AN INTRODUCTION TO THE MATHEMATICAL THEORY

OF THE POROUS MEDIUM EQUATION

BY

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ABSTRACT

These notes contain an introduction to the mathematical treatment of the porous medium equation $u_t = \Delta(u^m)$, one of the simplest examples of nonlinear evolution equation of parabolic type, which appears in the description of different natural phenomena related to diffusion, filtration or heat propagation. The notes begin with a discussion of the relevance of the equation and some of its applications. The main body of the notes is devoted to the study of the existence, uniqueness and regularity of a (generalized) solution for the two main problems, i.e. the initial-value problem and the Dirichlet boundary-value problem. Special attention is paid to the appearance of a free boundary, a consequence of the finite propagation property.

Keywords. Porous medium equation; existence and uniqueness; Dirichlet and Cauchy problems; classical, weak and strong solutions; finite propagation; free boundary.

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The aim of these lectures is to provide an introduction to the mathematical theory of the so-called **Porous Medium Equation** (in short PME), i.e. the equation

$$u_t = \Delta(u^m)\,, \tag{0.1}$$

where $u = u(x,t)$ is a scalar function and m is a constant larger than 1. The space variable x takes values in $\mathbf{R}^d$, $d \geq 1$, while $t \in \mathbf{R}$. Physical considerations lead to the restriction $u \geq 0$, which is mathematically convenient and currently followed, but not essential.

The PME is an example of nonlinear evolution equation, formally of parabolic type. In a sense it is the simplest possible nonlinear version of the heat equation. Written in divergence form

$$u_t = \text{div}\,\left(D(u)\nabla u\right), \tag{0.2}$$

we see that the diffusion coefficient $D(u)$ equals mu^{m-1}. It is then clear that the equation is parabolic only at those points where $u > 0$, while the vanishing of D implies that it degenerates wherever $u = 0$. We say that the PME is a degenerate parabolic equation.

There are a number of physical applications where this very simple model appears in a natural way to describe processes involving diffusion or heat transfer. Maybe the best known of them is the description of the flow of an isentropic gas through a porous medium [M]. Another important application refers to heat radiation in plasmas, [ZR]. Other applications have been proposed in mathematical biology, in water infiltration, lubrication, boundary layer theory, and other fields.

In spite of the simplicity of the equation and of its applications, and due perhaps to its nonlinear and degenerate character, a mathematical theory for the PME has been developed only very recently. Though the techniques depart strongly from the linear methods used in treating the heat equation, it is interesting to remark that some of the basic techniques are not very difficult nor need a heavy machinery. What is even more interesting, they can be applied in or adapted to the study of many other nonlinear PDE's of parabolic type. The study of the PME can provide the reader with an introduction to some interesting concepts and methods of nonlinear science, like the existence of free boundaries and the occurrence of regularity thresholds.

To begin with the mathematical treatment of the PME, a first and fundamental example of solution was obtained around 1950 in Moscow by Zel'dovich and Kompaneets [ZK] and Barenblatt [Ba], which found and analyzed a solution representing heat release from a point source, i.e. a *source-type solution.* In fact, the solution has the explicit formula

$$U(x,t) = t^{-\lambda}\left[C - k\,\frac{|x|^2}{t^{2\mu}}\right]_+^{\frac{1}{m-1}}\,, \tag{0.3.a}$$

where $[s]_+ = \max\{s, 0\}$,

$$\lambda = \frac{d}{d(m-1)+2}\,, \quad \mu = \frac{\lambda}{d}\,, \quad k = \frac{\lambda(m-1)}{2\,m\,d} \tag{0.3.b}$$

and $C > 0$ is an arbitrary constant. This solution was subsequently found by Pattle [Pa] in 1959, and is often referred to in the literature as the *Barenblatt*, and also as the *fundamental* or *source-type solution*, because it takes as initial data a Dirac mass: as $t \to 0$ we have $U(x,t) \to M\,\delta(x)$, where M is a function of the free constant C (and m and d).

An analysis of this example shows many of the important features which were encountered later in the general theory. Maybe the most important is the observation that the source-type solution has compact support in space for every fixed time, in physical terms that the disturbance propagates with finite speed. This is in strong contrast with one of the most contested properties of the classical heat equation, the infinite speed of propagation (a nonnegative solution of the heat equation is automatically positive everywhere in its domain of definition). In a sense the property of finite propagation supports the physical soundness of the equation to model diffusion or heat propagation. The occurrence of this phenomenon is a consequence of the degeneracy of the equation, i.e. the fact that the coefficient D vanishes at the level $u = 0$.

The phenomenon of finite propagation gives rise to the appearance of a **free boundary** separating the regions where the solution is positive (i.e. where "there is gas", according to the standard interpretation of u as a gas density, see below), from the "empty region" where $u = 0$. Precisely, we define the free boundary as

$$\Gamma = \partial P_u \cap Q\,, \tag{0.4}$$

where Q is the domain of definition of the solution in space-time, $P_u = \{(x,t) \in Q : u(x,t) > 0\}$ is the positivity set, and ∂ denotes boundary. This free boundary or propagation front is an important and difficult subject of the mathematical investigation.

A second (and related) observation is that, though the source-type solution is continuous in its domain of definition $Q = \mathbf{R}^d \times \mathbf{R}_+$, it is not smooth at the free boundary, again a consequence of the loss of the parabolic character of the equation when u vanishes. In fact, the function u^{m-1} is Lipschitz continuous in Q with jump discontinuities on Γ (i.e., there exists a regularity threshold). On the contrary, the solution is C^∞-smooth in P_u. And we are interested in noting that though u is not smooth on Γ, nevertheles the free boundary is a smooth surface given by the equation

$$t = c\,|x|^{d(m-1)+2}\,, \tag{0.5}$$

where $c = c(C, m, d)$.

The systematic theory of the PME was begun by Oleĭnik and her collaborators around 1958 [OKC], who introduced a suitable concept of generalized solution and analyzed both the Cauchy and the standard boundary value problems in one space dimension. The work was continued by Sabinina, who extended the results to several space dimensions, and Kamin, who began the analysis the asymptotic behaviour. Since the 70's the interest for the equation has touched many other scholars from different countries, notably Angenent, Aronson, Bénilan, Brezis, Caffarelli, Crandall, Dahlberg, di Benedetto, Friedman, Gilding, Kenig, Peletier and Pierre, to quote just a few names which made important contributions to the basic theory.

There exists today a relatively complete theory covering the subjects of existence and uniqueness of suitably defined generalized solutions, regularity, properties of the free boundary and asymptotic behaviour, for different initial and boundary-value problems. Many of these results have been extended to the natural generalizations of the equation, the simplest of them being the so-called *fast diffusion equation*, which is the same equation (0.1) with exponent $m < 1$. More generally, we can consider the general *Filtration Equation*, namely

$$u_t = \Delta\Phi(u)\,, \tag{0.6}$$

where Φ is an increasing function : $\mathbf{R}_+ \mapsto \mathbf{R}_+$. Another extension avenue consists in considering the PME with data of changing sign. Then we have to write the power in a convenient way to account for negative values. The usual choice is

$$u_t = \Delta(|u|^{m-1}u)\,. \tag{0.7}$$

Finally, we should mention that a parallel, sometimes divergent, sometimes convergent story applies to the other popular nonlinear degenerate parabolic equation:

$$u_t = \text{div}\,(|\nabla u|^{p-2}\nabla u)\,, \tag{0.8}$$

which has also attracted much attention from researchers.

These notes are an enlarged version of the course taught at the Université de Montréal in June-July of 1990, aimed at introducing the subject and its techniques to young researchers. The material has been also used for a graduate course at the Universidad Autónoma de Madrid. It is clearly impossible to cover the so many developments occurred in this field in a short set of lectures. Therefore, a selection of topics was necessary. The leading idea in writing these notes has been that of providing an elementary introduction to the questions of existence, uniqueness and the main properties of the solutions, whereby everything is derived from basic estimates using standard Functional Analysis and well-known PDE results. The exposé begins with a reminder of the main applications. Chapters II and III deal respectively with the Dirichlet and Cauchy Problems. For reasons of space the treatment is restricted to integrable data, a sound assumption on physical grounds.

We hope that the material will make it easier for the interested reader to delve into deeper or more specific literature. To guide this further study there exist some expository works. The reader will find an account of many of the main results on the equation in the excellent survey paper by Aronson [Ar], written in 1986. A previous survey was published by Peletier [Pe] in 1981. In his book on Variational Principles and Free-Boundary Problems [F], 1982, Friedman devotes a chapter to the PME. These works contain details of the proofs and techniques. Another contribution, more in the form of a summary but including a discussion of related equations and a very extensive reference list is due to Kalashnikov [Ka] in 1987.

There is now a feeling that maybe the time is ripe for a complete version of the mathematics of the PME and related nonlinear parabolic equations and free-boundary problems. Actually, the outline here described originated as part of a bigger project with Don Aronson to supply a comprehensive and elementary introduction to the PME, in a way a natural continuation of his survey [Ar]. I am very happy to acknowledge my indebtedness to him for sharing with me his expertise in the field and for so many other reasons. I am also grateful to Ph. Bénilan, H. Brezis, L. Caffarelli and S. Kamin for their advice and suggestions.

A comment about formula numbers. Inside the same chapter formulas are described by their section and number; thus, (3.12) refers to the formula 12 in section 3; when referring to a different chapter the chapter number is added, thus (II.3.12). A similar notation applies to the numbering of Theorems, Propositions, Lemmas and Corollaries.

Note on this version. A number of misprints have been eliminated with respect to the original version. The author welcomes any further information, as well as comments on the text.

1. Flow of a Polytropic Gas Through a Porous Medium. According to Muskat [M] the flow of an ideal gas through a porous medium can be described in terms of the variables *density*, which we represent by u, *pressure*, represented by v, and *velocity*, represented by V, which are functions of space x and time t. These quantities are related by the following laws:

(i) *Mass balance*

$$\rho\, u_t + \nabla \cdot (u\, V) = 0\,. \tag{1.1}$$

(ii) *Darcy's Law*, [Dr], an empirical law which describes the dynamics of flows through porous media

$$\mu\, V = -k\, \nabla v\,. \tag{1.2}$$

(iii) *Equation of State*, which for perfect gases states that

$$v = v_0\, u^\gamma\,, \tag{1.3}$$

where the exponent γ is 1 for isothermal processes and larger than 1 for adiabatic ones. The parameters ρ (the porosity), μ (the viscosity), k (the permeability) and v_0 (the reference pressure) are assumed to be positive and constant, which constitutes an admissible simplification in many practical instances. An easy calculation allows to reduce (1.1)-(1.3) to the form

$$u_t = c\,\Delta(u^m)\,,$$

with $m = 1+\gamma$ and $c > 0$ a constant, which can be easily scaled out, thus leaving us with the PME. Observe that in the above applications the exponent m is always equal or larger than 2. Mathematically constants that can be scaled out play no role, so it is now the custom to define the mathematical pressure by the expression

$$v = \tfrac{m}{m-1} u^{m-1}\,, \tag{1.4}$$

and write Darcy's law in the form

$$V = -\nabla v = -m\, u^{m-2} \nabla u\,. \tag{1.5}$$

Then the mass balance is just $u_t + \nabla \cdot (u\, V) = 0$. In all the formulas the operators $\nabla\cdot = \text{div}$, $\nabla = \text{grad}$ and Δ, the Laplacian, are supposed to act on the space variables $x = (x_1, \cdots, x_d)$.

Let us remark that the consideration of flows where ρ, μ and k are not constants, but functions of space and time, provides us with a natural generalization of the PME.

2. Heat transfer with temperature-dependent thermal conductivity. A second important application happens in the theory of heat propagation. The general equation describing such a process (in the absence of heat sources or sinks) takes the form

$$c\,\rho\frac{\partial T}{\partial t} = \text{div}\,(\kappa \nabla T)\,, \tag{2.1}$$

where T is the temperature, c the specific heat (at constant pressure), ρ the density of the medium (which can be a solid, fluid or plasma) and κ the thermal conductivity. In principle all these quantities are functions of $x \in \mathbf{R}^3$ and $t \in \mathbf{R}$. In the case where the variations of c, ρ and κ are negligible we obtain the classical heat equation. However, when the range of variation of the temperatures is large, say hundreds or thousands of degrees, such an assumption is not very reasonable. The simplest case of variable coefficients corresponds to constant c and ρ and variable κ, a function of temperature, $\kappa = \phi(T)$. We then write (2.1) in the form

$$T_t = \Delta\Phi(T)\,. \tag{2.2}$$

This is the generalized PME, called Filtration Equation in the Russian literature. The constitutive function Φ is given by

$$\Phi(T) = \frac{1}{c\rho}\int_0^T \phi(s)\,ds\,. \tag{2.3}$$

If the dependence is given by a power function

$$\kappa(T) = a\,T^n\,, \tag{2.4}$$

with a and $n > 0$ constants, then we get

$$T_t = b\,\Delta(T^m) \quad \text{with} \quad m = n+1\,, \tag{2.5}$$

and $b = a/(c\rho\,m)$, thus the PME but for the constant b which is easily scaled out. In case we also assume that $c\,\rho$ is variable, $c\,\rho = \psi(T)$, we still obtain a generalized PME though we have to work a bit more. Thus, we introduce a new variable T' by the formula

$$T' = \Psi(T) \equiv \int_0^T \psi(s)\,ds\,. \tag{2.6}$$

We then obtain (2.2) for the variable T' but now

$$\Phi(T') = \int_0^T \phi(s)\,ds\,, \tag{2.7}$$

where T is expressed in terms of T' by inverting (2.6), i.e. $T = \Psi^{-1}(T')$. Again, if the dependences are given by power functions we obtain the PME with an appropriate exponent.

Zel'dovich and Raizer [ZR, Chapter X] propose the above model to describe heat propagation by radiation occurring in ionized gases at very high temperatures. According to them, a good approximation of the process is obtained with the PME for an exponent m close to 6.

3. Other applications. The previous application shows how naturally the PME appears to replace the classical heat equation in processes of heat transfer (or diffusion of a substance) whenever the assumption of constancy of the thermal conductivity (resp. diffusivity) cannot be sustained, and instead it is reasonable to assume that it depends in a power-like fashion on the temperature (resp. density or concentration). Once the theory for the PME began to be known, a number of applications have been proposed.

A very interesting example concerns the spread of biological populations. The simplest law regarding a population consisting of a single species is

$$u_t = \text{div}\ (\kappa\, \nabla u) + f(u), \tag{3.1}$$

where the reaction term $f(u)$ accounts for the interaction with the medium, which is supposed to be homogeneous. According to Gurtin and McCamy [GMC] when populations behave so as to avoid crowding it is reasonable to assume that the diffusivity κ is an increasing function of the population density, hence

$$\kappa = \phi(u)\,, \quad \phi \quad \text{increasing.} \tag{3.2}$$

A realistic assumption in some particular cases is $\phi(u) = a\, u$. Disregarding the reaction term we obtain the PME with $m = 2$.

Of course, a complete study must take into account at least the reaction terms, and very often, the presence of several species. This leads to the consideration of nonlinear reaction-diffusion systems of equations of parabolic type containing lower order terms, whose diffusive terms are of PME type. Such equations and systems constitute therefore an interesting possibility of generalization of the theory of the PME. Among the many works on the subject let us mention the early papers of Aronson and Weinberger [AW] and Aronson, Crandall and Peletier [ACP].

CHAPTER II. THE HOMOGENEOUS DIRICHLET PROBLEM

In this chapter we consider the first boundary-value problem to the PME in a spatial domain $\Omega \subset \mathbf{R}^d$, $d \geq 1$, which is bounded and has a smooth boundary. We also consider homogeneous Dirichlet boundary conditions in order to obtain a simple problem for which a fairly complete theory can be easily developed.

A consequence of the degeneracy of the equation is that we do not expect to have classical solutions of the problem when the initial data take on the value $u = 0$, say, in an open subset of Ω.

Therefore we need to introduce an appropriate concept of *generalized* solution of the equation. At the same time we have to define in what sense the initial and boundary conditions are taken. In many cases this latter information can be built into the definition of generalized solution.

There are different ways of defining generalized solutions, the most usual idea being that of multiplying the equation by suitable test functions, integrating by parts some or all of the terms and asking from the solution a regularity that allows this expression to make sense. In this case we say that the solution is a *weak* solution.

In any case the concept of generalized solution changes the meaning of the term *solution*, so we have be careful to ensure that the new definition makes good sense. First of all, the new solutions must be so defined that they include all classical solutions whenever the latter exist. Moreover, a concept of generalized solution will be useful if the problem becomes well-posed for a reasonably wide class of data, i.e. if a unique such solution exists for each set of data in a given class and it depends continuously on the data in the appropriate topologies. As we will see, it can happen that several concepts of generalized solution arise naturally. It is then important to check that they agree in their common domain of definition (i.e. for data which are compatible with two or more of them). Selecting one them as the preferred definition depends of several factors, the most important being in principle that of having the largest domain. However, one could consider a more restrictive definition which still covers the applications in mind if it involves simpler statements or more natural concepts, or when it leads to simpler proofs of its basic properties.

In this chapter we introduce a suitable concept of weak solution and prove the existence and uniqueness of a weak solution for all initial data in $L^{m+1}(\Omega)$. We then extend our definition to encompass data in $L^1(\Omega)$. By means of appropriate estimates we also establish the main properties of these solutions. In particular, we show that the solution satisfies the equation in a strong sense.

Though a strong solution of the Dirichlet Probem will not be in general a classical solution, it is a continuous function in $Q = \Omega \times (0, \infty)$, with a uniform Hölder modulus of continuity away from $t = 0$.

Finally, we establish the existence of a special solution of the form $\tilde{U}(x,t) = f(x)\, t^{-\alpha}$ with decay rate $\alpha = 1/(m-1)$. This solution is unique and acts as an absolute upper bound for all solutions of the Dirichlet Problem. The existence of such a solution is a typical *nonlinear effect*, which is not possible in the linear theory.

Notations. They are rather standard. As usual, $\mathbf{R}_+ = (0, \infty)$. For a subset E of a metric space $\overline{E}$ denotes its closure. For vectors $\mathbf{u}$ and $\mathbf{v} \in \mathbf{R}^d$ the scalar product is denoted by $\mathbf{u} \cdot \mathbf{v}$. If $\Omega \subset \mathbf{R}^d$ is the domain where the spatial variable lives, then $\partial\Omega$ denotes its boundary, while Q is the cylinder $\Omega \times \mathbf{R}_+$ and for $0 < T < \infty$ we write $Q_T = \Omega \times (0, T)$ and $Q^T = \Omega \times (T, \infty)$. The lateral boundary of Q is denoted by $\Sigma = \partial\Omega \times [0, \infty)$, while $\Sigma_T = \partial\Omega \times [0, T]$. Integrals without limits are understood to extend to the whole domain under consideration, Ω, Q or Q_T.

Concerning functional spaces, $C(\Omega)$, $C^k(\Omega)$ and $C^\infty(\Omega)$ denote the spaces of continuous, k-times differentiable and infinitely differentiable functions in Ω, $C_c^\infty(\Omega)$ denotes the C^∞-smooth functions with compact support in Ω and $\mathcal{D}'(\Omega)$ the space of distributions. For $1 \le p \le \infty$ we denote the usual Lebesgue spaces by $L^p(\Omega)$ with norm $\|\cdot\|_p$, while $H^1(\Omega)$ and $H_0^1(\Omega)$ are the usual Sobolev spaces, and the subscript *loc* refers to local spaces. The same applies to functions defined in Q or Q_T or their closures. $C^{2,1}(Q)$ denotes those functions being twice differentiable in the space variables and once in time. For a function $u(x,t)$ we use the notation $u(t)$ to denote the function-valued map $t \mapsto u(\cdot, t)$.

Finally, the symbols $[s]_+$, $[s]^+$ mean $\max\{s, 0\}$, i.e. the positive part of the number s, and $[s]_- = [s]^- = \max\{-s, 0\}$, the negative part. The function sign_0^+ is defined as

$$\mathrm{sign}_0^+(s) = 1 \text{ for } s > 0, \quad \mathrm{sign}_0^+(s) = 0 \text{ for } s \le 0\,.$$

1. Weak solutions. We look for solutions $u = u(x,t)$ of the problem

$$u_t = \Delta(u^m) \quad \text{in } Q_T\,, \tag{1.1}$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega, \tag{1.2}$$

$$u(x,t) = 0 \quad \text{in } \Sigma_T, \tag{1.3}$$

where $m > 1$ and u_0 is a nonnegative, locally integrable function defined in Ω, a bounded domain in $\mathbf{R}^d$, $d \ge 1$, with boundary $\Gamma = \partial\Omega \in C^{2+\alpha}$, $\alpha \in (0,1)$. The time T can be finite or infinite. Though we will obtain solutions for all $T > 0$, i.e. with $T = \infty$, it is interesting for technical reasons to allow $T < \infty$.

First of all we introduce a suitable concept of weak solution for problem (1.1)–(1.3). Following [OKC] and [Sa] we propose

Definition 1. *A nonnegative function u defined in Q_T is said to be a weak solution of problem (1.1)–(1.3) if*

i) $u^m \in L^2(0, T : H_0^1(\Omega))$

ii) *u satisfies the identity*

$$\iint_{Q_T} \{\nabla(u^m) \cdot \nabla\varphi - u\varphi_t\} dx dt = \int_\Omega u_0(x)\varphi(x,0) dx \tag{1.4}$$

for any function $\varphi \in C^1(\overline{Q}_T)$ which vanishes on Σ and for $t = T$.

In the above definition u_0 should belong at least to $L^1(\Omega)$ for (1.4) to be well defined.

Observe that the equation is satisfied only in a weak sense since we do not assume that the derivatives appearing in equation (1.1) are actual functions, but merely exist in the sense of distributions. In fact, by specializing φ to the test function space $C_c^\infty(\Omega)$ we observe that $u_t = \Delta(u^m)$ in $\mathcal{D}'(Q_T)$. Moreover, the boundary condition (1.3) is hidden in the functional space $H_0^1(\Omega)$. Finally, the initial condition (1.2) is built into the integral formulation (1.4), and is actually satisfied in a very weak sense. Let us show as an example of the scope of the above definition how another natural way of defining a weak solution is included in Definition 1.

Proposition 1. *Let u be a nonnegative function defined in Q_T and such that*

i) $u^m \in L^2(0, T : H_0^1(\Omega))$

ii) *u satisfies the identity*

$$\iint \{\nabla(u^m) \cdot \nabla\varphi - u\varphi_t\} dxdt = 0 \tag{1.5}$$

for any function $\varphi \in C_c^\infty(Q_T)$.

iii) *for every $t > 0$ we have $u(t) \in L^1(\Omega)$ and $u(t) \to u_0$ as $t \to 0$ in $L^1(\Omega)$.*

Then u is a weak solution to (1.1)–(1.3) according to Definition 1.

Proof of Proposition 1. Suppose that u is as in the statement. We have to prove that (1.4) holds. It is very easy to see that (1.5) continues to hold when $\varphi \in C^1(Q_T)$ with $\varphi = 0$ on the boundary of Q_T (Hint: approximate φ with $\varphi_\varepsilon \in C_c^\infty$ and pass to the limit).

Now if φ is as in (1.4) we take a cut-off function $\zeta \in C^\infty(\mathbf{R})$, $0 \le \zeta \le 1$ such that $\zeta(s) = 0$ for $s < 0$, $\zeta(s) = 1$ for $s \ge 1$ and $\zeta' \ge 0$, and let $\zeta_n(t) = \zeta(nt)$. Applying (1.5) with test function $\varphi(x,t)\zeta_n(t)$ gives

$$\iint_Q \{\nabla(u^m) \cdot \nabla\varphi - u\varphi_t\}\zeta_n dxdt = \iint_Q u\varphi\zeta_{n,t} = \iint_{Q_{1/n}} u\varphi\zeta_{n,t}$$
$$= \iint_{Q_{1/n}} (u - u_0)\varphi\zeta_{n,t} + \iint_{Q_{1/n}} u_0(x)\varphi(x,t)\zeta_{n,t}(t)\,.$$

Fix $\varepsilon > 0$ and let n be so large that $\|u - u_0\|_1 \le \varepsilon$ for $0 \le t \le 1/n$. Then the first integral in the last member can be estimated as $\varepsilon\|\varphi\|_\infty \int \zeta_{n,t} dt = \varepsilon\|\varphi\|_\infty$ which vanishes as $n \to \infty$, $\varepsilon \to 0$. As for the last term we get

$$\iint_{Q_{1/n}} u_0(x)\varphi(x,t)\zeta_{n,t}(t)dxdt = \int_\Omega u_0(x)\varphi(x, \tfrac{1}{n})dx$$
$$- \iint_{Q_{1/n}} u_0\varphi_t\zeta_n dxdt \to \int_\Omega u_0(x)\varphi(x,0)dx$$

as $n \to \infty$, which proves (1.4). $\square$

Of course, a classical solution of problem (1.1)–(1.3) is automatically a weak solution of the problem. Moreover, though the explicit source-type solution $U(x,t) = U(x,t;C)$ of the Introduction is not a weak solution because of its singular initial data and because the boundary data are not necessarily 0, we can obtain from it weak solutions by the following method. Take $x_0 \in \Omega$, let $\tau > 0$ and let the constant C in U be small enough. Then the function

$$w(x,t) = U(x - x_0, t + \tau; C) \tag{1.6}$$

is a weak solution of the Dirichlet Problem (1.1)–(1.3) in any time interval $(0,T)$ in which the free boundary lies inside of Ω, i.e. if

$$T + \tau \le c\,\mathrm{dist}(x_0, \partial\Omega)^{d(m-1)+2},$$

cf. (0.5). Observe that w is not a classical solution.

2. Uniqueness of weak solutions.

2. Uniqueness of weak solutions. The uniqueness of weak solutions as defined above is very easily settled by means of an interesting trick, consisting in using a specific test function.

Theorem 2. *Problem (1.1)–(1.3) has at most one weak solution.*

Proof. Suppose that we have two such solutions u_1 and u_2. By (1.4) we have

$$\iint_{Q_T} (\nabla(u_1^m - u_2^m) \cdot \nabla\varphi - (u_1 - u_2)\varphi_t)\, dxdt = 0 \tag{2.1}$$

for all test functions φ. We want to use as a test function the one introduced by Oleĭnik,

$$\eta(x,t) = \begin{cases} \int_t^T (u_1^m(x,s) - u_2^m(x,s))ds, & \text{if } 0 < t < T \\ 0 & \text{if } t \ge T, \end{cases} \tag{2.2}$$

where $T > 0$. Even if η does not have the required smoothness we may approximate it (by mollification) with smooth functions η_ε for which (1.4) will hold. Since

$$\eta_t = -(u_1^m - u_2^m) \in L^2(Q_T), \tag{2.3}$$

$$\nabla\eta = \int_t^T (\nabla u_1^m - \nabla u_2^m)\, ds \in L^2(Q_T), \tag{2.4}$$

and moreover $\eta(t) \in H_0^1(\Omega)$ and $\eta(T) = 0$, we may pass to the limit $\varepsilon \to 0$ and (1.4) will still hold for η. Hence

$$\iint (u_1^m - u_2^m)(u_1 - u_2) + \iint (\nabla(u_1^m - u_2^m)) \cdot (\int_t^T (\nabla u_1^m - \nabla u_2^m)\, ds) = 0.$$

Integration of the last term gives

$$\iint\limits_{Q_T} (u_1^m - u_2^m)(u_1 - u_2)\, dx\, dt + \frac{1}{2} \int\limits_{\Omega} \Big(\int\limits_0^T (\nabla u_1^m - \nabla u_2^m)\, ds\Big)^2\, dx = 0\,. \tag{2.5}$$

Since both terms are nonnegative, we conclude that $u_1 = u_2$ a.e. in Q. $\square$

As a consequence of the uniqueness of weak solutions we have

Corollary 3. *There exist initial data for which the problem does not admit a classical solution.*

Proof. This is a rather standard argument. Firstly, we note that a classical solution of problem (1.1)–(1.3) is necessarily a weak solution in our sense. Secondly, we remark that $w(x,t)$ defined in (1.6) is a weak and nonclassical solution. By the uniqueness result there cannot be any other weak solution of (1.1)–(1.3) with the same data, hence no classical solution exists. $\square$

3. Existence of a weak solution. Energy estimate.

In a first approach we establish existence under the assumption that the data belong to the space $L^{m+1}(\Omega)$.

Theorem 4. *Suppose that $u_0 \in L^{m+1}(\Omega), u_0 \geq 0$. Then problem (1.1)–(1.3) has a weak solution with infinite time interval, $T = \infty$.*

The proof will be divided into several steps. First, we will consider the case of a smooth function u_0 vanishing on the border and prove the result, obtaining at the same time an important estimate. This estimate will allow us to solve the general problem by an approximation technique.

First step: We assume that u_0 is a nonnegative and C^∞-smooth function with compact support in Ω.

We begin by constructing a sequence of approximate initial data u_{0n} which does not take the value $u = 0$, so as to avoid the degeneracy of the equation. In our case we may simply put

$$u_{0n}(x) = u_0(x) + \frac{1}{n}\,. \tag{3.1}$$

We now solve the problem

$$(u_n)_t = \Delta(u_n^m) \quad \text{in} \quad Q, \tag{3.2}$$

$$u_n(x,0) = u_{0n}(x) \quad \text{in} \quad \overline{\Omega}, \tag{3.3}$$

$$u_n(x,t) = \frac{1}{n} \quad \text{on} \quad \Sigma. \tag{3.4}$$

In view of the data we expect the solution to be bounded by

$$\frac{1}{n} \le u_n(x,t) \le M + \frac{1}{n} \quad \text{in} \quad \overline{Q} \tag{3.5}$$

by the Maximum Principle, where $M = \sup(u_0)$. Therefore, we are dealing in practice with a uniformly parabolic problem. Actually, problem (3.2)–(3.4) has a unique solution $u_n \in C^{2,1}(\overline{Q})$. The rigorous justification uses a trick consisting in replacing equation (3.2) by

$$(u_n)_t = \operatorname{div}(a_n(u)\,\nabla u)\,, \tag{3.2'}$$

where $a_n(u)$ is a positive and smooth function, $a_n(u) \ge c > 0$, and $a_n(u) = m\,u^{m-1}$ in the interval $[1/n,\, M+1/n]$. This equation is not degenerate and a unique solution u_n of (3.2'), (3.3), (3.4) exists in the space $C^{2,1}(\overline{Q})$ by the standard quasilinear theory, cf. [LSU, Chapter 6], and it satisfies (3.5). Moreover, by repeated differentiation and interior regularity we are able to conclude that $u_n \in C^\infty(Q)$. Now, due to the definition of a_n, equations (3.2) and (3.2') coincide on the range of u_n. In this way problem (3.2)-(3.4) is solved in a classical sense and the degeneracy of the equation has been avoided.

Moreover, again by the Maximum Principle

$$u_{n+1}(x,t) \le u_n(x,t) \quad \text{in} \quad \overline{Q} \tag{3.6}$$

for all $n \ge 1$. Hence we may define the function

$$u(x,t) = \lim_{n\to\infty} u_n(x,t), \qquad (x,t) \in \overline{Q}. \tag{3.7}$$

Then u_n converges to u in $L^p(\Omega)$ for every $1 \le p < \infty$. In order to show that this u is the weak solution of problem (1.1)–(1.3) we will need some estimates. First of all, from (3.5) we get

$$0 \le u \le M \quad \text{in} \quad \overline{Q}. \tag{3.8}$$

We now control the spatial gradient $\nabla(u^m)$. Multiply equation (3.2) by $\varphi_n = u_n^m - \left(\frac{1}{n}\right)^m$ and integrate by parts in Q_T to obtain

$$\begin{aligned}
\iint\limits_{Q_T} |\nabla u_n^m|^2 dx dt &= \int\limits_\Omega \left(\frac{1}{m+1} u_{0n}^m(x) - \frac{1}{n^m}\right) u_{0n}(x)\, dx \\
&- \int\limits_\Omega \left(\frac{1}{m+1} u_n^m(x,T) - \frac{1}{n^m}\right) u_n(x,T) dx \\
&\le \frac{1}{m+1}\int \left(u_0(x) + \frac{1}{n}\right)^{m+1} dx + \frac{1}{n^m}\left(M + \frac{1}{n}\right)\int\limits_\Omega dx.
\end{aligned} \tag{3.9}$$

Since T is arbitrary, it follows that $\{\nabla u_n^m\}$ is uniformly bounded in $L^2(Q)$, and therefore a subsequence of it should converge to some limit ψ weakly in $L^2(Q)$. Since also $u_n^m \to u^m$

it follows that $\psi = \nabla u^m$ in the sense of distributions. The limit is uniquely defined so that the whole sequence must converge to it. Passing to the limit in (3.9) we get the following important estimate

$$(m+1) \iint_{Q_T} |\nabla u^m|^2 \, dx \, dt + \int u^{m+1}(x,T) \, dx \le \int u_0^{m+1}(x) \, dx \,, \tag{3.10}$$

called the **Energy Estimate**. On the other hand, since $u_n \in C(\overline{Q})$, $u_n(x,t) = \frac{1}{n}$ on Σ and $0 \le u \le u_n$, we have

$$\lim_{(x,t)\to\Sigma} u(x,t) = 0$$

with uniform convergence. Hence $u^m(\cdot,t) \in H_0^1(\Omega)$ for a.e. $t > 0$.

Finally, since u_n is a classical solution of (1.1), it clearly satisfies (1.4) with u_0 replaced by u_{0n}. Letting $n \to \infty$ we obtain (1.4) for u. Therefore, u is a weak solution of (1.1)–(1.3).

Let us remark, to end this step, that if we have two initial data $u_0, \hat{u}_0$ such that $u_0 \le \hat{u}_0$, then the above approximation process can produces ordered approximating sequences, $u_{0n} \le \hat{u}_{0n}$. By the classical maximum principle, [LSU], we have $u_n \le \hat{u}_n$ for every $n \ge 1$. Hence in the limit $u \le \hat{u}$.

Second step: We assume that u_0 is bounded and vanishes near the boundary.

The method of the previous step can still be applied. According to the quasilinear theory, cf. [LSU], now the approximate solutions $u_n \in C^\infty(Q) \cap C^{2,1}(Q \cup \Sigma)$ and are not continuous down to $t = 0$ unless the data are, but instead take the initial data in $L^p(\Omega)$ for every $p < \infty$. Passage to the limit offers no novelty. Comparison still applies.

Third step. General case.

In the general case $u_0 \in L^{m+1}(\Omega)$ we take an increasing sequence of cutoff functions ζ_k which vanish near Γ, and consider the sequence of approximations of the initial data

$$u_{0k}(x) = \min\{u_0(x)\zeta_k(x), k\} \,. \tag{3.11}$$

Using Step 2 we solve problem (1.1)–(1.3) with initial data u_{0k} and obtain a unique weak solution u_k. By the comparison remark $u_{k+1} \ge u_k$ in Q. On the other hand, by estimate (3.10) $\{u_k\}$ is bounded uniformly in $L^\infty(0,\infty : L^{m+1}(\Omega))$ and ∇u_k^m is likewise in $L^2(Q)$. Hence, $\{u_k\}$ converges a.e. to a function $u \in L^\infty(0,\infty; L^{m+1}(\Omega))$, ∇u_k^m converges weakly in $L^2(Q)$ to ∇u^m and (3.10) holds for u. It follows that $u^m \in L^2(0,\infty; H_0^1(\Omega))$. Finally, equation (1.4) is satisfied. $\square$

An alternative proof, where Steps 2 and 3 are replaced by a single approximation step using a stability result, will be given at the end of §6. On the other hand, we see from the proof that the choice of $L^{m+1}(\Omega)$ as the space for the initial data depends essentially on estimate (3.10). A priori estimates are one of the most powerful and widely used tools in the study of P.D.E. This approach will be stressed in our treatment of the existence, uniqueness and qualitative properties of solutions to the different problems.

4. Absolute Bound in Sup Norm. Before we proceed, we will derive another important estimate, the boundedness of the solutions for positive times. This bound will give us a needed control on u^m.

Proposition 5. *Every weak solution u of (1.1)-(1.3) is bounded in Q^τ for every $\tau > 0$. Moreover, we have an absolute decay estimate of the form*

$$u(x,t) \le C(m,d)\, R^{\frac{2}{m-1}}\, t^{-\frac{1}{m-1}}\,, \tag{4.1}$$

where $C(m,d) > 0$ and R is the radius of a ball containing Ω.

By *absolute* we mean that the bound does not depend on the data we are considering.

Proof. Let us first consider the case where u_0 is continuous and vanishes on $\partial\Omega$. We will construct an explicit super-solution $z(x,t)$ with which to compare the approximate solutions u_n to (3.2)–(3.4).

In fact we take a ball B of radius R strictly containing Ω, i.e. with $\Gamma \subset B$, and consider

$$z(x,t) = A(t+\tau)^{-\alpha}(1-bx^2)^\beta \tag{4.2}$$

for suitable constants $A, \tau, \alpha, \beta, b > 0$. Setting $b = R^{-2}$ will make the function positive in $B \times (0,\infty)$, hence for all large n

$$u_n(x,t) = \frac{1}{n} < z(x,t) \quad \text{in} \quad \Sigma \tag{4.3}$$

if A, τ, α, β are kept fixed. Moreover, we will have

$$u_{0n}(x) \le z(x,0) \tag{4.4}$$

whenever τ is small enough. Finally setting $\alpha = 1/(m-1)$ and $\beta = 1/m$ we obtain

$$z_t - \Delta(z^m) \ge 0 = (u_n)_t - \Delta(u_n^m) \tag{4.5}$$

whenever

$$A \ge R^{\frac{2}{m-1}}(2d(m-1))^{-\frac{1}{m-1}}. \tag{4.6}$$

With those choices, the classical Maximum Principle implies that $u_n(x,t) \le z(x,t)$ in Q, hence in the limit $u(x,t) \le z(x,t)$. Since τ could be arbitrarily small, we will let $\tau \to 0$ and finally get

$$u(x,t) \le At^{-\frac{1}{m-1}}\left(1 - \left(\frac{x}{R}\right)^2\right)^{1/m} \tag{4.7}$$

By approximation (4.7) holds for every weak solution. $\square$

5. Existence of classical solutions.

Once a solution of the equation is constructed in some generalized sense, it is an important point to decide if it is indeed a classical solution. Though we know that in general this will not be the case, it can happen under additional requirements on the data. We prove in this section that when the initial data are smooth and positive inside Ω, so that the equation is parabolic nondegenerate, we obtain a classical solution by essentially using the standard quasilinear theory.

Proposition 6. *Let $u_0 \in C(\overline{\Omega})$ be positive in Ω and vanish on its boundary Γ, and let u be the corresponding weak solution. Then $u \in C^\infty(Q) \cap C(\overline{Q})$, u is positive in Q and vanishes on Σ.*

Proof. The first step will be proving that for every point $x_0 \in \Omega$ where $u_0(x_0) > 0$ we will have $u(x_0, t) > 0$ for every $t > 0$. This is done by the classical method of *barriers*, comparing u with a suitable source-type solution. Actually, if $B = B_r(x_0)$ is a ball of radius r where u_0 is positive, say $u_0(x) \ge c > 0$ for $x \in B$, we consider the Barenblatt function

$$\bar{u} = U(x - x_0, t + 1; C).$$

We may choose C small enough so that $u_0(x) \ge \bar{u}(x, 0)$ in B, and also that the support of $\bar{u}$ is contained in Q_T for a given $T > 0$. This support is of the form $\mathcal{S} = \{(x,t) : c|x - x_0|^\gamma < (t+1)\}$ with $\gamma = d(m-1) + 2$ (cf. (0.5)), $\bar{u} \in C^\infty(\mathcal{S})$ and $\bar{u}$ vanishes on the lateral boundary of $\mathcal{S}$.

Hence, by the classical Maximum Principle applied in $\mathcal{S} \cap Q_T$ to $\bar{u}$ and a smooth approximation to u we conclude that $u \ge \bar{u}$ in $\mathcal{S}$, hence $u(x,t)$ is bounded uniformly away from 0 in a neighbourhood of the form $N = B_1 \times (0, T)$, $B_1 = B_r(x_0)$.

Therefore, when taking the limit $u_n \to u$ in the approximation process of Theorem 4, we can apply in N the regularity theory of quasilinear nondegenerate parabolic equations, and conclude that $u \in C^\infty(N)$ and the initial data are taken continuously in B_1.

The fact that u vanishes continuously on Σ is a simple consequence of the approximation process (3.1)–(3.4). In fact $u \le u_n$, $u_n \in C^\infty(\overline{Q})$ and $u_n(x,t) = \frac{1}{n}$ on Σ. $\square$

Of course, if moreover u_0 is smooth, e.g. if $u_0 \in C^k(\Omega)$ for some $k > 0$, this regularity is reflected in the regularity of u near $t = 0$.

6. The basic L^1-estimate.

This section contains another very important estimate which allows to develop an existence, uniqueness and stability theory in the space $L^1(\Omega)$.

Proposition 7 (L^1-Contraction Principle). *Let $u_0, \hat{u}_0$ be two initial data in $L^{m+1}(\Omega)$ and let $u, \hat{u}$ be their respective weak solutions. Then for every $t > \tau \ge 0$*

$$\int [u(x,t) - \hat{u}(x,t)]^+ dx \le \int [u(x,\tau) - \hat{u}(x,\tau)]^+ dx \le \int [u_0(x) - \hat{u}_0(x)]^+ dx\,, \tag{6.1}$$

so in particular

$$\|u(t) - \hat{u}(t)\|_1 \le \|u(\tau) - \hat{u}(\tau)\|_1 \le \|u_0 - \hat{u}_0\|_1. \tag{6.2}$$

Proof. Let $p \in C^1(\mathbf{R})$ be such that $0 \le p \le 1$, $p(s) = 0$ for $s < 0$, $p'(s) > 0$ for $s > 0$, and consider the approximate solutions $u_n, \hat{u}_n$ to problem (3.2)-(3.2) with same n. We have in Ω for $t > 0$

$$\int (u_n - \hat{u}_n)_t \, p(u_n^m - \hat{u}_n^m)\, dx = \int \Delta(u_n^m - \hat{u}_n^m) p(u_n^m - \hat{u}_n^m)\, dx$$
$$= - \int |\nabla(u_n^m - \hat{u}_n^m)|^2 p'(u_n^m - \hat{u}_n^m)\, dx \le 0$$

(Observe that $p(u_n^m - \hat{u}_n^m) = 0$ on Σ). Therefore, letting p converge to the sign function sign_0^+, and observing that $\frac{d}{dt}[u_n - \hat{u}_n]^+ = (u_n - \hat{u}_n)_t \ \mathrm{sign}_0^+(u_n - \hat{u}_n)$, cf. [GT], we get

$$\frac{d}{dt} \int [u_n - \hat{u}_n]^+ dx \le 0,$$

which implies (6.1) for $u_n, \hat{u}_n$. Passing to the limit we obtain (6.1). To obtain (6.2), combine (6.1) applied first to u and $\hat{u}$ and then to $\hat{u}$ and u. $\square$

The above proof establishes the uniqueness of solutions of problem (1.1)–(1.3) by a technique (the L^1 technique) which is completely different from that of Theorem 4. It is interesting to remark that estimate (6.1) not only implies L^1-dependence of solutions on data, but also a *comparison* theorem.

Corollary 8. *If $u_0 \le \hat{u}_0$ a.e. in Ω, then $u \le \hat{u}$ a.e. in Q.*

Another consequence of the estimate is

Corollary 9 (Continuity in L^1). *The weak solution of (1.1)-(1.3) can be viewed as a continuous curve in $L^1(\Omega)$, i.e. $u \in C([0,\infty) : L^1(\Omega))$.*

Proof. When $u_0 \in C(\overline{\Omega})$ is positive in Ω and vanishes on $\partial\Omega$, we have shown in §5 that $u \in C(\overline{Q})$, hence $u \in C([0,T] : L^1(\Omega))$. For general u_0, we approximate with functions $\hat{u}_0$ as above and write, using Proposition 7,

$$\|u(\tau) - u_0\|_1 \le \|u(\tau) - \hat{u}(\tau)\|_1 + \|u_0 - \hat{u}_0\|_1 + \|\hat{u}(\tau) - \hat{u}_0\|_1 \le$$
$$\le 2\|u_0 - \hat{u}_0\|_1 + \|\hat{u}(\tau) - \hat{u}_0\|_1.$$

Therefore, as $\hat{u}_0 \to u_0$ and $\tau \downarrow 0$ we get $u(\tau) \to u_0$. This settles the continuity at $t = 0$. To settle it at any other time $t > 0$ we may displace the origin of time and argue as before at the times t and $t + \tau$. $\square$

Remark (The proof of Theorem 4 revisited). To end this section we give an alternative to steps 2 and 3 of the existence proof of Theorem 4, §3. Now that we know that solutions depend continuously on the data in $L^1(\Omega)$, we may approximate a general data $u_0 \in L^{m+1}(\Omega)$ with $u_{0n} \in C_c^\infty(\Omega)$, apply step 1 and pass to the limit using Proposition 7 and the Energy Estimate (3.10).

7. Solutions with data in $L^1(\Omega)$.

The continuous dependence with respect to the L^1 norm allows us to extend our existence result and consider as data any nonnegative function $u_0 \in L^1(\Omega)$. The idea is to approximate the initial data with a sequence $u_{0n} \in L^\infty(\Omega)$ such that $u_{0n} \to u_0$ in $L^1(\Omega)$, obtain solutions u_n with these data and use Proposition 7 to pass to the limit in u_n as $n \to \infty$. We thus obtain a function $u \in C([0,\infty) : L^1(\Omega))$ such that $u(0) = u_0$. The question is, is u a weak solution of (1.1)-(1.3) according to Definition 1?

To begin with, it turns out that in general u does not satisfy the condition $u^m \in L^2(0,\infty : H_0^1(\Omega))$, which is important in giving a sense to identity (1.4), therefore we must change our definition of weak solution. It happens that, thanks to the absolute bound, in passing to the limit $n \to \infty$ in the sequence u_n considered above we encounter difficulties in checking that u is a weak solution only near $t = 0$. A convenient definition to cover solutions with L^1 data is

Definition 2. *A nonnegative function $u \in C([0,\infty) : L^1(\Omega))$ is said to be a weak solution of problem (1.1)–(1.3) if*

i) $u^m \in L^2_{\rm loc}(0,\infty : H_0^1(\Omega))$

ii) *u satisfies the identity*

$$\iint_Q \{\nabla u^m \cdot \nabla\varphi - u\varphi_t\}\,dxdt = 0 \tag{7.1}$$

for any function $\varphi \in C_c^1(Q)$.

iii) $u(0) = u_0$.

We immediately see that a weak solution in the sense of Definition 1 is a also a weak solution in the present sense if we can ensure that it belongs to the class $C([0,\infty) : L^1(\Omega))$. We will come back to the relation between both definitions later.

Theorem 10. *There exists a unique weak solution of Problem (1.1)–(1.3) with given initial data $u_0 \in L^1(\Omega)$, $u_0 \geq 0$, where weak solution is understood in the sense of Definition 2. The Contraction Principle holds for these solutions.*

Proof. (i) *Existence.* We construct approximations u_n as before and pass to the limit using the L^1 and L^∞ estimates derived in Propositions 5 and 7 plus the energy estimate (3.10).

(ii) *Uniqueness.* Let u_1, u_2 be weak solutions of the problem with same initial data u_0. Given $\varepsilon > 0$ there exists $\tau > 0$ such that $\|u_1(t) - u_0\|_1$, $\|u_2(t) - u_0\|_1 < \varepsilon$ for $0 \leq t \leq \tau$. Consider now the functions $v_1(x,t) = u_1(x,t+\tau)$, $v_2(x,t) = u_2(x,t+\tau)$. Both v_1 and v_2 satisfy the assumptions of Proposition 1, hence they are weak solutions of the same problem with initial data $u_1(x,\tau)$, $u_2(x,\tau)$ resp. In particular these initial data are bounded functions. Hence, by Proposition 7 we get for $t > \tau$.

$$\begin{aligned}\|u_1(t) - u_2(t)\|_1 &= \|v_1(t-\tau) - v_2(t-\tau)\|_1 \\ &\leq \|v_1(0) - v_2(0)\|_1 = \|u_1(\tau) - u_2(\tau)\|_1 < \varepsilon\,.\end{aligned}$$

We may now let $\varepsilon \to 0$ to get $u_1(t) = u_2(t)$ a.e. for every $t > 0$.

(iii) The validity of the Contraction Principle (Proposition 7) is just a consequence of the limit process. $\square$

We can now establish the relationship between both definitions.

Proposition 11. *A weak solution in the sense of Definition 1 such that $u \in C([0,T) : L^1(\Omega))$ is also a weak solution in the sense of Definition 2. Conversely, a weak solution in the new sense is also a weak solution according to Definition 1 if and only if $u_0 \in L^{m+1}(\Omega)$.*

Proof. The first statement is obvious. This applies in particular whenever $u_0 \in L^{m+1}(\Omega)$. Thus, both definitions coincide in this case.

Now let u be a solution in the sense 2 with $u_0 \in L^1(\Omega)$. Then Definition 1 is satisfied if $\nabla u^m \in L^2(0,T : H_0^1(\Omega))$. It can be proved that the Energy Estimate holds with equality sign. But then the bound on ∇u^m is equivalent to the condition $u_0 \in L^{m+1}(\Omega)$. $\square$

Definition 2 has the advantage over Definition 1 that both the comparison and stability results proved in the last section are immediately seen to hold. It is also obvious according to this definition that, if $u(x,t)$ is a solution with data $u_0(x)$ and $\tau > 0$, then $v(x,t) = u(x,t-\tau)$ is the solution corresponding to data $v_0(x) = u(x,\tau)$ (this is usually known as the *semigroup property*).

8. Further regularity.

Though solutions with data which are not stricly positive need not be classical solutions, they enjoy some interesting regularity properties that we derive as a consequence of new estimates. We recall that, as a consequence of estimate (3.10) the L^{m+1}-norm of a solution u is nonincreasing in time. By the same method we can obtain monotonicity in all the L^p norms, $1 \le p < \infty$, by using other powers of u as test functions in the calculation. Thus we obtain the following monotonicity statement for the L^p-norms of the solutions.

Proposition 12. *For every $u_0 \in L^p(\Omega), p \ge 1$, we have $u(\cdot,T) \in L^p(\Omega)$ for any $T > 0$ and*

$$\|u(\cdot,t)\|_p \le \|u_0\|_p \,. \tag{8.1}$$

Proof. It is based on the estimate

$$\frac{4qm}{(q+m)^2} \iint_{Q_T} |\nabla \left(u^{\frac{q+m}{2}}\right)|^2 + \frac{1}{q+1}\int u^{q+1}(x,T)dx \le \frac{1}{q+1}\int u_0^{q+1}(x)dx \,, \tag{8.2}$$

valid for $q > 0$. To get the case $p = 1$ we pass to the limit as $q \to 0$.

The preceding estimate is not very important in our situation since we have a very precise decay result thanks to Proposition 5. More important is the investigation on regularity of the derivative u_t appearing as one of the members of the equation. Our results will lead in the next section to the concept of strong solution. We begin with an "energy" estimate for the time derivative.

Proposition 13. *$(u^q)_t \in L^2(Q^\tau)$ for every $q \ge (m+1)/2$ and every $\tau > 0$. The basic estimate is*

$$\frac{8m}{(m+1)^2}\iint\limits_{Q_T} t\Big|\frac{d}{dt}(u^{(m+1)/2})\Big|^2 + T\int\limits_{\Omega}|\nabla u^m(x,T)|^2dx \le \iint\limits_{Q_T}|\nabla u^m|^2dxdt\,, \tag{8.3}$$

valid for all $T>0$. From this we obtain the decay rate

$$\iint_{Q^t}|(u^q)_t|^2\,dxdt = O(t^{-\frac{2(q-1)}{m-1}})\iint_{Q^t}|\nabla u^m|^2dxdt\,. \tag{8.4}$$

Proof. Let $w = u^{\frac{m+1}{2}}$ with $u = u_n$ smooth solution of (3.2)–(3.4). We have

$$\Big|\frac{dw}{dt}\Big|^2 = (\tfrac{m+1}{2})^2u^{m-1}|u_t|^2 = \tfrac{(m+1)^2}{4m}(u^m)_t\,\Delta u^m,$$

and

$$\frac{1}{2}\frac{d}{dt}\int|\nabla u^m|^2dx = \int\nabla u^m\cdot(\nabla u^m)_t\,dx = -\int\Delta u^m(u^m)_t\,dx,$$

since $(u^m)_t = 0$ on Σ. Therefore, integration in Q_T gives

$$\iint t\,\Big|\frac{dw}{dt}\Big|^2dxdt = -\frac{(m+1)^2}{8m}\int\limits_0^T t\left(\frac{d}{dt}\int|\nabla u^m|^2dx\right)dt =$$

$$= -\frac{(m+1)^2}{8m}T\int|\nabla u^m(x,T)|^2dx + \frac{(m+1)^2}{8m}\iint|\nabla u^m|^2dxdt.$$

This is estimate (8.3) for $u = u_n$. When u is any weak solution we proceed by approximation. We notice that the same argument applies in any time interval (τ, T) with $\tau > 0$. To estimate $(u^q)_t$ with $q > (m+1)/2$ we observe that

$$(u^q)_t = (2q/(m+1))u^{q-(m+1)/2}(u^{m+1)/2})_t$$

and recall that u is bounded in Q^τ by Proposition 5. Combining inequality (8.3) in (τ, T), with $T\to\infty$, with the L^∞-estimate (4.1), we get (8.4). $\square$

Remark. (8.3) combined with (3.10) gives estimates of the first member in terms of $\int u_0^{m+1}dx$. Thus,

$$\iint\limits_{Q} t\Big|\frac{d}{dt}(u^{(m+1)/2})\Big|^2\,dxdt \le \frac{m+1}{8m}\int_\Omega u_0^{m+1}(x)\,dx\,.$$

Unfortunately, the preceding estimates do not allow for a direct control of the derivative u_t appearing in the equation. We obtain next an estimate for u_t.

Proposition 14. *We have in* $\mathcal{D}'(Q)$

$$u_t \geq -\frac{u}{(m-1)t}\,. \tag{8.5}$$

Proof. (i) *First proof.* Let $u = u_n$ be one of the approximate solutions to problem (1.1)–(1.3). Consider the function

$$z = (m-1)tu_t + u.$$

It is a simple computation to show that z is a solution in Q of the equation

$$z_t = \Delta(mu^{m-1}z). \tag{8.6}$$

that $z(x,t) = u(x,t) \geq 0$ on Σ and $z(x,0) \geq 0$ for all $x \in \Omega$. Hence, by the standard Maximum Principle $z(x,t) \geq 0$, which is equivalent to (8.5). In this case we obtain a pointwise inequality.

We now pass to the limit in (8.5) to obtain the estimate for any weak solution of (1.1)–(1.3). This can only be done on the weak or distributional form of the inequality, which is obtained by multiplying by a test function $\varphi \in C_c^\infty(Q)$, $\varphi \geq 0$, and integrating by parts, i.e.

$$\int\!\!\int \left(\frac{1}{(m-1)t}u\varphi - u\varphi_t\right) dxdt \geq 0.$$

Therefore, the fact that (8.5) holds in the sense of distributions does not mean that u_t is a function. At least, since u is the limit of a sequence $\{u_n\}$ for which $(u_n)_t$ is locally bounded below uniformly in n, u_t is a Radon measure.

(ii) *Second proof.* The reader may wonder how did we find the precise combination

$$z = u + (m-1)tu_t$$

to which the Maximum Principle can be applied. There is a beautiful and simple argument based on scaling which produces such magic function. It is as follows: given a smooth solution u and a constant $\lambda > 1$, we consider the function

$$\tilde{u}(x,t) = \lambda u(x, \lambda^{m-1}t)\,. \tag{8.7}$$

This is again a solution of the PME. Moreover, for $\lambda > 1$ we have $\tilde{u}(x,0) = \lambda u(x,0) \geq u(x,0)$, hence by the Maximum Principle $\tilde{u} \geq u$ in Q. Now differentiate (8.7) with respect to λ at $\lambda = 1$. We get

$$0 \leq \frac{d}{d\lambda}\tilde{u}(x,t)|_{\lambda=1} = u(x,t) + (m-1)tu_t(x,t)\,,$$

namely (8.5). $\square$

Remark. We can extend (8.5) to an L^∞-estimate down $t = 0$ if Δu_0^m is conveniently controlled from below. In fact, if

$$(m-1)\Delta u_0^m \geq -au_0 \tag{8.8}$$

for some constant $a > 0$, we may compare the functions $z_1 = (m-1)(at+1)u_t + au$ and $z_2 = 0$: both are solutions of (8.6) in Q and $z_1 \geq z_2$ on the parabolic boundary of Q. Hence, by the maximum principle which is again justified by approximation, we obtain $z_1 \geq z_2$, i.e.

$$u_t \geq -\frac{au}{(m-1)(at+1)}. \tag{8.9}$$

Condition (8.8) is implied for instance by the pressure bound

$$m\Delta u_0^{m-1} \geq -a.$$

In order to prove that u_t is actually an integrable function we have to translate the estimate for $(u^{(m+1)/2})_t$ into an estimate for u_t. This is rather technical. We use the following result.

Lemma 15. *Let K be a subset of $\mathbf{R}^d$ with finite measure, let $I = [t_0, t_1]$ and assume that v is a function defined in $K \times I$ that satisfies*

i) $v \in L^\infty(I : L^1(K))\,, v \geq 0\,, v_t \geq 0$,

ii) v^λ *and* $\dfrac{d}{dt}(v^\lambda) \in L^r(K \times I)$ *for some* $\lambda, r > 1$.

Then $\dfrac{d}{dt}v \in L^p(K \times I)$ *for every* $p \in [1, p_1)$, *where*

$$p_1 = \frac{r\lambda}{r(\lambda-1)+1} \in (1, r).$$

Proof. Without loss of generality we may assume that $v \geq \varepsilon > 0$ in $K \times I$ by replacing v by $v + \varepsilon$ since our estimates will not depend on ε. Now, for any $p \in (1, r)$ and $\nu \in (0, p)$ we have

$$|\frac{dv}{dt}|^p = \left[\frac{1}{\lambda}\frac{dv^\lambda}{dt}\right]^\nu \left(v^{\sigma-1}\frac{dv}{dt}\right)^{p-\nu}$$

where $1 - \sigma = \nu(\lambda-1)/(p-\nu)$. We choose ν such that $p - \nu + (\nu/r) = 1$, that is

$$\nu = \frac{(p-1)r}{r-1}.$$

Clearly, $0 < \nu < p$. Moreover, we obtain for σ the value

$$\sigma = 1 - \frac{r(p-1)(\lambda-1)}{r-p}$$

so that $\sigma > 0$ if $p < p_1$. With the assumption we have in $K \times I$

$$\iint |\frac{dv}{dt}|^p \leq \frac{1}{\lambda^\nu}\left(\iint |\frac{dv^\lambda}{dt}|^r\right)^{\nu/r} \left(\iint v^{\sigma-1}\frac{dv}{dt}\right)^{p-\nu}.$$

Finally, the last integral is estimated as

$$\frac{1}{\sigma}\int v^\sigma dx \leq \frac{1}{\sigma}(\operatorname{meas} K)^{1-\sigma}\left(\int v dx\right)^\sigma \qquad \square$$

Corollary 16. *For any weak solution of problem (1.1)–(1.3)* $u_t \in L^p_{\rm loc}(Q)$ *for any* $p \in [1,(m+1)/m)$ *and* $T > 0$.

Proof. Again we may restrict ourselves to classical solutions by approximation. If u is the solution, then

$$v(x,t) = tu(x, t^{m-1})$$

satisfies the conditions of Lemma 15. Observe in particular that $v_t \geq 0$ is a consequence of (8.5). By Proposition 13 we may take $\lambda = \frac{m+1}{2}$, $r = 2$, hence $p_1 = (m+1)/m$. □

As a consequence of Propositions 12, 14, and Corollary 16 we have

Corollary 17. *For any weak solution* $tu_t \in L^\infty(0,\infty : L^1(\Omega))$ *and*

$$\int u_t\,dx \leq 0\,, \qquad t\|u_t(t)\|_1 \leq \frac{2}{m-1}\|u_0\|_1\,. \tag{8.10}$$

Proof. In case u is smooth the first inequality follows from (8.1) for $p = 1$. Since $u_t = [u_t]^+ - [u_t]^-$, and $|u_t| = [u_t]^+ + [u_t]^-$, we have

$$\int [u_t]^+ \leq \int [u_t]^- \quad \text{and} \quad \int |u_t|\,dx = \int ([u_t]^+ + [u_t]^-)\,dx \leq 2\int [u_t]^-\,dx.$$

We now use (8.5) to obtain (8.10). □

Remark. Again if Δu_0^m is bounded below as in (8.8) the bound (8.10)-right for u_t can be improved and $u_t \in L^\infty(0,\infty : L^1(\Omega))$.

9. Strong solutions.

A nonnegative, locally integrable function u for which all the derivatives which appear in an equation are functions rather than distributions and such that the equation is satisfied a.e. in its domain is called a *strong* solution of that equation. For equation (1.1) these requirements amount to the following:

i) u, u^m, u_t, $\Delta(u^m) \in L^1_{\rm loc}(Q)$,
ii) $u_t = \Delta(u^m)$ as locally integrable functions in Q.

A precise definition of strong solution for a problem, like (1.1)–(1.3), asks for functional spaces which allow to define in what sense the initial and boundary data are taken. Again a convenient choice of spaces should allow both for existence for a suitable class of data and, on the other side, for uniqueness.

In our case the estimates obtained in the previous section imply the following result.

Theorem 18. *For every* $u_0 \in L^1(\Omega)$ *the weak solution to problem (1.1), (1.3) is a strong solution in the following sense:*

i) $u^m \in L^2(\tau,\infty : H^1_0(\Omega))$ *for every* $\tau > 0$
ii) u_t *and* $\Delta u^m \in L^1_{\rm loc}(0,\infty : L^1(\Omega))$ *and* $u_t = \Delta(u^m)$ *a.e. in* Q
iii) $u \in C([0,\infty) : L^1(\Omega))$ *and* $u(0) = u_0$

Proof. Conditions (i) and (iii) have been already established. As for ii), we have proved that $u_t \in L^\infty(\tau, \infty : L^1(\Omega)) \cap L^p_{\rm loc}(Q)$ for every $\tau > 0$ and $1 < p < p_1 = (m+1)/m$. Returning to equation (7.1) with $\varphi \in C_c^\infty(Q)$ we conclude that $\nabla(u^m)$ has u_t as its divergence, hence $\Delta(u^m) \in L^p_{\rm loc}(Q)$ for p as above. By standard theory, all the second spatial derivatives of u^m belong to $L^p_{\rm loc}(Q)$. Moreover, (1.1) is satisfied in Q. $\square$

We give next a summary of the additional properties of the solution

Theorem 19. *The strong solution of problem (1.1)-(1.3) also satisfies*

(a) $u \in L^\infty(Q^\tau)$ *and (4.1) holds.*
(b) $\nabla(u^\gamma) \in L^2(Q^\tau)$ *for every* $\gamma > m/2$ *and (3.10), (8.1) and (8.2) hold.*
(c) $tu_t \in L^\infty(0, \infty : L^1(\Omega)) \cap L^p_{\rm loc}(\Omega)$ *for* $1 \le p < p_1$ *and (8.3), (8.4), (8.5) and (8.10) hold.*
(d) *For every two solutions* $u, \hat{u}$ *we have (6.1), (6.2). In particular* $u_0 \le \hat{u}_0$ *implies* $u \le \hat{u}$ *in* Q.
(e) *For every* $t \ge \tau \ge 0$ *and every* $1 \le p \le \infty$ *we have* $\|u(t)\|_p \le \|u(\tau)\|_p$.
(f) *If* $u_0 \in C(\Omega), u_0(x) > 0$ *for* $x \in Q$ *and* $u_0(x) = 0$ *for* $x \in \Gamma$ *then* u *is a classical solution, positive in* Q.

Remark. The condition $u \in C([0, \infty) : L^1(\Omega))$ does not look essential in the definition, in the sense that we could replace it by $u \in L^1_{\rm loc}(Q)$ (and then write $u(t) \to u_0$ in $L^1(\Omega)$ instead of $u(0) = u_0$ in iii)) and get the same uniqueness result. Nevertheless, it is natural since we want to view our solution as a *continuous* curve in some functional space, in this case $t \in [0, \infty) \to u(t) \in L^1(\Omega)$. This view leads to the concept of *semigroup of transformations* with interesting consequences. Anyway, in our case it does not mean any extra condition since $u^m \in L^2_{\rm loc}(0, \infty : H^1_0(\Omega))$ clearly implies $u \in L^2_{\rm loc}(0, \infty : L^1(\Omega))$ which together with $u_t \in L^1_{\rm loc}(0, \infty : L^1(\Omega))$ gives $u \in C((0, \infty) : L^1(\Omega))$. We make the assumption of continuity at $t = 0$ in order to satisfy the initial condition $u(0) = u_0$.

10. A comment on continuity.

A typical result of the quasilinear elliptic and parabolic theories says that solutions of such equations which are bounded in some Lebesgue space, say L^2, and satisfy in a weak sense an equation with certain structural assumptions, are in fact Hölder continuous with Hölder exponents and constants depending only on the L^2 norm of the solution and the bounds in the structure assumptions. This is also true for the PME, notwithstanding the degeneracy of the equation: using essentially variants of the Moser iteration technique and some of the estimates established above, one can establish Hölder continuity in x and t for the solutions of the Dirichlet problem for the PME. The classical theory of quasilinear equations allows moreover to prove that the solution is C^∞-smooth in the open set where it is positive. Finally, if the initial data have certain smoothness properties, these are inherited by the solution at $t = 0$. We do not have time in these lectures to go into this interesting chapter of the theory, which can be found [GP]. Continuity of the weak solutions was first established for the Cauchy Problem in [CF].

11. The special solution in separated variables. In Proposition 5 we have obtained an absolute bound for the solutions of the Cauchy- Dirichlet Problem (1.1)-(1.3). This bound will be improved in this section. Indeed, we show that there exists a function $\tilde{U}$ which is the largest element in the class of functions which are weak solutions of the Dirichlet Problem in Q. We will call this solution the *maximal solution* of the Cauchy-Dirichlet Problem. Moreover, $\tilde{U}$ is a solution in separated-variables form. We have

Theorem 20. *There exists a unique function $\tilde{U}(x,t)$ which is a solution of the Dirichlet problem in Q_τ for every $\tau > 0$ and takes initial values $\tilde{U}(x,0)=\infty$. This function has the form*

$$\tilde{U}(x,t) = t^{-\frac{1}{m-1}} f(x)\,. \tag{11.1}$$

Then $\tilde{U}$ is the maximal solution of the PME in Q with zero Dirichlet conditions and $g=f^m$ is the unique positive solution of the nonlinear eigenvalue problem

$$\Delta g + \tfrac{1}{m-1} g^{\frac{1}{m}} = 0\,, \quad g \in H_0^1(\Omega)\,. \tag{11.2}$$

Proof. (i) For every integer $n \geq 1$ we solve the problem

$$(P_n) \quad \begin{cases} (u_n)_t & = \Delta(u_n^m) \quad \text{in } Q\,, \\ u_n(x,0) & = n \quad \text{in } \Omega\,, \\ u_n(x,t) & = 0 \quad \text{on } \Sigma\,. \end{cases}$$

Let u_n be the solution to this problem. Clearly, the sequence $\{u_n\}$ is monotone: $u_{n+1} \geq u_n$. We also know from Proposition 5 that for every n

$$u_n(x,t) \leq C t^{-1/(m-1)} \quad \text{in} \quad Q\,. \tag{11.3}$$

Therefore, we may pass to the limit and find a function

$$\tilde{U}(x,t) = \lim_{n\to\infty} u_n(x,t)$$

also satisfying (11.3). Now fix $\tau > 0$ and observe that, by (11.3), there exists $n_1 = n_1(\tau)$ such that for every $n \geq n_1$

$$u_n(x,\tau) \leq n_1 = u_{n_1}(x,0)\,.$$

By the Maximum Principle we conclude that

$$u_n(x,t+\tau) \leq u_{n_1}(x,t) \quad \text{in} \quad Q\,,$$

so that

$$\tilde{U}(x,t+\tau) \leq u_{n_1}(x,t) \quad \text{in} \quad Q\,. \tag{11.4}$$

As a monotone limit of bounded solutions u_n in Q_τ such that the u_n^m are bounded above by a function in $L^2(\tau,\infty : H_0^1(\Omega))$, it is straightforward to conclude that $\tilde{U}$ is a strong solution of the Cauchy-Dirichlet problem for the PME in any time interval (τ, ∞). It is also clear that it takes on the value $\tilde{U}(x, 0) = \infty$.

(ii) Let us now prove that $\tilde{U}$ has the form (11.1). To do that we introduce the transformation

$$(\mathcal{T}u)(x,t) = \lambda u(x, \lambda^{m-1}t)\,, \quad \lambda > 0\,. \tag{11.5}$$

This transformation leaves the equation invariant. It is also clear that when applied to our latter sequence $\{u_n\}$ we get

$$(\mathcal{T}u_n)(x,t) = u_{\lambda n}(x,t) \quad \text{in} \quad Q \tag{11.6}$$

(check the initial and boundary values). Passing to the limit $n \to \infty$ in (11.6) we get

$$(\mathcal{T}\tilde{U})(x,t) = \tilde{U}(x,t) \tag{11.7}$$

which holds for every $(x,t) \in Q$ and every $\lambda > 0$. Fixing (x,t) and setting $\lambda = t^{-1/(m-1)}$ we get (11.1) with $f(x) = \tilde{U}(x,1)$. The fact that $g = f^m$ satisfies (11.2)-(11.3) is also obvious.

(iii) Let us prove now that $\tilde{U}$ is larger than any solution of the Cauchy-Dirichlet Problem in Q. By Proposition 5 we know that every such solution satisfies

$$u(x,\tau) \le C(\tau) < \infty\,.$$

It follows from the Maximum Principle that

$$u(x, t+\tau) \le \tilde{U}(x,t) \quad \text{in} \quad Q\,.$$

Letting now $\tau \to 0$ we get $u(x,t) \le \tilde{U}(x,t)$ in Q as desired.

(iv) Finally, we prove the uniqueness of the solution with $u(x,0) = \infty$. Assume that v is another such solution. Since we assume that $v(x, t+\tau)$ is a weak solution of problem (1.1)-(1.3), $v(x,\tau)$ must be an element in $H_0^1(\Omega)$, hence $v(x, 2\tau)$ is bounded (by Proposition 5), and by comparison with the sequence u_n we conclude that $v(x, t+2\tau) \le u_n(x,t)$ in Q for some n large enough. Letting $\tau \to 0$ we get

$$v(x,t) \le \tilde{U}(x,t)\,. \tag{11.8}$$

On the other hand, a function v which has infinity initial values is of course larger than the solutions u_n, hence $v \ge \tilde{U}$. Therefore, $v = \tilde{U}$. $\square$

COMMENTS

As we have explained above, solutions for the Cauchy, Dirichlet and Neumann Problems were first announced by Oleĭnik [O] and explained in detail in [OKC], published in 1958. The case of one space dimension was considered and a class of so-called generalized solutions was introduced. The uniqueness result, Theorem 2, follows the proof in [OKC]. A study of the properties of weak solutions to the Dirichlet Problem was done by Aronson and Peletier in [AP], who use a definition similar to our Definition 1. Hölder continuity of the solutions of this problem is proved by Gilding and Peletier in [GP]. Some of the estimates are more or less classical in nonlinear parabolic equations. The first proof of the control of u_t from below follows the proof of Caffarelli and Friedman in [CF], while an argument close to the second proof was used in [CVW] in the study of the regularity of the Cauchy Problem. Proposition 13 and Lemma 15 are due to Bénilan [Be]. These estimates are crucial in establishing that the solution is strong. The existence of the special solution (11.1) is established in [AP] by a different method, consisting in studying the elliptic equation (11.3). A more general result can be found in Dahlberg and Kenig [DK].

In this chapter we study the initial-value problem for the PME in d-dimensional space, $d \geq 1$, with integrable and nonnegative initial data, $u_0 \in L^1(\mathbf{R}^d)$, $u_0 \geq 0$. We establish existence and uniqueness of a strong solution for the Cauchy Problem as well as its main properties, among them the conservation of mass, the boundedness of the solutions for $t \geq \tau > 0$ and a version of the finite propagation property.

In this chapter we will use the symbols $Q = \mathbf{R}^d \times \mathbf{R}_+$ and $Q_T = \mathbf{R}^d \times (0, T)$.

1. Definition of strong solutions. Uniqueness. We consider the problem:

$$u_t = \Delta(u^m) \quad \text{in } Q \tag{1.1}$$

$$u(x,0) = u_0(x) \qquad \text{for } x \in \mathbf{R}^d, \tag{1.2}$$

where $m > 1$ and $u_0 \in L^1(\mathbf{R}^d)$, $u_0 \geq 0$. No difficulties arise in restricting time to the interval $0 \leq t \leq T$ and replacing Q by Q_T. Following the motivation of the previous chapter we will first give a suitable definition of strong solution for our initial-value problem and then prove existence, uniqueness and a series of basic properties of such solutions. Here we restrict ourselves to solutions which are integrable with respect to the space variables, or solutions with *finite mass*, and develop the corresponding L^1-theory.

Definition 1. *We say that a nonnegative function $u \in C([0,\infty) : L^1(\mathbf{R}^d))$ is a strong L^1-solution of problem (1.1), (1.2) if*

i) $u^m, u_t, \Delta(u^m) \in L^1_{\text{loc}}(0, \infty : L^1(\mathbf{R}^d))$
ii) $u_t = \Delta(u^m)$ *a.e. in* Q
iii) $u(0) = u_0$.

In the rest of the chapter strong solution will always mean strong L^1-solution. Our first step in the study of strong solutions will be to establish the crucial L^1-order-contraction property similar to Proposition II.7.

Lemma 1. *Let u_1, u_2 be two strong solutions of (1.1), (1.2) in Q_T. For every $0 < t_1 < t_2$ we have*

$$\int [u_1(x,t_2) - u_2(x,t_2)]_+ dx \leq \int [u_1(x,t_1) - u_2(x,t_1)]_+ dx. \tag{1.3}$$

Proof. Let $p \in C^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$ be such that $p(s) = 0$ for $s \leq 0$, $p'(s) > 0$ for $s > 0$ and $0 \leq p \leq 1$, and let $j(r) = \int_0^r p(s)ds$ be a primitive of p. Since p will be an approximation to the sign function

$$\text{sign}_0^+(r) = 1 \text{ if } r > 0, \quad \text{sign}_0^+(r) = 0 \text{ if } r \leq 0, \tag{1.4}$$

j will approximate the function $s \mapsto [s]_+$. Moreover, consider a cutoff function $\zeta_0 \in C_c^\infty(\mathbf{R}^d)$ such that $0 \leq \zeta_0 \leq 1$, $\zeta_0(x) = 1$ if $|x| \leq 1$, $\zeta_0(x) = 0$ if $|x| \geq 2$ and let $\zeta = \zeta_n(x) = \zeta_0(x/n)$. As $n \to \infty, \zeta_n \uparrow 1$.

We subtract the equations satisfied by u_1 and u_2, multiply by $p(u_1^m - u_2^m)\zeta$ and integrate on $S = \mathbf{R}^d \times [t_1, t_2]$ to obtain, with $w = u_1^m - u_2^m$,

$$\int\int (u_1 - u_2)_t \, p(w)\zeta = \int\int \Delta w \, p(w)\zeta \,. \tag{1.5}$$

Now, approximate w by means of a smooth kernel sequence ρ_n. If $w_n = w * \rho_n$ ($*$ denotes convolution) we have $w_n \to w$, $\nabla w_n \to \nabla w$ and $\Delta w_n \to \Delta w$ in $L^1_{\text{loc}}(Q)$ and almost everywhere for a subsequence, so that $p(w_n) \to p(w)$ a.e. Moreover,

$$\int\int p(w_n)\Delta w_n \zeta + \int\int p'(w_n)|\nabla w_n|^2 \zeta + \int\int p(w_n)\nabla w_n \cdot \nabla\zeta = 0 \,.$$

Letting $n \to \infty$ we observe that the second integral is uniformly bounded above. Moreover, by Fatou's lemma

$$\int\int p'(w)|\nabla w|^2 \zeta \le -\int\int p(w)\Delta w \zeta - \int\int p(w)\nabla w \cdot \nabla\zeta \,. \tag{1.6}$$

Hence, returning to (1.5) we get

$$\begin{aligned} &\int\int (u_1 - u_2)_t \, p(w)\zeta \le -\int\int p'(w)|\nabla w|^2\zeta - \int\int p(w)\nabla w \cdot \nabla\zeta \\ &\le -\int\int p(w)\nabla w \cdot \nabla\zeta = -\int\int \nabla j(w)\cdot\nabla\zeta = \int\int j(w)\Delta\zeta \le \int\int |w| \, |\Delta\zeta|, \end{aligned} \tag{1.7}$$

where integration is understood on S. Letting now p tend to sign_0^+ and observing that $\frac{d}{dt}[u_1 - u_2]_+ = \text{sign}_0^+(u_1 - u_2)\frac{d}{dt}(u_1 - u_2)$, we get

$$\begin{aligned} &\int [u_1(x,t_2) - u_2(x,t_2)]_+ \zeta \, dx \le \int [u_1(x,t_1) - u_2(x,t_1)]_+ \zeta \, dx \\ &\qquad + \|\Delta\zeta\|_\infty \iint_{S\cap\{|x|>n\}} |w(x,t)| \, dx \, dt. \end{aligned} \tag{1.8}$$

We let now $n \to \infty$ to obtain (1.3), since $w \in L^1(t_1, t_2 : L^1(\mathbf{R}^d))$ and $\|\Delta\zeta_n\|_\infty = \|\Delta\zeta_0\|_\infty / n^2$. $\square$

Again, as in Chapter II, we obtain *uniqueness* and *comparison* as simple consequences of this result.

Theorem 2. *Problem (1.1), (1.2) has at most one strong solution. If u_1, u_2 are strong solutions with initial data u_{01}, u_{02} resp. and $u_{01} \le u_{02}$ are in $\mathbf{R}^d$, then $u_1 \le u_2$ a.e. in Q. In particular, if $u_{01} = u_{02}$ a.e. then $u_1 = u_2$ a.e.*

Remark. The proof of Lemma 1 actually uses the following requirements on u : u, u^m, $u_t, \Delta u^m \in L^1_{\text{loc}}(Q)$ and

$$\iint_{S_n} u_1^m(x,t)dxdt = o(n^2) \text{ as } n \to \infty \,, \tag{1.9}$$

where $S_n = \{(x,t) : n \le |x| \le 2n, t_1 \le t \le t_2\}$ with $0 < t_1 < t_2$, which are weaker than our Definition 1. Therefore, Theorem 2 also holds under the above hypotheses, if (1.9) holds uniformly for $0 \le t \le t_2$ and if the initial data are taken continuously in $L^1_{\rm loc}(\mathbf{R}^d)$. We shall use this remark later on.

Example. Though the source-type solution $U(x,t)$ fails to be a strong solution of the Cauchy Problem because of the singularity of its initial data, any time-delayed version $u(x,t) = U(x, t+\tau)$ with $\tau > 0$ is indeed a strong solution.

2. Existence of Solutions. Conservation of mass. We begin this section by constructing solutions for bounded initial data by using an approximation process and the results of the previous chapter. The existence result for general initial data in $L^1(\mathbf{R}^d)$ will follow once we show that every solution is bounded for $t \ge \tau > 0$, which will be done in Section 4.

Theorem 3. *For every nonnegative function $u_0 \in L^1(\mathbf{R}^d) \cap L^\infty(\mathbf{R}^d)$ there exists a strong solution of problem (1.1), (1.2). Moreover $u_t \in L^p_{\rm loc}(Q)$ for $1 \le p < (m+1)/m$ and*

$$u_t \ge -\frac{u}{(m-1)t} \quad \textit{in} \quad \mathcal{D}'(Q), \tag{2.1}$$

$$\|u_t(\cdot,t)\|_1 \le \frac{2\|u_0\|_1}{(m-1)t}. \tag{2.2}$$

If $u_0 \in L^p(\mathbf{R}^d)$ for $1 \le p \le \infty$ then $u(t) \in L^p(\mathbf{R}^d)$ and

$$\|u(t)\|_p \le \|u_0\|_p\,. \tag{2.3}$$

Moreover, the map $u_0 \mapsto u(t)$ is an ordered contraction in $L^1(\mathbf{R}^d)$.

Proof. i) We begin by assuming that u_0 is not only bounded and integrable over $\mathbf{R}^d$, but also that it is strictly positive, C^∞-smooth and all its derivatives are bounded in $\mathbf{R}^d$, and finally (8.8) that holds. Under these conditions we construct a strong and classical solution. For that we consider the Dirichlet problems

$$(P_n) \quad \begin{cases} u_t = \Delta(u^m) \quad \text{in} \quad Q_n = B_n(0) \times (0,\infty), \\ u(x,0) = u_{0n}(x) \quad \text{for} \quad |x| \le n, \\ u(x,t) = 0 \quad \text{for} \quad |x| = n,\ t \ge 0, \end{cases}$$

where $u_{0n} = u_0\zeta_n$, $\{\zeta_n\}$ being a cutoff sequence with the following properties: $\zeta_n \in C^\infty(\mathbf{R}^d)$, $\zeta_n(x) = 1$ for $|x| \le n-1$, $\zeta_n(x) = 0$ for $|x| \ge n$, $0 < \zeta_n(x) < 1$ for $n-1 < |x| < n$, the derivatives of the ζ_n up to second order are bounded uniformly in $x \in \mathbf{R}^d$, and $n \ge 2$. Finally, $\Delta(\zeta_n^{m-1})$ is uniformly bounded below.

By the results of Chapter II (Theorem 4 and Proposition 6), (P_n) admits a unique classical solution $u_n \in C^\infty(Q_n) \cap C(\overline{Q}_n)$ and $u_n > 0$ in Q_n. In particular, u_{n+1} will be a classical solution of the PME in Q_n with positive boundary data and initial data larger than u_{0n}. We conclude from the classical Maximum Principle that $u_{n+1} \ge u_n$ in Q_n, i.e. , the sequence $\{u_n\}$ is monotone. Moreover, we get from Chapter II uniform estimates for

(a) u_n in $L^\infty(0,\infty : L^p(B_n(0)),\ 1 \le p \le \infty,$
(b) $(u_n)_t$ in $L^\infty(0,\infty : L^1(B_n(0))) \cap L^p_{\rm loc}(Q_n)$ for $1 \le p < p_1,$
(c) u_n^m in $L^2(0,\infty : H^1_0(B_n(0)))$.

Since all of these estimates involve bounds which are independent of n, we may pass to the limit $n \to \infty$ and obtain a positive function $u \in L^\infty(0,\infty : L^p(\mathbf{R}^d))$ for every $p \in [1,\infty)$, such that $u_t, u^m, \Delta u^m$ belong to the same spaces to which $(u_n)_t, u_n^m, \Delta(u_n^m)$ belonged, and equation (1.1) holds in Q.

To check the smoothness of u, we first observe that in a neighbourhood $N \subset \overline{Q}$ of any point $(x_0,t) \in \overline{Q}$, $u_n(x,t)$ is defined and positive, say $u_n(x,t) \geq c > 0$ for every $(x,t) \in N$ if $n > n_0$. Since the sequence $\{u_n\}$ is monotone nondecreasing and bounded, the interior regularity theory for uniformly parabolic quasilinear equations gives uniform bounds for all the derivatives of $u_n, n \geq n_0$, in a smaller neighbourhood of (x_0,t). In the limit we conclude that $u \in C^\infty(\overline{Q})$. Moreover for $t = 0$ we get $u(x,t) = u_0(x), x \in \mathbf{R}^d$.

We have proved that u is classical solution of problem (1.1), (1.2). To comply with our definition of strong solution we still have to check the continuity of $u = u(t) : [0,\infty) \to L^1(\mathbf{R}^d)$. It is a consequence of the fact that $u \in L^\infty(0,\infty : L^1(\mathbf{R}^d))$ (by (II.8.1)) and $u_t \in L^\infty(0,\infty : L^1(\mathbf{R}^d))$ (cf. remark to Corollary II.17) so that u is absolutely continuous from $[0,\infty)$ into $L^1(\mathbf{R}^d)$.

Estimates (2.1), (2.2), (2.3) are a consequence of similar estimates for the Dirichlet problem. In particular we have $0 \leq u(x,t) \leq \|u_0\|_\infty$.

ii) If $u_0 \in L^1(\mathbf{R}^d) \cap L^\infty(\mathbf{R}^d)$ does not fulfill the above requirements we approximate it by a sequence $\{u_{0n}\}$ of such functions. We may always do in such a way that $\|u_{0n}\|_1 \leq \|u_0\|_1$, $\|u_{0n}\|_\infty \leq \|u_0\|_\infty$, $u_{0n} \to u_0$ in $L^1(\mathbf{R}^d)$. Let u_n be the solution with data u_{0n}. It follows from Lemma 1 that u_n converges in $C([0,\infty) : L^1(\mathbf{R}^d))$ to a function u and $u(0) = u_0$.

Again estimates (a), (b), (c) of the previous step will hold uniformly in n so that passing to the limit $n \to \infty$ produces a strong solution of (1.1), (1.2), which satisfies the estimates (2.1), (2.2), (2.3). $\square$

The solutions of the Cauchy problem (1.1), (1.2) have an important conservation property, not enjoyed by the solutions of the Dirichlet problem.

Proposition 4 (Conservation of total mass). *For every $t > 0$*

$$\int u(x,t)dx = \int u_0(x)dx\,. \tag{2.4}$$

Proof. We take a cutoff function ζ_n, as in Theorem 3, and integrate by parts as follows:

$$\begin{aligned}\int u(x,t)\zeta_n(x)dx - \int u_0(x)\zeta_n(x)dx &= \iint u_t\zeta_n\,dxdt \\ = \iint \Delta u^m \zeta_n dxdt = \iint u^m \Delta\zeta_n dxdt &\to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

The calculation is justified if u is smooth and bounded. For general u it follows by approximation, using Lemma 1. $\square$

3. The fundamental estimate for the Cauchy Problem.

Perhaps the most significant novelty of the Cauchy problem is the existence of a lower bound for the Laplacian of the pressure. Indeed, we have

Proposition 5. *Let* $v = mu^{m-1}/(m-1)$. *Then*

$$\Delta v \geq -\frac{\lambda}{t} \quad \text{with} \quad \lambda = \frac{d}{d(m-1)+2}\,. \tag{3.1}$$

The inequality is understood in the sense of distributions in Q. This bound will be used so often that we consider it the *fundamental* estimate for the Cauchy problem. Estimates of the form $\Delta v \geq -C$ play a role in the theory of Hamilton-Jacobi equations, cf. e.g. [Li]. Such functions are called semi-superharmonic functions. Let us also remark that (3.1) is optimal in the sense that equality is actually attained by the source-type or Barenblatt solutions, which are a kind of worst case with respect to this bound, a fact which has interesting consequences.

As a consequence of (3.1) we have the following improvement of (2.1), (2.2).

Corollary 6. $u_t \in L^\infty_{\text{loc}}(0, \infty : L^1(\mathbf{R}^d))$ *and*

$$u_t \quad \geq -\frac{\lambda u}{t} \quad \text{in} \quad \mathcal{D}'(Q)\,, \tag{3.2}$$

$$t\|u_t\|_1 \leq 2\lambda \|u_0\|_1. \tag{3.3}$$

Proof. The first inequality is a consequence of

$$v_t = (m-1)v\Delta v + |\nabla v|^2 \geq (m-1)v\Delta v,$$

together with $v_t/v = (m-1)u_t/u$ and (3.1). For the second one argue as in Corollary II.17. Again the calculations are justified for smooth solutions and hold in the limit for every solution. $\square$

Proof of Proposition 5. (i) The formal derivation of the estimate is very simple. We first write the PDE satisfied by the pressure, v, i.e.

$$v_t = (m-1)v\Delta v + |\nabla v|^2. \tag{3.4}$$

Then we write the equation satisfied by $p = \Delta v$ by differentiating (3.4) twice. We have

$$p_t = (m-1)v\Delta p + 2m\nabla v \cdot \nabla p + (m-1)p^2 + 2\sum_{i,j}\left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)^2.$$

Since

$$\sum_{i,j}(a_{ij})^2 \geq \sum_i (a_{ii})^2 \geq \frac{1}{d}\left(\sum_i a_{ii}\right)^2,$$

we get

$$\mathcal{L}(p) \equiv p_t - (m-1)v\Delta p - 2m\nabla v \cdot \nabla p - \left(m - 1 + \frac{2}{d}\right)p^2 \geq 0.$$

Here $\mathcal{L}$ is a quasilinear parabolic operator with smooth, variable coefficients, since we consider v as a fixed function of x and t. We now apply $\mathcal{L}$ to the trial function

$$P(x,t) = -\frac{C}{t+\tau} \tag{3.5}$$

and observe that $\mathcal{L}(P) \le 0$ if and only if $C \ge \lambda = 1/\left(m-1+\frac{2}{d}\right)$. We fix $C=\lambda$. By choosing τ small enough we may also obtain

$$p(x,0) \equiv \Delta v(x,0) \ge P(x,0) \equiv -\frac{C}{\tau}, \tag{3.6}$$

from which the classical maximum principle should allow us to conclude that $p \ge P$ in Q. Letting $\tau \to 0$ we would then obtain a pointwise inequality $\Delta v \ge -\lambda/t$.

(ii) The application of the maximum principle is justified when considering classical solutions of (3.4) such that $v, \nabla v$ and $p = \Delta v$ are bounded and v is bounded below away from 0 so that the equation is uniformly parabolic. Therefore, we need to construct *new approximate solutions*. This we do as follows. We may always restrict ourselves to initial data u_0 which are bounded, smooth and positive, thanks to Lemma 1. Consider now initial data

$$u_{0\varepsilon}(x) = u_0(x) + \varepsilon, \quad \varepsilon > 0. \tag{3.7}$$

According to [LSU] there exists exactly one function $u_\varepsilon \in C^\infty(\overline{Q})$ that solves (1.1) with initial data $u_{0\varepsilon}$, and $\varepsilon \le u_\varepsilon \le M + \varepsilon$, where $M = \|u_0\|_\infty$. Moreover, by interior regularity results all the derivatives of u_ε are bounded in Q. In particular, equation (1.1) is uniformly parabolic on u_ε. It follows that the fundamental estimate (3.1) holds for v_ε, the pressure of u_ε.

Now, if we prove that $v_\varepsilon \to v$ as $\varepsilon \to 0$ in $L^1_{\mathrm{loc}}(Q)$, then (3.1) will still hold in the limit for v, though only in distribution sense, i.e.

$$\iint \left(v\Delta\varphi - \frac{\lambda}{t}\varphi \right) dxdt \ge 0 \tag{3.8}$$

for every $\varphi \in C_c^\infty(Q)$, $\varphi \ge 0$. Therefore, the proof is complete with the following convergence result.

Lemma 7. *As $\varepsilon \to 0$ $u_\varepsilon \to u$ locally uniformly in Q.*

Proof. We first observe that by the maximum principle the family $\{u_\varepsilon\}$ is nonincreasing as $\varepsilon \downarrow 0$. It is also easy to establish that every u_ε is above the solution u with initial data u_0 (Hint: compare u_ε with the approximations u_n to u constructed in step 1 of Theorem II.4 in the domain Q_n and let $n \to \infty$). Since u is strictly positive in $\overline{Q}$ and $u_\varepsilon \ge u$, and thanks again to the interior regularity results, not only $\{u_\varepsilon\}$ converges to a function $\hat{u}$, but also the derivatives converge, so that $\hat{u}$ is a C^∞ solution of (1.1) in Q, $\hat{u}(\cdot,0) = u_0$ and $\hat{u} \ge u$.

To conclude that $\hat{u} = u$ we still need some control of u^m as $|x| \to \infty$, as in (1.9), to be able to apply Theorem 2. We use the following result

Lemma 8. *For every ε and $t > 0$ we have*

$$\int (u_\varepsilon(x,t) - \varepsilon) dx \leq \int u_0(x) dx \,. \tag{3.9}$$

Proof. Formally, we have $\int u_{\varepsilon,t}\, dx = \int \Delta u_\varepsilon^m\, dx = 0$, hence

$$\int (u_\varepsilon(x,t) - \varepsilon)\, dx = \int (u_{0\varepsilon}(x) - \varepsilon)\, dx = \int u_0(x)\, dx \,.$$

More rigorously, we approximate u_ε with the solution $u_{\varepsilon n}$ of the following Dirichlet problem

$$\begin{cases} u_t = \Delta(u^m) \quad \text{in } Q_n \\ u(x,0) = u_{0n}(x) + \varepsilon \quad \text{for } |x| \leq n \\ u(x,t) = \varepsilon \quad \text{for } |x| = n \text{ and } t \geq 0 \,, \end{cases}$$

for which we argue as in Section II.3 and get a contraction formula as (II.6.1), which we apply to $u_{\varepsilon n}$ and $\hat{u}_n = \varepsilon$ to get (3.9) for $u_{\varepsilon n}$. Letting $n \to \infty$ we obtain that $u_{\varepsilon n}$ converges (the sequence is compact by the interior regularity theory) to a solution of (1.1) which is u_ε by uniqueness. In the limit (3.9) holds.

Going back now to the main argument, we let $\varepsilon \to 0$ to obtain

$$\int \hat{u}(x,t) dx \leq \int u_0(x) dx.$$

It follows that $\hat{u}(t) \in L^\infty(0,\infty : L^1(\mathbf{R}^d)) \cap L^\infty(Q)$, hence by the Remark to Theorem 2 we conclude that $\hat{u} = u$ in Q. This ends the proof of the fundamental estimate. $\square$

4. Boundedness of the solutions. Existence with general data.

We are now in a position to prove that all solutions are bounded for $t \geq \tau > 0$, the so-called L^1-L^∞ *smoothing effect.*

Theorem 9. *For every $t > 0$ we have*

$$u(x,t) \leq c\, \|u_0\|_1^\alpha\, t^{-\lambda} \,, \tag{4.1}$$

where $\alpha = 2/(d(m-1)+2)$, $\lambda = d/(d(m-1)+2)$ and $c > 0$ depends only on m and d.

The Theorem can be derived as a consequence of the fundamental estimate (3.1), thanks to the following result.

Lemma 10. *Let g be any nonnegative, smooth, bounded and integrable function in $\mathbf{R}^d$ such that*

$$\Delta(g^r) \geq -K \tag{4.2}$$

for some r and $K>0$. Then $g \in L^\infty(\mathbf{R}^d)$ and $\|g\|_\infty$ depends only on r,K, d and $\|g\|_1$ in the form

$$\|g\|_\infty \le C(r,d)\,\|g\|_1^\rho\, K^\sigma\,, \tag{4.3}$$

with $\rho = 2/(2+rd)$ and $\sigma = d/(2+rd)$.

Given Lemma 10, it suffices to fix $t>0$, and put

$$r = m-1, \quad g(x) = u(x,t) \quad \text{and} \quad K = \frac{\lambda(m-1)}{mt}.$$

to obtain Theorem 9 in the case where the solution u is positive everywhere, hence smooth. The general case is done by approximation.

Proof of Lemma 10. Let $f(x) = g^r$. Then $\Delta f \ge -K$. Therefore, for every $x_0 \in \mathbf{R}^d$ the function

$$F(x) = f(x) + \frac{K}{2d}|x-x_0|^2$$

is subharmonic in $\mathbf{R}^d$. It follows that

$$F(x_0) \le \fint_B F(x)dx, \tag{4.4}$$

where $B = B_R(x_0)$, $R>0$ and $\fint_B$ denotes average on B. The argument will continue in a different way for $r>1$ and for $0<r\le 1$.

(i) In the latter case, $r \le 1$, we can use (4.4) to estimate f at an arbitrary point x_0 as follows:

$$\begin{aligned} f(x_0) &\le \fint_B f(x)dx + \frac{K}{2d}\fint_B |x-x_0|^2 dx \le (\fint_B f^{1/r}\,dx)^r + \frac{K\,R^2}{2(d+2)} \\ &\le \|g\|_1^r \left(\frac{1}{\omega_d\,R^d}\right)^r + \frac{K\,R^2}{2(d+2)}. \end{aligned} \tag{4.5}$$

(ω_d denotes the volume of the unit ball). Minimization of the last expression with respect to $R>0$ gives

$$f(x_0) \le C\,\|g\|_1^{\frac{2r}{rd+2}}\,K^{\frac{rd}{rd+2}}\,,$$

which is equivalent to (4.3).

ii) For $r>1$ we modifiy the calculation as follows: for every $x_0 \in \mathbf{R}^d$

$$\begin{aligned} f(x_0) &\le \fint_B g^r(x)\,dx + \frac{K}{2d}\fint_B |x-x_0|^2 dx \le \|g\|_\infty^{r-1}\fint_B g\,dx + \frac{K\,R^2}{2(d+2)} \\ &\le \|g\|_\infty^{r-1}\|g\|_1 \frac{1}{\omega_d\,R^d} + \frac{K\,R^2}{2(d+2)}\,. \end{aligned} \tag{4.6}$$

Taking the supremum in $\mathbf{R}^d$ in the first member and putting $y = \|g\|_\infty$, we can write (4.6) in the form

$$y^r \le Ay^{r-1} + B \quad \text{with} \quad A = c_1\|g\|_1 R^{-d},\ B = c_2 K R^2\,,$$

which after an elementary calculation gives

$$y \le A + B^{1/r} = c_1\|g\|_1 R^{-d} + (c_2 K R^2)^{1/r}\,. \tag{4.7}$$

Minimization of this expression in R gives (4.3). $\square$

Formula (4.1) not only asserts that solutions with L^1 data are bounded for positive times, but also gives a very precise quantitative estimate of the bound. In fact, the exponents appearing in the formula can be derived from the general boundedness statement thanks to a scaling argument. Since this kind of argument has wider applicability we give here a proof of this implication, as a small diversion.

Lemma 11. *Suppose that for all solutions of (1.1), (1.2) with $\|u_0\|_1 \le 1$ we have $\|u(\cdot,1)\|_\infty \le C = C(m,d) > 0$. Then (4.1) necessarily holds.*

Proof. Let u be any solution of (1.1), (1.2) with $\|u_0\|_1 = M > 0$. Now, if we consider the rescaled function

$$\hat{u}(x,t) = Ku(Lx, Tt),$$

with constants $K, L, T > 0$, $\hat{u}$ is again a solution of (1.1) if

$$K^{m-1}L^2 = T.$$

On the other hand $\|\hat{u}_0\|_1 = 1$ if

$$KM = L^d.$$

Both equalities are satisfied for T arbitrary, $K = M^{-\alpha}T^{\lambda}$, $L = M^{\beta}T^{\mu}$ with $\beta = (m-1)\lambda/d$, $\mu = \lambda/d$. Under these conditions our assumptions say that $\hat{u}(x,1) \le C$. Then

$$u(x,T) = K^{-1}\hat{u}(L^{-1}x, 1) \le C/K = CM^{\alpha}T^{-\lambda} \quad \square$$

It is interesting to remark that if we calculate the decay rate of the Barenblatt solution in *sup* norm we find that formula (4.1) holds with a certain precise constant. It can be shown that the constant corresponding to the Barenblatt solution is the *optimal* constant in inequality (4.1). This means that the Barenblatt solutions solve an extremal problem, that of maximizing $\sup u(x,t)$ for given $t > 0$ and given $\|u_0\|_1 = M$. See in this respect [Va1].

The same techniques can be used to prove a more general version of the smoothing effect

Proposition 12. *For every $t > 0$ and $1 \le p < q \le \infty$ we have*

$$\|u(t)\|_q \le C\|u_0\|_p^{\gamma} t^{-\sigma} \tag{4.8}$$

whenever $u_0 \in L^p(\mathbf{R}^d)$. The constants C, γ and σ depend only m, p, q and d.

We leave it to the reader to fill in the details and also to calculate the explicit values of γ and d, which are given again by a scaling argument.

We may now give our complete existence result

Theorem 13. *There exists a strong solution u of problem (1.1), (1.2) for every $u_0 \in L^1(\mathbf{R}^d)$, $u_0 \geq 0$, $u \in C([0,\infty) : L^1(\mathbf{R}^d)) \cap L^\infty(\mathbf{R}^d \times (\tau,\infty))$ for every $\tau > 0$, and satisfies the estimates (2.1)–(2.4) and (3.1)–(3.3). If u_0 is strictly positive and continuous then u is a classical solution of (1.1).*

Proof. We only need to approximate u_0 with a sequence of functions $u_{0n} \in L^1(\mathbf{R}^d) \cap L^\infty(\mathbf{R}^d)$ converging to u_0, say $u_{0n}(x) = \max(u_0(x), n)$, apply the previous results to the solutions u_n and observe that since $\|u_{0n}\|_1 \leq \|u_0\|_1$, the sequence $\{u_n(\cdot,t)\}$ is bounded in $L^\infty(\mathbf{R}^d)$ uniformly in n and $t \geq \tau > 0$. Therefore, uniform estimates hold for u^m, u_t and Δu^m similar to the ones in Theorem 3 for $t \geq 2\tau > 0$, and we may pass to the limit $n \to \infty$ and obtain a strong solution u, which satisfies the above estimates. If u_0 is continuous and positive u is classical by local regularity theory as in Proposition II.5. □

Proposition 14. *Let $u_0 \in C(\mathbf{R}^d) \cap L^1(\mathbf{R}^d)$ be strictly positive and let u be the strong solution of (1.1), (1.2). Then $u \in C^\infty(Q) \cap C(\overline{Q})$ and is strictly positive in Q.*

Moreover, if u_0 is smooth this is reflected in the smoothness of u down to $t = 0$.

5. Finite speed of propagation. The free boundary.

We have already remarked that the diffusivity $D(u) = mu^{m-1}$ vanishes in the PME at the level $u = 0$. This degeneracy causes an important phenomenon to occur, i.e. finite speed of propagation of disturbances from 0, or more briefly, *Finite Propagation* (F.P.). We have observed this phenomenon on the source-type solutions in the form of compact support of the solution at any time $t > 0$. We are now in a position to establish the same result for a wide class of solutions of the PME we have

Theorem 15. *Let u be the strong solution to (1.1), (1.2) with initial data $u_0 \in L^1(\mathbf{R}^d) \cap L^\infty(\mathbf{R}^d)$, $u_0 \geq 0$ and such that u_0 is supported in a bounded set of $\mathbf{R}^d$. Then for every $t > 0$ the support of $u(\cdot,t)$ is a bounded set.*

The proof consists merely of noting that we can find a Barenblatt solution $U(x - x_0, t + \tau; M)$ such that

$$u_0(x) \leq U(x - x_0, \tau; M)$$

by suitably choosing x_0, τ and M. By Theorem 2 we get $u(x,t) \leq U(x - x_0, t + \tau; M)$, hence, if we denote by $r(t)$ the radius of the support of $U(x,t)$ at time t, cf. (0.3), (0.5),

$$\operatorname{supp}(u(t)) \subset x_0 + B_{r(t+\tau)}. \quad \square \tag{5.1}$$

Much more precise versions of the Finite Propagation Property can be established.

It can be proved that all the solutions are in fact continuous, cf. [CF]. Then the *positivity* set

$$P = P_u = \{(x,t) \in Q : u(x,t) > 0\} \tag{5.2}$$

is an open set in $\mathbf{R}^{d+1}$ an so are its sections

$$P(t) = \{x \in \mathbf{R}^d : u(x,t) > 0\} \tag{5.3}$$

in $\mathbf{R}^d$. Of course, the support of $u(t)$ is the closure of $P(t)$ in $\mathbf{R}^d$. We have

Proposition 16. *The family $\{P(t)\}_{t>0}$ is expanding, i.e. $P(t_1) \subset P(t_2)$ for every $0 < t_1 < t_2$.*

Proof. It follows from estimate (3.2), which just means that the function $z(t) = u(x,t)t^{\lambda}$ is nondecreasing for every fixed $x \in \mathbf{R}^d$. Hence if $z(t_1) > 0$ and $t_2 > t_1$ we have $z(t_2) > 0$, i.e. $x \in P(t_2)$. □

Remark. This property of expanding supports is sometimes called *retention* property, since u retains its positivity at any given point when time increases.

We may obtain a lower estimate for $P(t)$ similar to (5.1) as follows: we fix $\tau > 0$. Since u is continuous there exist $x_1 \in \mathbf{R}^d$ and M_1 and $\tau_1 > 0$ such that

$$u(x,\tau) \geq U(x - x_1, \tau_1; M_1).$$

By the comparison theorem it follows that for every $t \geq \tau$ we have $u(x,t) \geq U(x - x_1, t + \tau_1 - \tau; M_1)$, hence

$$P(t) \supset x_1 + B_{r(t+\tau_1-\tau)}, \tag{5.4}$$

which gives the desired lower bound.

The boundary of the positivity set in $Q, \Gamma = \partial P \cap Q$, called the *free boundary* or *interface*, is a very important object since it represents the region separating the "occupied region", $[u > 0]$, from the "empty region", $[u = 0]$. As a first result on the behaviour of the interfaces we can combine estimates (5.1) and (5.4) to get the following asymptotic expression

Proposition 17. *There exist constants $c_1, c_2 > 0$ such that*

$$c_1 t^{\lambda/d} \leq |x| \leq c_2 t^{\lambda/d} \tag{5.5}$$

holds for every $(x,t) \in \Gamma$ if t is large enough.

Proposition 17 implies in particular that every point of the space is eventually reached by the diffusing substance, a property that was not a priori obvious. It also gives an estimate of the speed of penetration of the substance into the empty region with exact exponent in the dependence of the radius on time. An exact asymptotic value of the constant can also be obtained, i.e we have $|x| \sim ct^{\lambda/d}$ with c depending only on m, d and the mass $\|u_0\|_1$. Actually, it can be proved that as $t \to \infty$ not only the solution but also its interface converges to the unique source-type solution $U(x,t;C)$ which has the same mass as u, i.e. C is determined from $\int U(x,t;C)\,dx = \int u(x,t)\,dx$, see [FK], [Va2] and [KV]. In this way the asymptotic sizes can be determined in first approximation by just copying from an explicit formula.

6. Local comparison.

Theorem 2 allows us to compare solutions of the Cauchy problem. However, in many cases we will be interested in corresponding functions which either are defined in a subdomain of Q or are not exact solutions of (1.1). We will give here a variant of Lemma 1 which covers such situations.

A function u defined in a subdomain S of Q is called a (strong) *supersolution* of (1.1) in S if u, u^m, u_t and $\Delta u^m \in L^1_{\text{loc}}(Q)$ and $u_t \geq \Delta u^m$ a.e. in S. A *subsolution* is defined in a similar way, only $u_t \leq \Delta u^m$. We have

Lemma 18. *Let Ω be a bounded subset of $\mathbf{R}^d$ with C^1 boundary, let $S = \Omega \times I \subset Q$, with $I = (t_1, t_2)$, and let u_1 be a subsolution, u_2 a supersolution of (1.1) in S. Assume moreover that u_1 and u_2 are continuous in $\overline{S}$ and $u_1 \le u_2$ on $\partial\Omega \times I$. Then for every $t \in [t_1, t_2]$*

$$\int [u_1(x,t) - u_2(x,t)]_+ dx \le \int [u_1(x,t_1) - u_2(x,t_1)]_+ dx \tag{6.1}$$

In particular, if $u_1(\cdot, t_1) \le u_2(\cdot, t_1)$ in Ω we have $u_1 \le u_2$ in S.

Proof. It follows the main lines of Lemma 1. We first select functions p and ζ as follows: $p \in C^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$, $0 \le p \le 1$, $p(s) = 0$ for $s \le \varepsilon$ and $p'(s) > 0$ for $s > \varepsilon$ with ε small and positive. $\zeta \in C_c^\infty(\Omega)$ is a cutoff function such that $0 \le \zeta \le 1$. Moreover, we may choose ζ in such a way that whenever we have $x \in \Omega$, $\zeta(x) < 1$ and $t \in I$ then $u_1^m(x,t) \le u_2^m(x,t) + \varepsilon$.

We subtract the inequations satisfied by u_1 and u_2 multiply by $p(u_1^m - u_2^m)\zeta$ and integrate over S. Arguing as in Lemma 1 and observing that $p(w)\nabla\zeta$ vanishes identically we get

$$\iint (u_1 - u_2)_t \, p(u_1^m - u_2^m)\zeta \le 0$$

from which we easily obtain (6.1) in the limit $\zeta \to 1$, $p(s) \to$ sign (s). □

Remark. We may combine Lemmas 1 and 18 to provide comparison for a subsolution and a supersolution defined in unbounded domains, for instance when $\Omega = (-\infty, 0)$ in one space dimension, or $\Omega = \mathbf{R}^d - B$, where B is a ball. We need to impose conditions on the initial and lateral boundary as in Lemma 18 plus integrability on the supersolution as $|x| \to \infty$, $t > 0$, like (1.9).

COMMENTS

As explained in the preceding chapter, pioneering work is due to Oleĭnik and collaborators. Sabinina [Sa] made the extension to several dimensions. The fundamental estimate is due to Aronson and Bénilan [AB]. The authors point out its optimality by checking it on the Barenblatt solutions and use the estimate in establishing existence of a strong solution of the Cauchy Problem with L^1-data. The boundedness of the solutions was first obtained by Véron [Ve] and Bénilan [Be]. The proof given here and based on the fundamental estimate is new (and considerably shorter). The control of the growth of the support as $t \to \infty$ (formula (5.5)) was first obtained in $d = 1$ by Knerr [Kn]. Sharp results are due to Vazquez [Va2] in $d = 1$ and [CVW] for $d > 1$, while the large-time behaviour of the solution was described by Friedman and Kamin [FK]; cf. [KV] for recent results.

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