sliver of initial data uniting the initial supports along a thin strip $\{x = (x_1, x') : |x'| < \varepsilon\}$. It can be proved that the connected support splits in two after a short time which is a continuous function of the size of the connecting sliver. Detailed proofs will appear elsewhere. A complete understanding of these phenomena is still needed.

14 Asymptotic behaviour, extinction and focusing

We will describe next two asymptotic situations, known as extinction and focusing. Both are terminal situations for a flame, though for opposite reasons. Complete statements and proofs are given in [GHV].

• As we have said, when u_0 is a compactly supported function the solution vanishes in a finite time $\tau = \tau(u_0)$ called the *extinction time*, i.e.,

$$u(x,\tau) \equiv 0 \quad \text{and} \quad u(x,t) \not\equiv 0 \ \text{ for all } t \in (0,\tau). \tag{80}$$

In combustion terms the burnt zone spreads to cover the whole space and the reaction stops by depletion of reactant. The paper [GHV] describes the asymptotic behaviour of the solution as $t \to \tau^-$ near an *extinction point*, x_0, i.e., a point of the set

$$E(u_0) = \{x \in \mathbf{R}^N : \exists\ \{x_n\} \to x \text{ and } \{t_n\} \to \tau^- \text{ such that } u(x_n,t_n) > 0\}, \tag{81}$$

called the *extinction set*, which is nonempty under our hypotheses. Radial symmetric solutions, i.e., spheric flames, are considered. The finite-time extinction process splits into three different cases:

(i) Single-point extinction of radial symmetric solutions, $u = u(r,t)$, $r = |x|$. This happens for solutions whose initial support is a ball, but it can also be an annulus with a small inner hole. The limit profile is given by the self-similar solutions (46). In one space dimension single-point extinction can be analyzed without the assunption of radial symmetry. The symmetric one-dimensional case had been analyzed in [HH2].

(ii) Extinction on a sphere $\{|x| = r_0 > 0\}$ for initial support in the form of a thin annulus. The asymptotic profile corresponds to the one-dimensional problem, the transversal directions do not count in the limit.

(iii) The two previous asymptotic behaviours are stable (under perturbation of the data). There is a still a different type of asymptotics, in the form of a self-similar solution with the same time dependence as (46) but with a support in the form of an annulus. In other words, we have single-point extinction by means of an annular self-similar solution with converging size of the order of $O(\tau - t)$. This solution separates the basins of attraction of the two previous modes of extinction. For details and proof of these results we refer to [GHV].

• A completely different extinction mode happens when the solution loses heat and the flame is frozen in finite time. In more mathematical terms, we mean that the support (fresh zone) expands to cover the whole space. Again, working in a radial situation we consider as paradigm the case of initial data which are positive and increasing outside of a ball and leaves a hole near the origin that shrinks to zero at a time $\tau > 0$. We call this phenomenon *focusing*, because the flame front focuses at the origin.

For every $N \geq 2$ there is a self-similar solution of the form

$$u(x,t) = (t_1 - t)^{1/2} f_1(|x|/(t_1 - t)^{1/2}). \tag{82}$$

where now f_1 is an increasing function with support in the interval $0 < \eta_1 \leq \eta < \infty$ and $f_1(\eta) \sim a_1 \eta$ as $\eta \to \infty$. In 3 dimensions we have explicit values:

$$f_1(\eta) = \frac{1}{2}\left(\eta - \frac{2}{\eta}\right) \quad \text{and } \eta_1 = \sqrt{2},\ \ a_1 = \frac{1}{2}. \tag{83}$$

We can show that this self-similar solution gives the behaviour at focusing for more general (radially symmetric) data. Here is the result proved in [GHV]. Let $N \geq 2$ and let $u_0(r)$ be a smooth, bounded function satisfying

$$u_0 > 0, \;\; u_0' > 0 \;\text{ on } (1,\infty); \;\; u_0(1) = 0; \quad |u_0'| \leq M, \;\; |u_0''| \leq M. \tag{84}$$

Then, there exists a unique solution $u(r,t)$ of the radially symmetric problem with initial data u_0 which exists for a time $0 < t < \tau$ and exhibits *finite-time focusing:* there exists $\tau = \tau(u_0) > 0$ such that the unique interface $r = \zeta(t)$ reaches the origin

$$\lim_{t\to\tau} \inf\, \zeta(t) = 0 \quad \text{and } \zeta(t) > 0 \text{ for } t < \tau. \tag{85}$$

We can establish the following focusing description.

Theorem 5 *As* $t \to T^-$

$$w(\eta,t) \equiv (T-t)^{-1/2} u(\eta(T-t)^{1/2}, t) \to f_1(\eta) \tag{86}$$

uniformly on compact subsets in η*. The interface also converges,*

$$\zeta(t) = \eta_1 (T-t)^{1/2}(1+o(1)) \quad \text{as } t \to T^-, \tag{87}$$

where $\eta_1 = \eta_1(N) > 0$ *is the unique vanishing point of the function* $f_1(\eta)$.

In terms of the original variables $\{u, r, t\}$ we have

Theorem 6 *Under the above assumptions*

$$u(r,\tau) = a_1 r(1+o(1)) \quad \text{as } r \to 0, \tag{88}$$

where $a_1(N) > 0$ *is the constant in the expansion for* f_1 *at infinity.*

Focusing in one dimension is quite different, since then we have two independent support components located initially at $\{x \geq 1\}$ and $\{x \leq -1\}$. Generally, they arrive at the focusing time with a positive, finite speed, so that

$$\zeta(t) = O(t_1 - t), \tag{89}$$

which is in sharp contrast with (87). The methods of proof used in [GHV] combine a priori estimates and the precise knowledge of the self-similar solutions with dynamical systems ideas and non-standard comparison arguments. These methods can have wide applicability to different free boundary problems for other semilinear and quasilinear heat equations admitting finite time extinction or blow-up. They have two main drawbacks: they have not allowed to study asymmetrical evolution (but for $N = 1$) and they are not well suited to tackle systems.

15 Concluding remarks

We have presented a new type of free boundary problem for the heat equation and advanced some of its mathematical properties. Fundamental problems are still open in several dimensions regarding e.g. uniqueness and singularity formation. Further developments presently discussed are viscosity solutions, stability considerations and a general theory for two-phase problems.

We have derived the model in the context of flame propagation at high activation energy and also in groundwater filtration and we have pointed out the main approximations involved. In combustion future and more realistic analyses must involve the (T, Y) system obtained by removing the conditions of Lewis number unity and the constant enthalpy, as well as studying the convergence as $E \to \infty$. It is also of interest to consider nonlinear diffusion instead of the linear heat equation.

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physics of combustion and porous media, as well as the mathematics of free boundaries. Of course, all faults in the presentation lie with the author, who apologizes for any omissions or inaccuracies. It is to be hoped that the mathematical novelty of the subject will attract the attention of future researchers.

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