Symmetrization and Mass Comparison for Degenerate Nonlinear Parabolic and related Elliptic Equations

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Abstract

We consider the solutions to various nonlinear parabolic equations and their elliptic counterparts and prove comparison results based on two main tools, symmetrization and mass concentration comparison. The work focuses on equations like the porous medium equation, the filtration equation and the p-Laplacian equation.

The results will be used in a companion work in combination with a detailed knowledge of special solutions to obtain sharp a priori bounds and decay estimates for wide classes of solutions of those equations.

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1 Introduction

A modern approach to the existence and regularity theory of partial differential equations relies on obtaining suitable a priori estimates in terms of the information available on the data, typically the norms of the data in appropriate functional spaces. Following such ideas, this work is oriented to obtaining basic estimates for nonlinear parabolic equations, suitable to derive qualitative and quantitative aspects of the theory. We will focus our attention on the so-called Filtration Equation,

$$u_t = \Delta\varphi(u) + f \tag{1.1}$$

where usually φ is a C^1 real function with nonnegative derivative; more generally, φ can be a maximal monotone graph (see more about max. mon. graphs in Subsection 3.4). Equation (1.1) includes as particular cases several basic models in nonlinear diffusion, like the Porous Medium Equation, where $\varphi(u) = u^m$, $m > 1$; the Fast Diffusion Equation, where $m < 1$; or the Stefan equation, where $\varphi(u) = \max\{u - 1, 0\}$. The Classical Heat Equation is included as the case $\varphi(u) = u$. Many nonlinear diffusion variants can be treated by similar methods, like the p-Laplacian equation, where $\Delta\varphi(u)$ is replaced by $\nabla \cdot (|Du|^{p-2}Du)$; we will discuss such an extension. It can also be applied to the mean curvature evolution equation, where $\Delta\varphi(u)$ is replaced by $\nabla\cdot((|1+|Du|^2)^{-1/2}Du)$, and to other models. Here, Du stands for the spatial gradient of u, often written in the literature ∇u.

The use of symmetrization to obtain a priori estimates for the solutions of parabolic equations can be traced to the work of C. Bandle in the late 1970's for linear equations with smooth coefficients, cf. [Ba3]. In order to treat nonlinear, possibly degenerate equations like (1.1), we proposed in our paper [V1] to use the technique of Implicit Time Discretization (ITD for short); in this way the original problem is reduced to obtaining similar estimates for elliptic equations of a definite type. In fact, by replacing the time derivative by an increment quotient and using a partition of the time interval $[0, T]$ of the form $t_0 = 0 < t_1 < \cdots < t_N = T$, we are reduced to solve a sequence of elliptic problems with zero-order term of the form

$$-h_i\Delta v(t_i) + u(t_i) = u(t_{i-1}) + h\, f(t_{i-1}), \quad v(t_i) = \varphi(u(t_i)), \tag{1.2}$$

by means of which we calculate $u(t_i)$ in terms of the value of u in the previous step, $u(t_{i-1})$. Here, i runs from 1 to N, $h_i = t_i - t_{i-1} > 0$; $f_i = f_i(x)$ is a suitable discretization of the function $f = f(x,t)$ at the mesh times; the notation $u(t_i)$ means $u(x, t_i)$, seen as a function of x for fixed $t = t_i$, and likewise $v(t_i)$; more precisely, $v(t_i)$ is related to $u(t_i)$ by $v(x,t_i) = \varphi(u(x,t_i))$ a.e. in x (or by $v(x,t_i) \in \varphi(u(x,t_i))$ a.e. if φ is multi-valued). The method is described in more detail in Subsection 7.1. Therefore, every step reduces to solving the elliptic equation

$$-h\Delta\varphi(u) + u \ni F, \tag{1.3}$$

where $F(x)$ a different, but known function in each step. In the study we can drop the inessential constant $h > 0$.

Important tools to attack these elliptic problems are comparison theorems of three types: (i) the standard Maximum Principle; (ii) symmetrization results; and (iii) concentration comparison results for radially symmetric solutions. While the first two are well documented in the literature, the latter was used as a basic tool in our paper; in our opinion, it is quite natural for the theory that we want to develop. Besides, we want to recall that symmetrization for elliptic equations is a widely researched subject, see next section, but the type of results needed for the parabolic theory relate to equations with the special structure (1.2), (1.3), whose main term for our purposes is the zero-order term; the results are not standard, as we will see. As a final tool, Implicit Discretization in Time allows to translate the results to the parabolic setting by means of an iterative routine and a passage to the limit in the time discretization.

In this paper we will present in full detail the method which was introduced in two short papers published in 1982, expanding the results as necessary. The method was first described in the 4-page paper [V1] by applying it to the well-known filtration equation (1.1). It was extended to the p-Laplacian equation in [V2], but a full development of the ideas implicit in the papers was not done and will only be attempted here.

The outline of the work is as follows: we present the main ideas and motivation of the comparison results in Section 2, and devote Section 3 to gather a number of functional preliminaries and notation.

Elliptic block: we study the theorems of concentration comparison in Section 4. In Section 5 we review the main results of symmetrization and combine them with concentration to obtain our main comparison result in the stationary case, Theorem 5.2. As a complement of the elliptic study, we recall in Section 6 the main facts of the theory of elliptic equations of type (1.3), containing in particular work by the author and collaborators in these years.

Evolution block: we turn then to the application of the method to evolution problems. In Section 7 Implicit Time Discretization is reviewed and the evolution comparison result, Theorem 7.2, is stated and proved. We present the adaptation of this general result to p-Laplacian evolution equation and related models with gradient-dependent diffusion in Section 8.

We add an appendix, Section 9, on the different formulations of the concept concentration, elaborating on Hardy-Littlewood-Pólya's relation, and state a general result that says that solutions of the elliptic equations considered in our study are less concentrated than the data, and the solutions of evolution problems become less concentrated as time advances, even if no symmetry is present.

The present paper serves also as an introduction to the companion paper [V7]; indeed, the method we describe turns out to be quite effective in establishing so-called smoothing effects for equations like the filtration equation (1.1) and the p-Laplacian equation. Actually, exact decay rates and optimal constants are obtained by this method.

While developing the ideas of symmetrization and mass concentration comparison of [V1], [V2], the present paper also includes a fair number of related results on the theory of nonlinear diffusion developed in this time. This is done because, as said above, the development of the nonlinear elliptic and parabolic theory has been the motivation of the comparison techniques. We did not strive however at a complete presentation of symmetrization for elliptic and parabolic equations, which is very wide and active topic. We comment on several extensions in Section 10. For a survey of work on symmetrization and mass comparison up to 1990 see [Di2], where many developments and improvements of the ideas of [V1, V2] are considered. We add a final Section 11 containing some bibliographical notes concerning the issues raised in the paper and the sources.

2 Precedents and detailed outline

The idea of using *symmetrization* for the purpose of obtaining a priori estimates is rather old, it takes its origin in the famous isoperimetric problems, and goes back in its modern form to the works of H. A. Schwarz and J. Steiner, more than one hundred years ago, [S, St], hence the names of *Schwarz symmetrization* and *Steiner symmetrization*. In this paper we will be concerned with the former one. In the last century it has been made popular by the classical book of Hardy, Littlewood and Pólya [HLP] which introduces the concept of one-dimensional rearrangement. The application of (Schwarz) symmetrization to obtaining a priori estimates for elliptic problems is described by Weinberger in [W], 1962. When we consider the classical second-order elliptic equation of the form $Lu = f$ in Ω with Dirichlet data $u = 0$ on the boundary of Ω, it is proved that the solution $u(x)$ of the original problem is pointwise bounded above in the following way: first, u is symmetrized into a radial function $u^*(|x|)$ defined in the symmetrized domain B, as explained in Section 3. This function is then compared with the radial solution $v = v(|x|)$ of a suitably symmetrized problem, essentially of the form $-\Delta v = g$, with $g = f^*/\lambda$, and it is proved that

$$u^*(|x|) \le v(|x|)$$

everywhere in the ball B. Finally, the latter solution, $v(|x|)$, is easier to calculate, and bounds for it are obtained which apply by comparison to the original problem. Most of the results in this area strive to obtain pointwise bounds for the solution of a problem by comparison with the symmetrized problem. Comparison of L^p norms is also obtained, see [T1]. The application of these ideas to treat nonlinear elliptic problems can be found in the work of G. Talenti, cf. [T3], 1979.

However, such a simple scheme fails to produce the same results when applied to parabolic equations, even for the heat equation, where the pointwise comparison is not generally true, and only a *sup* estimate is obtained, plus a comparison of Orlicz norms of the solutions, cf. [Ba1], 1976, or the book [Ba3, pages 206-218], 1980. Indeed, Bandle's book gives a wide account of symmetrization techniques applied to elliptic and parabolic problems under a number of regularity assumptions. She also introduces as a technical tool the result about comparison of integrals in balls, see (2.2), that we have developed and called concentration comparison.

Indeed, elaborating on the difference between elliptic and parabolic equations, I presented in 1982 the method to treat general parabolic quasilinear equations, possibly of singular or degenerate type, based on the following steps which will be fully discussed here. The steps are:

(i) discretization in time to reduce parabolic problems to a cascade of elliptic problems, where the theory was more advanced and the technical difficulties have an easier treatment,

(ii) symmetrization for the corresponding elliptic equations, which have peculiar lower order terms like (1.3) that have to be taken into account; and

(iii) concentration comparison for the same type of equations under conditions of radial symmetry.

The comparison results of the two latter steps must be formulated as hereditary properties, i.e,., they should be conserved in all the steps of the iteration process of the Implicit Time Discretization scheme, and then in the limit by the solution of the evolution problem. Let us note that ITD was quite popular in 1980's in treating nonlinear evolution problems associated with dissipative operators after the work of Crandall et al. [CL], [Cr]. The main novelty resided thus in the idea of the iterative use of symmetrization together with *concentration comparison*, also called *mass concentration comparison*, to arrive at estimates for general nonlinear parabolic equations of dissipative type. This is given full prominence: once comparison of concentrations is established, the relevant estimates follow.

The comparison obtained from the symmetrization technique is *not* pointwise as we said above. Instead, there is a natural comparison of integrals, termed *concentration comparison*, a concept that makes sense for radially symmetric functions, cf. Section 3 for definitions and notations. Let us state at this moment the main result that expresses the established comparison following paper [V1], where we consider the Cauchy Problem for equation (1.1) posed in the space-time domain $Q = \mathbf{R}^n \times (0, \infty)$, with initial data

$$u(x, 0) = u_0(x), \quad u_0 \in L^1(\mathbf{R}^n). \tag{2.1}$$

Theorem 2.1 *Let u be the unique solution of the problem* (1.1)-(2.1) *with initial data $u_0 \in L^1(\mathbf{R}^n)$, $u_0 \geq 0$, and right-hand side $f \in L^1(0, T : L^1(\mathbf{R}^n))$, $f \geq 0$. Let u_0^* be the spherical rearrangement of u_0 (as defined in Section* 3*) and f^* the spherical rearrangement of f with respect to the space variable (defined for almost every $t > 0$). Let v be the solution of* (1.1)-(2.1) *with data u_0^*, f^*. Then, for every $t > 0$ the function $v(\cdot, t)$ is spherically rearranged, $v(\cdot, t)^* = v(\cdot, t)$, and we have*

$$\int_{B_r(0)} u^*(x, t)\, dx \leq \int_{B_r(0)} v(x, t)\, dx. \tag{2.2}$$

The concept of generalized solution that applies in such a general setting will be discussed in Section 7. In order to better understand the result, it should be borne in mind that for a, say, integrable function f in $\mathbf{R}^n$ we have the important property

$$\int_{B_r(0)} f^*(x)\, dx = \sup \left\{ \int_E f(x)\, dx : E \subset \mathbf{R}^n, \mathrm{meas}\,(E) = \mathrm{meas}\,(B_r(0)) \right\}. \tag{2.3}$$

Moreover, the mass-concentration comparison (2.2) can be translated into comparison of all L^p-norms, even all Orlicz norms, giving thus Bandle's type of results. In particular, we get in the limit $p \to \infty$ an estimate of the L^∞-norm, and we arrive at the smoothing effect that we will study below in detail in [V7]. Comparison of norms in Marcinkiewicz spaces will also play a big role in deriving estimates.

The steps of the method of Implicit Time Discretization can all be written in the form

$$-\Delta v + \beta(v) \ni F, \tag{2.4}$$

where β is a monotone function, or more generally a m.m.g., the inverse of φ. Since here F stands for the previous step and $\beta(v)$ for the new step $u(t_i)$ in the iterative calculation of the discretized version of the evolution solution $u(x,t)$, the estimate in the elliptic equation must be precisely done on the zero-order term, and here lies the refinement of the usual symmetrization results and the difficulty. We are able to show that symmetrization and mass concentration comparison hold in Equation (2.4) for $u = \beta(v)$. Together, these results produce a full elliptic comparison result, see Theorem 5.2, that also includes the possibility of comparing different nonlinearities, a main novelty of the method that was developed in parallel by Bénilan and collaborators, [AB, BB] using different comparison techniques. In the final sections, the elliptic results are used to solve the parabolic comparison problem, Theorem 7.3.

We inform the reader that a direct parabolic approach to these subjects, without reduction to elliptic theory, is also possible and appears in the literature. It is done for instance in [Ba3, MR, AB, BB]. The advantage of using ITD depends on the type of application. Since it allows to treat problems with reduced regularity assumptions, it has in principle a wide range of application.

3 Functional Preliminaries

This section collects the main general ideas that we shall be using. For the expert reader, it can be skimmed or skipped, and used later as the needed concepts appear.

Let Ω a domain in $\mathbf{R}^n$, not necessarily bounded, possibly $\mathbf{R}^n$. We denote by $|\Omega|$ the Lebesgue measure of Ω and by $\mathcal{L}(\Omega)$ the set of [classes of] Lebesgue measurable real functions defined in Ω up to a.e. equivalence.

For very function f defined and measurable in Ω we define the distribution function μ_f of f by the formula

$$\mu_f(k) = \text{meas}\ \{x : |f(x)| > k\}, \tag{3.1}$$

where meas means the Lebesgue measure in $\mathbf{R}^n$. We denote by $\mathcal{L}_0(\Omega)$ the space of measurable functions in Ω such that $\mu_f(k)$ is finite for every $k > 0$. If Ω has finite measure, then $\mathcal{L}_0(\Omega) = \mathcal{L}(\Omega)$, otherwise $\mathcal{L}_0(\Omega)$ contains the measurable functions that tend to zero at infinity in a weak sense. All $L^p(\Omega)$ spaces with $1 \le p < \infty$ are contained in $\mathcal{L}_0(\Omega)$.

3.1. Rearrangement
A measurable function f defined in $\mathbf{R}^n$ is called *radially symmetric* (or radial for short) if $f(x) = \tilde{f}(r)$, $r = |x|$. It is called *rearranged* if it is nonnegative, radially symmetric, and $\tilde{f}$ is a non-increasing function of $r > 0$. For definiteness, we also impose that $\tilde{f}$ be left-continuous at every jump point. We denote by $\mathcal{R}(\mathbf{R}^n)$ the set of all rearranged functions in $\mathbf{R}^n$. We will often write $f(x) = f(r)$ by abuse of notation.

A similar definition applies to functions defined in a ball $B = B_R(0) = \{x \in \mathbf{R}^n : |x| < R\}$, and we get the family $\mathcal{R}(B)$.

3.2. Schwarz Symmetrization

For every bounded domain Ω the *symmetrized domain* is the ball $\Omega^* = B_R(0)$ having the same volume as Ω, i.e.,

$$|\Omega| := \text{meas}\,(\Omega) = \omega_n R^n. \tag{3.2}$$

The precise value ω_n of the volume of the unit ball in $\mathbf{R}^n$ is $\omega_n = 2\pi^{n/2}/(n\Gamma(n/2))$, where Γ is Euler's Gamma function. We put $(\mathbf{R}^n)^* = \mathbf{R}^n$. For a function $f \in \mathcal{L}_0(\Omega)$ we define the *spherical rearrangement* of f (also called the symmetrized function of f) as the unique rearranged function f^* defined in Ω^* which has the same distribution function as f, i.e., for every $k > 0$

$$\mu_f(k) := \text{meas}\,\{x \in \Omega : |f(x)| > k\} = \text{meas}\,\{x \in \Omega^* : |f^*(x)| > k\}. \tag{3.3}$$

The quantity is finite for every $k > 0$ by the assumption $f \in \mathcal{L}_0(\Omega)$. Then,

$$f^*(x) = \inf\{k > 0 : \text{meas}\{y : |f(y)| > k\} < \omega_n |x|^n\}. \tag{3.4}$$

A rearranged function coincides with its spherical rearrangement. Sometimes the name symmetric decreasing rearrangement is used. The following Hardy-Littlewood formula is well-known and illustrates the relation between f and f^*:

$$\int_{B_R(0)} f^*\,dx = \sup\{\int_E |f|\,dx : \ E \subset \Omega, \text{meas}\,(E) \le \text{meas}\,(B_R)\}. \tag{3.5}$$

There is also an immediate relation between distribution functions and L^p integrals given by the formulas

$$\int_\Omega |f|^p\,dx = -\int_0^\infty k^p\,d\mu(k) = p\int_0^\infty k^{p-1}\mu(k)\,dk, \tag{3.6}$$

and

$$\int_{\Omega\cap\{|f|\ge a\}} |f|^p\,dx = -\int_a^\infty k^p\,d\mu(k) = p\int_a^\infty k^{p-1}\mu(k)\,dk + a\mu(a). \tag{3.7}$$

Since the distribution functions of f and f^* are identical, conservation of integrals $\int_\Omega |f|^p\,dx = \int_\Omega (f^*)^p\,dx$ holds for every $p \in [1,\infty)$. Moreover, for every convex, nonnegative and symmetrical real function

$$\int_\Omega \Phi(f)\,dx = \int_\Omega \Phi(f^*)\,dx. \tag{3.8}$$

Note finally that f^* is continuous if f is. There is another related function often used in the proofs, namely the one-dimensional symmetric representation, defined by means of the formula

$$f_*(s) = f^*(r), \quad s = \omega_n r^n. \tag{3.9}$$

Then, f_* is defined in the interval $[0, |\Omega|]$, with $|\Omega| = \text{meas}\,(\Omega)$. Notice that

$$f_*(s) = \inf\{t \ge 0 : \mu(t) < s\}, \tag{3.10}$$

which makes f_* a *generalized inverse* of μ_f.

The topics of rearrangement and symmetrization are covered in many classical texts, for more details cf. e.g. the books [Ba3], [BS], [Kw], [LL], or the articles [T1] and [T6].

3.3. Mass Concentration

The comparison of mass concentrations is a basic notion in our approach to getting estimates for elliptic and parabolic equations. The precise definition that was introduced in [V1] is as follows:

Definition 3.1 *Given two radially symmetric functions $f, g \in L^1_{loc}(\mathbf{R}^n)$ we say that f is more concentrated than g, $f \succ g$, if for every $R > 0$,*

$$\int_{B_R(0)} f(x)\ dx \geq \int_{B_R(0)} g(x)\ dx, \tag{3.11}$$

i.e.,

$$\int_0^R f(r) r^{n-1}\ dt \geq \int_0^R g(r) r^{n-1}\ dt. \tag{3.12}$$

The partial order relationship $\succ$ will be called *comparison of mass concentrations*. We can also write $f \succ g$ in the form $g \prec f$. A similar definition applies to radially symmetric and locally integrable functions defined in a ball $B = B_R(0)$. In the case of rearranged functions this notion coincides with the comparison introduced by Hardy and Littlewood, [HLP], which is also used by Bandle in her book; but the present definition does not ask for the condition of rearrangement, only radial symmetry, and the difference is used below.

In fact, the natural way of looking at the concept is to view it as a comparison between two radially symmetric measures, $d\mu_f = f(x)dx$ and $d\mu_g = g(x)dx$. Then the comparison reads,

$$\mu_f(B_R(0)) \geq \mu_g(B_R(0)) \quad \text{for every} \quad R > 0. \tag{3.13}$$

In this formulation, comparison can be considered for general radially symmetric Radon measures. Measures are natural data for elliptic and parabolic equations.

The comparison of concentrations can be formulated in an equivalent way when the functions are rearranged, thanks to a powerful equivalence result, which seems to be essentially due to Hardy and Littlewood. This is the precise formulation for which we will give a proof in the Appendix, Section 9.

Lemma 3.2 *Let $f, g \in L^1(\Omega)$ be rearranged functions defined in $\Omega = B_R(0)$ and let $g \to 0$ as $|x| \to R$. Then $f \succ g$ if and only if for every convex nondecreasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ we have*

$$\int_\Omega \Phi(f(x))dx \geq \int_\Omega \Phi(g(x))dx. \tag{3.14}$$

The result is also valid when $R = \infty$ and $f, g \in L^1_{loc}(\mathbf{R}^n)$, $g \to 0$ as $|x| \to \infty$.

The topic of concentration will be further discussed in Section 9. Let us mention an elementary property that we will use later: if f and $g \in L^1(\Omega)$ and are nonnegative, then

$$(f+g)^* \prec f^* + g^*.$$

The proof follows immediately from the characterization (3.5). Actually, the relation is true for all combinations of the form $af + bg$ with $a, b > 0$ constant.

3.4. Maximal monotone graphs

We will study nonlinear parabolic equations like $u_t = \Delta\varphi(u) + f$, and their elliptic counterparts, like $-\Delta v + \beta(v) = f$. To simplify, we may assume that φ is a continuous and monotone increasing function of its argument $u \in \mathbf{R}$ and then β is its inverse function. Making the requirement of parabolicity on the first equation leads to the condition $\varphi'(s) > 0$ for all s. However, the second equation, which is used to solve the first, does not need such a strong requirement. It is then possible and useful to consider a greater generality in which φ and β can be any *maximal monotone graph* in $\mathbf{R}^2$.

For the concept and applications of maximal monotone graph (m.m.g. for short) we may refer the reader to Brezis' treatise [Br], which covers the much more general theory of maximal monotone operators in Hilbert spaces. Let us remark that this generality has been introduced into nonlinear analysis because of its interest in modeling a number of physical applications, most notably to formulate variational inequalities.

Here is a summary of the main facts that we need: a m.m.g φ in $\mathbf{R}^2$ is the natural generalization of the concept of monotone nondecreasing real function to treat in an efficient way the cases where there are discontinuities; since we are dealing with monotone functions, they must be jump discontinuities. We want to fill in these 'gaps' for the benefit of obtaining existence of solutions of the equations where φ appears. Then, the function must become multi-valued and contain vertical segments (corresponding to the jumps). The multi-valued function φ is defined in a maximal interval $D(\varphi)$ which is not necessarily $\mathbf{R}$, and can be open or closed on either end. If one of the ends of $D(\varphi)$ is finite and not included in $D(\varphi)$, then there is a vertical asymptote at this end; if it is included, there is a semi-infinite vertical segment in the graph. Typical maximal monotone graphs appearing in the nonlinear ODEs and PDEs of Mathematical Physics are the *signum* function

$$\mathrm{sign}\,(s) \begin{cases} = 1 & \text{for } s > 0, \\ = -1 & \text{for } s < 0, \\ = [-1, 1] & \text{for } s = 0; \end{cases}$$

its positive part, denoted by $\mathrm{sign}^+(s)$, where we modify the signum so that $\mathrm{sign}^+(s) = 0$ for $s < 0$ and $\mathrm{sign}^+(0) = [0, 1]$; the Stefan graph, defined by $H(s) = c\,s + L\,\mathrm{sign}^+(s)$ with constants $c, L > 0$; and the angle graph, $A(s) = 0$ for $s \geq 0$, $A(0) = (-\infty, 0]$, which is defined in $D(A) = [0, \infty)$.

One of the main advantages of this generality, which will be used here, is the fact that the inverse of a m.m.g. is again a m.m.g.; actually, both graphs are symmetric with respect to the main bisectrix in $\mathbf{R}^2$.

The standard and somewhat awkward notation when using multi-valued operators is set inclusion, so that when (a, b) is a point in the graph φ we write $b \in \varphi(a)$ instead of $b = \varphi(a)$, since generally $\varphi(a)$ is not a singleton.

3.5. Comparison of maximal monotone graphs

In the study of the Filtration Equation (1.1) we will be interested in comparing the concentrations of solutions of two equations with different nonlinearities φ. This final goal will be prepared with a result for elliptic equations. We introduce the following concepts.

Definition 3.3 *We say that a max. mon. graph φ_1 is weaker than another one φ_2, and we write $\varphi_1 \prec \varphi_2$, if they have the same domains, $D(\varphi_1) = D(\varphi_2)$, and there is a contraction $\gamma : \mathbf{R} \to \mathbf{R}$ such that*

$$\varphi_1 = \gamma \circ \varphi_2 \,. \tag{3.15}$$

By contraction we mean $|\gamma(a)-\gamma(b)| \le |a-b|$. This implies in particular φ_1 must have horizonal points (or horizontal intervals) at the same values of the argument as φ_2, and maybe some more. We also assume that φ_1 does not accept vertical intervals (i.e., it is one-valued). Note that for smooth graphs condition (3.15) just means that

$$\varphi_1'(s) \le \varphi_2'(s), \quad \text{for every } s \in D(\varphi_2), \tag{3.16}$$

which is easier to remember or to manipulate. We will see that φ' is interpreted as the diffusivity in many parabolic problems, so that relation (3.16) can be phrased as: φ_1 *is less diffusive than* φ_2. This explains why it will be important in the evolution analysis.

In the development of the corresponding elliptic theory we will need to rephrase this condition in terms of the inverse graphs β_i entering the equations like (4.3) or (4.7) below: it then means that there is a contraction $\gamma : \mathbf{R} \to \mathbf{R}$ such that

$$\beta_2 = \beta_1 \circ \gamma. \tag{3.17}$$

To be precise, we also have to specify the relation of the domains, $D(\beta_1) = \gamma(D(\beta_2))$. But, as a general rule, we will prefer to stick to comparisons of diffusivities, $\varphi = \beta^{-1}$.

3.6. Marcinkiewicz spaces

Different classes of functional spaces are natural in the study of symmetrization, for instance the Lebesgue spaces $L^p(\Omega)$. Also the Marcinkiewicz spaces play a role. The Marcinkiewicz space $M^p(\mathbf{R}^n)$, $1 < p < \infty$, is defined as set of $f \in L^1_{loc}(\mathbf{R}^n)$ such that

$$\int_K |f(x)|\, dx \le C|K|^{(p-1)/p}, \tag{3.18}$$

for all subsets K of finite measure, cf. [BBC]. The minimal C in (3.18) gives a norm in this space, i.e.,

$$\|f\|_{M^p(\mathbf{R}^n)} = \sup \left\{ \text{meas}\,(K)^{-(p-1)/p} \int_K |f|\, dx : \; K \subset \mathbf{R}^n, \; \text{meas}\,(K) > 0 \right\}. \tag{3.19}$$

Since functions in $L^p(\mathbf{R}^n)$ satisfy inequality (3.18) with $C = \|f\|_{L^p}$ (by Hölder's inequality), we conclude that $L^p(\mathbf{R}^n) \subset M^p(\mathbf{R}^n)$ and $\|f\|_{M^p} \le \|f\|_{L^p}$. The Marcinkiewicz space is a particular case of Lorentz space, precisely $L^{p,\infty}(\mathbf{R}^n)$, and is also called weak L^p space.

Marcinkiewicz spaces will be important in paper [V7] tied to the idea of "worst case strategy" that plays an important role in our study of smoothing effects.

4 Concentration theory for elliptic equations

4.1. The Equations

We apply here the idea of concentration comparison to nonlinear elliptic equations with symmetric data and solutions. Our model equation is

$$-\Delta u + \beta(u) = f, \tag{4.1}$$

posed in a domain $\Omega \subset \mathbf{R}^n$. Here, we may assume to simplify that β is a continuous and monotone increasing function of its argument $u \in \mathbf{R}$. In the generality of m.m.g.'s we have to write the equation in the form of set inclusion,

$$-\Delta u + \beta(u) \ni f. \tag{4.2}$$

For normalization we usually take $\beta(0) \ni 0$ (but see exceptions later). Note that we have $v(x) := \Delta u(x) + f(x) \in \beta(u(x))$ a.e. We may use the inverse graph $\varphi = \beta^{-1}$ to write $u(x) \in \varphi(v(x))$ a.e., so that the equation looks formally

$$-\Delta\varphi(v) + v \ni f. \tag{4.3}$$

Actually, we would like to replace the Laplace operator in the first term of the equation by a whole class of operators of interest both in the theory of Nonlinear Analysis and in the applications to diffusive phenomena. We will consider operators $\mathcal{A}$ of the form

$$\mathcal{A}(u) = -\mathrm{div}\,(A(\nabla u)) \tag{4.4}$$

(and some variants thereof). We need the vector function $A : \mathbf{R}^n \to \mathbf{R}^n$ to be strictly monotone. This means that for two different vectors $w_1, w_2 \in \mathbf{R}^n$, $w_1 \ne w_2$, the following property holds

$$\langle A(w_1) - A(w_2), w_1 - w_2\rangle > 0. \tag{4.5}$$

In this section we deal with radial functions and we specialize A to the isotropic form

$$A(\nabla u) = \sigma(|\nabla u|)\nabla u, \tag{4.6}$$

which for radial functions becomes $\mathcal{A}\,u = -r^{1-n}\left(r^{n-1}\,\sigma(|u'(r)|)u'(r)\right)'$. The function $\sigma : [0,\infty) \to \mathbf{R}$ is assumed to be continuous and positive (except maybe at $s = 0$). A particular case is the famous *p-Laplace operator* where $\sigma(s) = s^{p-2}$ for $s \ge 0$, and then we write $\mathcal{A}u = -\Delta_p(u)$, $p > 1$. Also the *mean curvature operator* is included, with $\sigma(s) = (1+s^2)^{-1/2}$. Summing up, the more general equation we consider here is

$$\mathcal{A}u + v = f, \quad \text{and} \quad v \in \beta(u) \ a.e., \tag{4.7}$$

which we will often write in a more informal way as $\mathcal{A}u + \beta(u) \ni f$.

We will call f the *forcing term* of the equation. We assume that $f \in L^1_{loc}(\Omega)$ (exceptionally, it can be a measure). A *solution* of (4.2) is then a pair $(u, v) \in W^{1,1}_{loc}(\Omega) \times L^1_{loc}(\Omega)$ such that $|A(\nabla u)| \in L^1_{loc}(\Omega)$ and $\mathcal{A}u = f - v$ in $\mathcal{D}'(\Omega)$ and $v(x) \in \beta(u(x))$ a.e. Well-posedness of the problem needs a more definite setting, including boundary conditions. Typical options will be discussed in the next section. It must be observed that the results and applications we have in mind lead us to focus more on v than on u, contrary to standard elliptic theory and practice.

4.2. Solutions are less concentrated than their data

Our first result in the area of concentration shows that the solutions v of equations like (4.7) are less concentrated than the forcing term f from which they originate, under certain conditions that are often fulfilled, though not always. In physical terms, this reflects the spreading effect due to diffusion represented by the Laplace operator.

We recall that, in our definition, concentration is a concept that applies to radially symmetric functions (but see Section 9). We work in a space Ω that is either a finite ball $B_R(0)$ or the whole space $\mathbf{R}^n$ (i.e., $R = \infty$). There exist some differences in the results, but the argument is the same. We consider radial solutions of equation (4.7) under condition (4.6), with σ as stated.

Proposition 4.1 *Let $f(r)$ be a radially symmetric and decreasing function in $L^1_{loc}(\Omega)$ and let $(u(r), v(r))$ be a solution pair of the general equation* (4.7). *There is an alternative:*
(i) Standard case: $f \succ v$, *and then v and u are monotone non-increasing;*
(ii) Increasing case: $f \succ v$ *does not hold, and then u is increasing in r in an interval of the form $I = (a, R)$, v is nondecreasing with $v > f$ in I, and*

$$\int_\Omega (v(x) - f(x))\, dx > 0. \tag{4.8}$$

Moreover, in this case the function $X(r) := \int_0^r s^{n-1}\,(v(s) - f(s))\, ds$ is positive and grows in I.

Proof. For $0 < r < R$ we define the continuous functions

$$V(r) := \frac{1}{n\omega_n}\int_{B_r(0)} v(x)dx = \int_0^r v(s)s^{n-1}\, ds,$$
$$F(r) := \frac{1}{n\omega_n}\int_{B_r(0)} f(x)dx = \int_0^r f(s)s^{n-1}\, ds.$$

Integration of the equation $v = f - \mathcal{A}u$ in $B_r(0)$ gives the basic relation:

$$V(r) - F(r) = r^{n-1}A(u'(r)), \tag{4.9}$$

where we write $A(u'(r)) = \sigma(|u'(r)|)\, u'(r)$.

(i) The first case of the alternative happens when $V(r) \le F(r)$ for all $r \ge 0$, which is an equivalent way of expressing that $f \succ v$. Then, we can show that $v(r)$ and $u(r)$ are monotone non-increasing. Argument for u: it follows from (4.9) and the form of A, (4.6), that $u'(r) \le 0$. If β is single-valued or u is strictly decreasing then we immediately conclude that $v(r)$ is also monotone. We have to establish monotonicity for v in the case where β is multi-valued. Now, we can see that the only possibility for v to be non-monotone is when u fails to be strictly decreasing, say $u(r) =$ constant in an interval $J = a_1 < r < a_2$. But then the equation implies that $v = f$ in J, hence it must be non-increasing also there.

(ii) Assume now that $f \succ v$ does not hold. Then the set

$$G_\varepsilon := \{r \ge 0 \;:\; V(r) > F(r) + \varepsilon\}$$

is a nonempty open set of $(0, R)$ for some small $\varepsilon > 0$. In principle, it is a union of disjoint intervals of the form $G_\varepsilon = \bigcup_i (a_i, b_i)$. Formula (4.9) implies that $u(r)$ is a strictly increasing function inside G_ε, $\sigma(|u'(r)|)\, u'(r) \ge \varepsilon\, r^{1-n} > 0$. By the monotonicity of β and the relation $v(r) \in \beta(u(r))$ a.e. we conclude that $v(r)$ is also nondecreasing. Since f is non-increasing, it follows that

$$r^{1-n}(V - F)' = v - f$$

is nondecreasing in G_ε. Now, $V - F$ is continuous and $V(0) - F(0) = 0$, We conclude that whenever G_ε is non-empty it can only be an open interval of the form $I = (a_\varepsilon, R)$ for some $a_\varepsilon \in (0, R)$ and then $V - F$ is positive and nondecreasing in I. It follows that $(V - F)'$, hence $v - f$ must be positive at a certain point, say $c \in I$. Since $v - f$ is nondecreasing, this means that it does not tend to zero as $r \to R$. We conclude that both $u(r)$ and $V - F$ are increasing in $G_\varepsilon = (a_\varepsilon, R)$.

Remark. In the increasing case, if $R = \infty$ we can estimate the growth of $u(r)$ and $V(r) - F(r)$. Suppose that $v - f$ tends to a finite limit $C > 0$ as $r \to \infty$; then we have $V - F \sim Cr^n/n$ so that, using (4.9), $A(u'(r)) \sim C_1\, r$. If the limit $C = \infty$ the previous estimates become lower bounds.

We can be more precise in specific cases. Thus, in the Laplacian case, $\sigma = 1$, u grows at least quadratically $r \to \infty$. In any case $A(u') \sim r$ implies that $u'(r)$ is bounded below by $u'(r_1)$ if $r > r_1$, hence $u(r) \to \infty$ as $r \to \infty$.

In the applications we will consider below and in [V7] we want to work in the standard case. As a consequence of the preceding analysis, this case holds under several additional assumptions that are usually found in practice. Recall that f was assumed to be radially symmetric and non-increasing, but we not necessarily assume that $f \ge 0$ (in which case it is rearranged).

Corollary 4.2 *Under the conditions of Proposition* 4.1*, the standard case happens if one of the following situations occur:* (a)

$$\int_\Omega (v - f)\, dx \le 0, \tag{4.10}$$

(b) *R is finite and u takes Neumann data $u'(R) \le 0$,*
(c) *R is finite, $f \ge 0$, and v takes a value $v(R) \le 0$,*
(d) *$R = \infty$, and u is bounded, or*
(e) *$R = \infty$, $f \ge 0$, $v \in \mathcal{L}_0(\mathbf{R}^n)$ (in particular, if $v \in L^p(\mathbf{R}^n)$ for any $p < \infty$). In this last case, v is rearranged.*

Proof. The first assertions are easy. As for the last, we have seen in the proof of the Proposition that in part (ii) $v - f$ is positive in I. If $f \ge 0$ then v is positive and nondecreasing, hence it goes to a positive or infinite limit, which contradicts $v \in \mathcal{L}_0(\mathbf{R}^n)$. We are thus in the standard case, so that v is non-increasing and goes to zero as $r \to \infty$. See Remark 1) below in this respect.

Remarks. 1) It often happens that the integral of formula (4.10) vanishes, see Section 6 where it appears as a form of conservation of mass. Such an equality is a way of showing that the pointwise inequality $v \le f$ does not hold in general, for then it would imply that $v = f$ and $\mathcal{A}u = 0$. On the other hand, if we have equality of integrals and also $f(r) \to 0$ as $r \to \infty$, then $v(r) \to 0$, i.e., $v \in \mathcal{L}_0(\mathbf{R}^n)$. Then, both f and v are rearranged.

2) The results are false if f is not monotone. Imagine for instance that $n = 1$ and f is radially symmetric, nonnegative and compactly supported in the interval $[1, 3]$ and assume moreover that the part contained in $\{r > 0\}$ is rearranged around the point $r = 2$. Assume that $\beta(u) = u$, so u solves $u - u'' = f$ with $u \ge 0$. It follows from the strong maximum principle that $u > 0$ everywhere, hence $v \prec f$ is not true.

3) The increasing case of Proposition 4.1 cannot be discarded without some assumptions. Consider for instance the case $n = 1$, $\sigma = 1$, $f = 0$ and $\beta(u) = u$. We have an increasing solution of the form $u(r) = \mathrm{Cosh}(r)$, which is symmetric and increasing in the half-line. Examples in several dimensions, or having nontrivial f, are left as exercises.

4) Even if f is nonnegative, it is not proved here that in the standard case $u(r)$ and/or $v(r)$ are always nonnegative. If this happens, usually as a consequence of the Maximum Principle and the fact that $0 \in \beta(0)$, then the conclusion of the standard case is that they are rearranged. Generally speaking, we could have $v(r)$ converging to a $C \in \mathbf{R}$ or to minus infinity. If $f \to 0$, then the condition $f \succ v$ implies that $C \le 0$.

5) We are assuming that $v(r)$ and $u(r)$ are radially symmetric functions. In the general elliptic theory, this property follows from the radial symmetry of the data, the fact that the problem is invariant under rotations, plus the uniqueness of solutions in a certain class.

6) The above considerations are easier when β is smooth and strictly monotone $0 < \beta'(u) < \infty$. An argument of approximation justified by the theory allows to pass to the general case. But we have shown that the direct argument works easily in the present case.

4.3. Integral super- and subsolutions

The proof of the preceding result does not need to deal with an exact solution. It is sufficient that u is a *subsolution* in the sense that $\mathcal{A}u + \beta(u) \le f$. We can go one step further. We define a **radial integral subsolution** of equation (4.7) as a pair of radial functions $(u,v)\in(L^1_{loc}(\Omega))^2$ such that

$$f\succ \mathcal{A}u+v,\quad v(r)\in\beta(u(r))\ a.e. \tag{4.11}$$

Sometimes we refer to the subsolution only as u, since v is then defined in terms of u and f. In the same way we define a **radial integral supersolution**, using $f\prec\mathcal{A}u+v$ instead of $f\succ\mathcal{A}u+v$.

Proposition 4.3 *Let $f(r)$ be a radial function in $L^1_{loc}(\mathbf{R}^n)$ and let $(u(r),v(r))$ be a radial integral subsolution of equation* (4.7)*. Then, either*
(i) *$f\succ v$, or*
(ii) *the second option of Proposition* 4.1 *holds.*

The proof is the same with very minor changes, once we remark that the new assumption reads as

$$r^{n-1}A(u'(r))\geq V(r)-F(r), \tag{4.12}$$

which implies that u is strictly increasing in G_ε (the reader should at this point re-do that part of the proof of Proposition 4.1). Note that weak subsolutions in the sense of formula (4.11) need not be rearranged or even monotone if we do not impose the condition such a priori. The second part of the alternative is in any case true.

4.4. Comparison of solutions

Our next result is the comparison for different data. We still work with radial solutions and with nonlinearities A as above.

Proposition 4.4 *Let $f_i(r)$, $i=1,2$ be two radially symmetric functions in $L^1_{loc}(\Omega)$, let $u_i(r)$ be the respective solutions of* (4.7)*, and let $v_i=f_i-\mathcal{A}u_i$. Assume that $f_1\succ f_2$. Then either*
(i) (standard case) *$v_1\succ v_2$, and then*

$$\int_\Omega (v_2(x)-v_1(x))\,dx\leq 0;$$

or (ii) *$v_1\succ v_2$ does not hold, and then the functions $u(r)=u_2(r)-u_1(r)$ and*

$$\int_{B_r(0)}(v_2(x)-v_1(x))\,dx$$

are both positive and increasing for all large r. The same result holds when u_1 is an integral supersolution and u_2 is an integral subsolution.

Proof. We argue as before. We introduce the functions $u=u_2-u_1$, $v=v_2-v_1$, $f=f_2-f_1$,

$$V(r):=\int_0^r v(r)\,r^{n-1}dr,\quad F(r):=\int_0^r f(r)\,r^{n-1}dr,$$

and derive from the equation the relation

$$V(r) - F(r) \le r^{n-1}\{A(u_2'(r)) - A(u_1'(r))\}. \tag{4.13}$$

If we assume that u_1 and u_2 are exact solutions then equality holds in this formula, for super- and subsolutions we have inequality. By assumption, $F \le 0$. In the standard case we have $V \le 0$. To explore what happens otherwise, we consider the sets

$$G = \{r : V(r) > 0\}, \quad G_\varepsilon = \{r : V(r) > \varepsilon\}.$$

In case (ii) G_ε is non-empty for some small ε, we conclude that in G_ε $A(u_2') > A(u_1')$, hence $u_2' > u_1'$ and $u = u_1 - u_2$ is strictly increasing in G. Since $V(0) = 0$ and V becomes positive, it must be increasing somewhere in G, say at $c \in G$, hence $v(c) = v_2(c) - v_1(c) > 0$ and by the monotonicity of β, $u_2(c) \ge u_1(c)$, so that $u_2 > u_1$ for $r > c$, $r \in G$. It follows that $v = v_2 - v_1 \ge 0$ for $r \ge c$, $r \in G$, hence V is nondecreasing in G, which must be an interval going to infinity or to R, and so are the sets G_ε for small $\varepsilon > 0$.

Summing up, in the nonstandard case we have as $r \to R \le \infty$ that (a) $V(r) \ge \varepsilon > 0$ and is nondecreasing, (b) $u = u_2 - u_1$ is positive and increasing.

Remarks. As remarked above, we are interested in the standard case, hence we discuss several variants of the theorem where the assumption $u_i \in \mathcal{L}_0(\Omega)$ is replaced by other conditions. Thus,

(1) The standard case holds if (a) $\int_\Omega (v_2(x) - v_1(x))\,dx \le 0$, or (b) $\Omega = \mathbf{R}^n$ and $u_i \in \mathcal{L}_0(\mathbf{R}^n)$, or (c) $\beta^{-1}(0) = \{0\}$ and $v_i \in \mathcal{L}_0(\mathbf{R}^n)$.

(2) In many instances we know (see Section 6) that f_i and v_i are integrable and $\int v_i = \int f_i$. Then we have $V(R) - F(R) \le 0$, and the standard case follows.

(3) In dimensions $n = 1, 2$ and with $\sigma = 1$ (Laplacian case) the assumption that G is non-empty leads to the conclusion that $u_2 - u_1 \to \infty$ (hint: integrate (4.13)). Thus, a condition that u_i be bounded is enough to get the standard case. This is in particular the case if $v_i \in \mathcal{L}_0(\mathbf{R}^n)$ and $\beta^{-1}(0)$ is a bounded interval, $[a, b]$. For $n \ge 3$ such condition does not work, see Section 6.

(4) The special cases where β is trivial in an infinite interval: $\beta^{-1}(0) \supset (-\infty, b)$, or where $\beta(s) > 0$ for every $s \in D(\beta)$, are specially interesting in dimensions $n = 1, 2$ and will be also discussed in Section 6.

The result has an extension when the relation of concentration is not exactly satisfied.

Corollary 4.5 *Let $f_i(r)$, $i = 1, 2$ be two radially symmetric functions in $L^1_{loc}(\Omega)$, let $u_i(r)$ be the respective solutions of* (4.7), *and let $v_i = f_i - \mathcal{A}u_i$. Assume that there is a constant $C > 0$ such that*

$$\int_\Omega f_2(x)\,dx \le \int_\Omega f_1(x)\,dx + C.$$

Then either we have (i) (standard case)

$$\int_\Omega v_2(x)\,dx \le \int_\Omega v_1(x)\,dx + C;$$

or otherwise (ii): *the function $u(r) = u_2(r) - u_1(r)$ is positive and increasing for all large r and*

$$\int_{B_r(0)} (v_2(x) - v_1(x))\, dx$$

is larger than C and increasing for all large r. The same result holds when u_1 is an integral supersolution and u_2 is an integral subsolution.

The proof is exactly the same as in Proposition 4.4, fixing our attention on the set G_C when it is nonempty.

4.5. Comparison with different nonlinearities

The previous result can be strengthened to cover different nonlinearities β. We need a criterion to compare the graphs entering equations (4.2) or (4.7). Actually, it is more convenient to think of the equation as written in the form

$$\mathcal{A}\varphi(v) + v \ni f \tag{4.14}$$

(which is the form directly derived from the filtration equation), and define the comparison for the inverse graphs $\varphi = \beta^{-1}$. This is what we have done at the end of Section 3.

The following is our more general comparison result for equation (4.7).

Theorem 4.6 *Let $f_i(r)$, $i = 1, 2$, be two radial functions in $L^1_{loc}(\Omega)$, let β_i be two m.m.g.'s and let $u_1(r)$ be an integral supersolution and $u_2(r)$ an integral subsolution of the respective equations*

$$\mathcal{A}u_i + \beta_i(u_i) \ni f_i, \tag{4.15}$$

$v_i = f_i - \mathcal{A}u_i$. If $f_1 \succ f_2$ and $\beta_1^{-1} \prec \beta_2^{-1}$, and either v_1 or v_2 is monotone nondecreasing, then an alternative holds: either (i) *$v_1 \succ v_2$, or* (ii) *the quantity*

$$V(r) = \int_{|x| \le r} (v_2(x) - v_1(x))\, dx$$

is positive and increasing for all large r, $v_2(r) - v_1(r)$ is nonnegative and $u_2 - u_1$ is strictly increasing.

Roughly speaking, the result states a reasonable fact: when f_1 is more concentrated than f_2 and $\varphi_1 = \beta_1^{-1}$ is less diffusive than $\varphi_2 = \beta_2^{-1}$, than the first solution is more concentrated than the second one.

Proof. We try to repeat the proof of Proposition 4.4. If G is empty we are done. If it is non-empty, then we have $V > \varepsilon > 0$ in a maximal interval $I = (a, b) \subset (0, R)$ with $a > 0$. In that interval we have $r^{n-1}\{A(u_2') - A(u_1')\} \ge \varepsilon > 0$, and we conclude that $u = u_2 - u_1$ is strictly increasing. On the other hand, $V(a) = 0$ and $V(r) > 0$ in I, hence there are points $c > a$ as close as we wish to a, such that $V'(c) = w(c) > 0$.

The novelty in the argument lies in the proof of the fact that V is increasing all the way to $r = R$. This is an immediate consequence of the following **Claim:** *V cannot have a maximum in I.*

This claim is proved as follows. If V has a positive maximum at a point $c \in I$, formally $V'(c) = v(c) = 0$. Even if v is not continuous, we know that either V is constant on an interval $[c_1, c_2]$ containing c, or there are points $c_1, c_2 \in I$, $c_1 < c < c_2$, such that

$$v(c_1) = v_2(c_1) - v_1(c_1) > 0, \quad v(c_2) = v_2(c_2) - v_1(c_2) < 0.$$

We know that $u(c_2) - u(c_1) > 0$. Now,

$$u(c_2) - u(c_1) = \varphi_2(v_2(c_2)) - \varphi_1(v_1(c_2)) - \varphi_2(v_2(c_1)) + \varphi_1(v_1(c_1)).$$

If we know that v_2 is monotone, $v_2(c_1) \geq v_2(c_2)$, and, using the fact that $\varphi_1(s) = \gamma(\varphi_2(s))$, we get

$$\varphi_2(v_2(c_1)) - \varphi_2(v_2(c_2)) \geq \varphi_1(v_2(c_1)) - \varphi_1(v_2(c_2)).$$

We may then estimate the increment of u in terms of only φ_1:

$$u(c_2) - u(c_1) \leq \varphi_1(v_2(c_2)) - \varphi_1(v_1(c_2)) - \varphi_1(v_2(c_1) + \varphi_1(v_1(c_1)).$$

Using the inequalities $v_2(c_2) < v_1(c_2)$ and $v_2(c_1) > v_1(c_1)$, and the monotonicity of φ_1 we get $u(c_2) - u(c_1) \leq 0$. This contradicts the fact that u is strictly increasing in I.

On the other hand, if we know that v_1 is monotone, $v_1(c_1) \geq v_1(c_2)$, and, using the fact that $\varphi_1(s) = \gamma(\varphi_2(s))$, we get

$$\varphi_1(v_1(c_1)) - \varphi_1(v_1(c_2)) \leq \varphi_2(v_2(c_1)) - \varphi_2(v_2(c_2)).$$

We may then estimate the increment of u in terms of only φ_1:

$$u(c_2) - u(c_1) \leq \varphi_2(v_2(c_2)) - \varphi_2(v_1(c_2)) - \varphi_2(v_2(c_1)) + \varphi_2(v_1(c_1)).$$

Using the inequalities $v_2(c_2) < v_1(c_2)$ and $v_2(c_1) > v_1(c_1)$, and the monotonicity of φ_2 we get $u(c_2) - u(c_1) \leq 0$. This again contradicts the fact that u is strictly increasing in I.

In case V is constant in $[c_1, c_2]$, then $v = 0$ inside and the argument is even simpler.

Using this result, if G is not empty then $v_2 \geq v_1$ for $r > a$ and we have V increasing in G_ε, hence $I = (a, R)$ with $\lim_{r \to R} V(r) > 0$.

Remarks (1) If both graphs are normalized in the form $0 \in \beta_i(0)$, and $u_2 \geq 0$, then we can prove that in case (ii) $u_2 - u_1$ is positive and increasing for all large r: Indeed, we know that $u_2 - u_1$ is strictly increasing in I, so that either $u_2 - u_1 > 0$ for $r \in I$ or $0 > u_1 > u_2$ in a subinterval J. In this case we have,

$$v_2 = \beta_2(u_2) = \beta_1(\gamma(u_2)) \leq \beta_1(u_2) \leq \beta_1(u_1) = v_1,$$

which means that V is non-increasing in $J \subset I$, a contradiction. Therefore, we have $u_2 > u_1$ for all $r > a$ and $\lim_{r \to \infty}(u_2 - u_1) > 0$.

(2) It is maybe interesting to consider a different proof of the Claim when φ is smooth. If there is a maximum at a point $c \in I$ we have $V_1(c) > V_2(c)$, hence

$$0 \geq u_2'(c) > u_1'(c), \tag{4.16}$$

where we have also used the fact that u_2 is rearranged (so that $u_2' \leq 0$). Since $u = u_1 - u_2$ is C^1 and the β_i are C^1, so is $v = v_1 - v_2$. At the point of maximum we have $V'(c) = 0$, and $V''(c) \leq 0$, hence

$$v_1(c) = v_2(c) = v, \quad v_2'(c) \leq v_1'(c).$$

Moreover, $v_i' \leq 0$ in I. Therefore, we get

$$u_2'(c) = (\varphi_2(v_2))'(c) = \varphi_2'(v)v_2'(c) \leq \varphi_1'(v)v_2'(c) \leq \varphi_1'(v)v_1'(c) = u_1'(c),$$

a contradiction with (4.16).
(3) The theorem applies in particular when $\beta_1 \equiv 0$ for every β_2 such that $\beta_2(0) \ni 0$. But in that case concentration comparison can be replaced by pointwise comparison, as is well known, cf. [T3].
(4) Again, the standard case is obtained under a number of additional assumptions listed after Proposition 4.4, like (a) $\int_\Omega (v_2(x) - v_1(x))dx \leq 0$, or (b) $u_i \in \mathcal{L}_0(\mathbf{R}^n)$.

Extension. The extended version of Corollary 4.5 with different maximal monotone graphs is also true. We leave the detail to the reader.

5 Symmetrization and comparison in the model elliptic problem

We tackle next the second main technique of our paper, Schwarz symmetrization for elliptic equations, a subject which is better known. Here, we review the basic theory since it leads at the end of the section to the presentation of the interaction between both techniques. Such interaction needs a different way of looking at the standard symmetrization inequality in terms of concentration comparison.

Let us consider the problem

$$-\sum_{i,j} \partial_i(a_{ij}\partial_j u) + b(x,u) = f \tag{5.1}$$

in a bounded open set $\Omega \in \mathbf{R}^n$, or in $\mathbf{R}^n$. The coefficients a_{ij} are bounded measurable functions in Ω satisfying the ellipticity hypothesis

$$\sum_{i,j} a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \tag{5.2}$$

for some constant $\lambda > 0$ all vectors $\xi \neq 0$. Without loss of generality we may take $\lambda = 1$ by replacing f by f/λ. We assume that the function $b(x,u)$ is measurable, continuous and nondecreasing in u for fixed x, and bounded in x uniformly for bounded u. We will also assume that

$$b(x,u)\,u \geq 0 \quad \text{for a.e. } x \text{ and all } u \tag{5.3}$$

The second member f is a measurable function in some Lebesgue L^p space, though other spaces like Marcinkiewicz spaces also appear in the literature. We take boundary conditions of Dirichlet type

$$u(x) = 0 \quad x \in \partial\Omega. \tag{5.4}$$

The *symmetrized problem* is posed in the ball $\Omega^* = B_R(0)$ and consists of the symmetrized equation

$$-\Delta \overline{u} = f^* \quad \text{in } \Omega^*, \tag{5.5}$$

where f^* is the spherical rearrangement of f, with boundary conditions

$$\overline{u}(x) = 0 \qquad \text{on } \partial\Omega^*. \tag{5.6}$$

5.1. Standard symmetrization result revisited. The classical result of symmetrization theory says that, in the absence of the zero-order term, the symmetric rearrangement of u, that we write u^*, can be compared pointwise with the solution of the symmetrized problem. Let us review that result, since forms the outline of the proof of our main comparison theorem for symmetrized functions in equations with lower-order terms.

Theorem 5.1 *Let us assume that $f \in L^2(\Omega)$, $f \geq 0$ and that $u \in H_0^1(\Omega)$ is a weak solution of equation* (5.1) *under the above hypotheses. Then,*

$$u^*(r) \leq \overline{u}(r) \qquad \text{for all } \; r \in (0,R), \tag{5.7}$$

Proof. The steps of the proof, as described e.g. in Talenti [T], are as follows:
(i) Write equation (5.1) in variational form as

$$\int_\Omega \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} = \int_\Omega g(x,u) v\, dx, \tag{5.8}$$

for test functions $v \in H_0^1(\Omega)$, where $g(x,u) = f(x) - b(x,u)$. Taking $f \geq 0$ we have $u \geq 0$ by the maximum principle. Let us now write $a(\nabla u, \nabla v) \equiv \sum a_{ij}\partial_i u \partial_j v$.
(ii) We calculate for a.e. $t > 0$ the derivative $\frac{d}{dt}\int_{\Omega(t)} a(\nabla u, \nabla u) dx$, where $\Omega(t) = \{u > t\}$. Taking as test function $v = (u-t)_+$ in (5.8) we get

$$\int_{\{u>t\}} a(\nabla u, \nabla u) dx = \int_{\{u>t\}} g(x,u) v(x) dx.$$

It is then proved that for a.e. $t \in (0, \sup\text{ess}(u))$ we have

$$\frac{d}{dt}\int_{\{u>t\}} g v dx = -\int_{\{u>t\}} g\, dx.$$

It follows that

$$-\frac{d}{dt}\int_{\{u>t\}} a(\nabla u,\nabla u)dx = \int_{\{u>t\}} g(x,u)dx. \tag{5.9}$$

(iii) A very elementary step using the ellipticity assumption allows to conclude that

$$-\frac{d}{dt}\int_{\{u>t\}} a(\nabla u,\nabla u)dx \geq -\frac{d}{dt}\int_{\{u>t\}} |\nabla u|^2 dx \geq 0.$$

We transform in this way equality (5.9) into

$$-\frac{d}{dt}\int_{\{u>t\}} |\nabla u|^2 dx \leq \int_{\{u>t\}} g(x,u)dx. \tag{5.10}$$

(iv) We need to transform the left-hand side. Using the Cauchy-Schwartz inequality, we get:

$$\frac{1}{h}\int_{\{t<u<t+h\}} |\nabla u| dx \leq \left(\frac{1}{h}\int_{\{t<u<t+h\}} |\nabla u|^2 dx\right)^{1/2} \left(\frac{1}{h}\int_{\{t<u<t+h\}} dx\right)^{1/2}.$$

from which, after recalling the definition of distribution function $\phi = \phi_u(t)$ of the function u, we get in the limit $h \to 0$

$$\left(-\frac{d}{dt}\int_{\{t<u\}} |\nabla u|\, dx\right)^2 \leq \left(\frac{d}{dt}\int_{\{t<u\}} |\nabla u|^2 dx\right)(-\phi'(t))$$

which by (5.10) is equal or less than $(-\phi'(t))\int_{\{u>t\}} g\, dx$, hence

$$\left(-\frac{d}{dt}\int_{\{t<u\}} |\nabla u|\, dx\right)^2 \leq (-\phi'(t))\int_{\{u>t\}} g\, dx. \tag{5.11}$$

(v) We now use heavier artillery: Fleming-Rishel's formula says that for a.e. t

$$P_\Omega(\{u>t\}) = -\frac{d}{dt}\int_{\{u>t\}} |\nabla u|\, dx, \tag{5.12}$$

while De Giorgi's isoperimetric inequality can be written as

$$P_\Omega(\{u>t\}) \geq n\omega_n^{1/n}\phi(t)^{\frac{n-1}{n}}. \tag{5.13}$$

Using both formulas, (5.11) becomes

$$n^2\omega_n^{2/n}\phi(t)^{2-\frac{2}{n}} \leq (-\phi'(t))\int_{\{u>t\}} g(x,u)\, dx. \tag{5.14}$$

(vi) Moreover, $g(x) = f(x) - b(x,u)$, and by the properties we have assumed on b, we have $\int_{\{u>t\}} b(x,u)dx \geq 0$, therefore

$$n^2\omega_n^{2/n}\phi(t)^{2-\frac{2}{n}} \leq (-\phi'(t))\int_{\{u>t\}} f(x)\, dx. \tag{5.15}$$

(vii) We use Hardy-Littlewood's theorem to estimate

$$\int_{\{u>t\}} f dx \le \int_{\{u^*>t\}} f^* dx = \int_{B_r} f^*(x)dx,$$

where $|B_r| = |\{u > t\}|$, i.e. $\omega_n r^n = \phi(t)$. Let us also introduce the notation $F(s) = \int_{B_r} f^*(x)dx$, with $s = \omega_n r^n$. Substituting into (5.15), we get the inequality

$$n^2 \omega_n^{\frac{2}{n}} \phi(t)^{2-\frac{2}{n}} \le (-\phi'(t)) \int_{\{u^*>t\}} f^*(x)dx = F(\phi(t)). \tag{5.16}$$

At this stage we recall that (5.16) is satisfied with equality by the symmetrized problem. Indeed, we have

$$-n\omega_n r^{n-1} \overline{u}'(r) = \int_{B_r} f^*(x)dx, \tag{5.17}$$

The comparison we are looking for follows easily since, using the fact that for a.e. r we have $\phi(\overline{u}(r)) = n\omega_n r^n$, it follows that (5.16) can be written as

$$-n\omega_n r^{n-1} (u^*)'(r) \le \int_{B_r} f^*(x)dx. \tag{5.18}$$

Using the boundary condition $u^*(R) = \overline{u}(R) = 0$ we conclude the inequality. For the sequel, we notice that (5.18) is a formulation as an integral subsolution.

5.2. General symmetrization-concentration comparison theorem. We deal now with the presence of the lower-order term. In the previous result its effect has been eliminated, but this leads to a poorer understanding and poorer estimates. This subject has been investigated by a number of authors. Briefly stated, the problem is that keeping track of this term changes the last part of the preceding proof and destroys the corresponding conclusion.

We have been first led to keeping track of the term in the study of parabolic problems by implicit discretization in time. It happens that, in the spirit of our end of the previous proof, there is a simple modification that naturally leads to concentration comparison. In this way we can compare the result of solving and then rearranging with the result of the reverse procedure, i.e., first rearranging and then solving the symmetrized problem. In fact, using the results of Section 4 we can do better: to compare the symmetrized problem at once with a radial problem for different forcing term f and different nonlinearity β. This is our main result.

Theorem 5.2 *Next to the assumptions of the preceding theorem on f and u, we suppose that $b(x,s)s \ge \beta(s)s$ for all s, where β is a maximal nondecreasing function with $\beta(0) = 0$ and let $v = \beta(u)$. Let $\overline{u}(r)$, $0 < r < R$ be an integral supersolution of the radial problem*

$$-\lambda \Delta \overline{u} + \beta_1(\overline{u}) \ni \overline{f}(r), \tag{5.19}$$

with boundary condition $\overline{u}(R) \ge 0$, where $\overline{f}$ is a radial function in $L^1(\Omega^*)$ such that $\overline{f} \succ f^*$, and β_1 is a maximal nondecreasing function such that $\beta_1^{-1} \prec \beta^{-1}$. Under *these assumptions, we conclude that the two radial functions $v^*(r)$ and $\overline{v}(r)$ are ordered:*

$$v^* \prec \overline{v}. \tag{5.20}$$

Proof. (i) We repeat the previous proof with the following modification at the end. In formula (5.15) we observe that

$$\int_{\{u>t\}} g(x,u)dx \geq \int_{\{u>t\}} f(x)\,dx - \int_{\{u>t\}} \beta(u)\,dx. \tag{5.21}$$

This means the needed result: u^* is a radial integral subsolution of equation

$$-\lambda\Delta u + \beta(u) = f^*(r). \tag{5.22}$$

(ii) The comparison is now a consequence of Theorem 4.6 of Section 4. The fact that we are in the standard case is ensured by the Dirichlet conditions $u = 0$ on $\partial\Omega$ and the nonnegative condition for $\overline{u}$.

Generalization. (1) We have refrained from stating the theorem with multi-valued graphs for simplicity. However, in view of what we have seen up to now, the same result holds if β and β_1 are maximal monotone graphs in $\mathbf{R}^2$ with the corresponding modifications: $0 \in \beta_i(0)$; $\overline{v}(r) \in \beta_1(\overline{u}(r))$ is specified as part of the definition of supersolution, and $v^*(r) \in \beta(u^*(r))$ a.e.
(2) Other boundary or integrability conditions would imply the same conclusion, see Section 4.

The concentration statement can be re-formulated in terms of standard norms by means of Lemma 3.2. After some elementary calculations we get

Corollary 5.3 *Under the assumptions of Theorem* 5.2*, for every convex nondecreasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ we have*

$$\int_{B_R} \Phi(v^*(x))\,dx \leq \int_{B_R} \Phi(\overline{v}(x))\,dx. \tag{5.23}$$

In particular, for every $1 \leq p \leq \infty$ we have

$$\|v^*\|_p \leq \|\overline{v}\|_p. \tag{5.24}$$

The same conclusion holds for other norms like the natural norm in the Marcinkiewicz spaces M^p, $1 < p < \infty$. These spaces will play an important role for the smoothing effects obtained in paper [V7] and will be discussed in detail there.

5.3. Problem in the whole space. The results of the Symmetrization-Concentration Theorem apply equally when we pose the problem in $\mathbf{R}^n$. In fact, we can use two approaches: either (i) to derive the estimates directly, or (ii) use approximation of solutions in the whole space by solutions in a sequence of bounded balls with radii $R \to \infty$.

When working in the whole space it is important to recall that the classical conclusion $u^* \le \overline{u}$, or the inequality $v^* \le \overline{v}$ cannot be true in general, since we have in most cases equality for the masses,

$$\int_{\Omega^*} v^*(x)dx = \int_{\Omega} v(x)dx = \int_{\Omega} f(x)dx, \quad \int_{\Omega^*} \overline{v}(x) = \int_{\Omega^*} f^*(x)dx.$$

It would follow in that case from $v^* \le \overline{v}$ that $v^* \equiv \overline{v}$. But this is not true in general.

6 Solving the elliptic problems. A partial survey

In order to complete the picture of the class of elliptic problems for which we are trying to obtain a priori estimates by the combined method of concentration and symmetrization, we discuss in this section the existence and uniqueness of solutions of the model elliptic equation

$$-\Delta u + \beta(u) \ni f, \tag{6.1}$$

or in its equivalent formulation, $-\Delta\varphi(v) + v \ni f$.

6.1. Problems in the whole space. To be specific, we concentrate on problems posed the whole space. We take as second member the classical choice $f \in L^1(\mathbf{R}^n)$. This is the setting proposed by Bénilan, Brezis and Crandall in their famous paper [BBC], 1975. This is their main result.

Theorem 6.1 *Let $f \in L^1(\mathbf{R}^n)$ and let β a maximal monotone graph in $\mathbf{R}^2$ with $\beta(0) = 0$. Then,*
(i) If $n \ge 3$, there exists a unique $u \in M^{n/(n-2)}(\mathbf{R}^n)$ which solves (6.1) in the weak sense. Moreover, $|\nabla u| \in M^{n/(n-1)}(\mathbf{R}^n)$.
(ii) If $n = 1, 2$, and under the additional assumption that 0 belongs to the interior of the $R(\beta)$, there exists a unique solution (but for a constant in some cases) which belongs to $W^{1,\infty}(\mathbf{R}^2)$ if $n = 1$, and to $W^{1,1}_{loc}(\mathbf{R}^2)$ with $|\nabla u| \in M^2(\mathbf{R}^2)$ if $n = 2$.
(iii) The possible non-uniqueness happens only for very special cases where $\beta^{-1}(0)$ is an interval $I = [a, b]$, and then $\beta(u) = 0$, $R(u)$ is strictly included in I and $\Delta u = f$.
(iv) In all cases the function $v = \Delta u + f$ is well-defined, and the map

$$T : f \mapsto v$$

is a contraction in $L^1(\mathbf{R}^n)$. More precisely, for functions $f_i \in L^1(\mathbf{R}^n)$, $i = 1, 2$, and corresponding $v_i = \Delta u_i + f_i$ we have

$$\int (v_1 - v_2)_+ dx \le \int (f_1 - f_2)_+ dx. \tag{6.2}$$

This is technically called a T-contraction.

Here, $f(x)_+ = \max\{f(x), 0\}$ denotes the positive part of a function. We recall that the notations $R(\beta)$, $R(u)$ denote the range or image set of the respective functions or graphs. On the other hand, $M^p(\mathbf{R}^n)$ denotes the Marcinkiewicz or weak L^p space, which is carefully described in that paper.

Indeed, this result can be stated as a result on the solvability of the equation

$$\mathcal{A}(v) + v = f,$$

where the nonlinear operator $\mathcal{A} : L^1(\mathbf{R}^n) \to L^1(\mathbf{R}^n)$ is roughly speaking $-\Delta\varphi(\cdot)$ with $\varphi = \beta^{-1}$. It is precisely defined for $n \geq 3$ in the domain

$$\begin{aligned} D(\mathcal{A}) = \Big\{ & v \in L^1(\mathbf{R}^n) : \exists u \in W^{1,1}_{loc}(\mathbf{R}^n) : \Delta u \in L^1(\mathbf{R}^n), \\ & |\nabla u| \in M^{n/(n-1)}(\mathbf{R}^n), \quad \text{and} \quad v(x) \in \beta(u(x)) \ a.e. \Big\} \end{aligned} \tag{6.3}$$

by the formula $\mathcal{A}(v) = -\Delta u$, with modifications on the space to which u belongs if $n = 1, 2$ as indicated in the Theorem. The previous Theorem amounts to say that the *resolvent operator*

$$J(\mathcal{A}) = (I + \mathcal{A})^{-1} \tag{6.4}$$

is a well-defined contraction in $L^1(\mathbf{R}^n)$ (more precisely, a T-contraction). Clearly, the same is true for the general resolvent, $J_\lambda(\mathcal{A}) = (I + \lambda\mathcal{A})^{-1}$ with $\lambda > 0$.

Remark. There is no difficulty in generalizing these results to the more general equation

$$-\sum_{i,j} \partial_i(a_{ij}\partial_j u) + \beta(x, u) = f \tag{6.5}$$

with coefficients a_{ij} which are bounded measurable functions in Ω satisfying the ellipticity hypothesis

$$\sum_{i,j} a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \tag{6.6}$$

for some constant $\lambda > 0$ and all vectors $\xi \neq 0$, when $\beta(x, u)$ is a Carathéodory function, i.e. measurable in x for all u and continuous in u for almost all x, with the conditions that it is monotone increasing in u (for a.e. x) and uniformly bounded in x for bounded u. More general conditions are sometimes useful and have been studied in the literature but need not concern us at this point, but cf. a very general case in [B6].

6.2. Positive β. Of interest for some of the applications in diffusion is the special possibility that arises in dimensions $n = 1, 2$ of considering equation (6.1) for maximal monotone graphs that do not satisfy the normalization condition $0 \in \beta(0)$. We take graphs such that $\beta > 0$ everywhere, and insist in keeping the condition that $v \in \beta(u)$ be still an integrable function, $v \in L^1(\mathbf{R}^n)$. This forces zero to be the infimum of $R(\beta)$. It follows that $\beta(s) \to 0$ as $s \to -\infty$, and also $u(x, t) \to -\infty$ as $r = |x| \to \infty$.

Dimension $n = 1$. It was studied in [CE]. The main result is

Theorem 6.2 *Let $\beta(\mathbf{R}) \subset (0,\infty)$. Then the problem*

$$-u'' + \beta(u) \ni f, \qquad u'(\pm\infty) = 0 \tag{6.7}$$

is solvable for every $f \in L^1_+(\mathbf{R}) = \{f \in L^1(\mathbf{R}), \int f(x)dx > 0\}$, iff β is integrable at $-\infty$:

$$\int_{-\infty} \beta(s)ds < \infty.$$

The map $f \to u'' + f \in \beta(u)$ is an L^1-contraction with domain $L^1_+(\mathbf{R})$.

The solution thus constructed is not the only possibility of a well-posed problem with $v \in L^1(\mathbf{R})$. Solutions with non-zero "Neumann conditions at infinity" can be constructed; they were investigated in [RV].

Theorem 6.3 *Let $\beta(\mathbf{R}) \subset (0,\infty)$ and $\int_{-\infty} \beta(s)ds < \infty$. Then, problem*

$$\begin{cases} -u'' + \beta(u) \ni f \\ u'(-\infty) = a, \quad u'(\infty) = -b, \end{cases} \tag{6.8}$$

is solvable for every pair of constants $a, b \geq 0$ if $f \in L^1_+(\mathbf{R})$ and $\int f(x)dx > a + b$. The solution constructed in [CE] *is maximal in this set and corresponds to $a = b = 0$.*

Dimension $n = 2$. It was studied in [V3] and [V4].

Theorem 6.4 *Let $\beta(\mathbf{R}) \subset (0,\infty)$. For every $f \in L^1(\mathbf{R}^2)$ with $\int f(x)dx > 0$ there is a solution of the problem*

$$-\Delta u + \beta(u) \ni f \tag{6.9}$$

in the class: $u \in W^{1,1}_{loc}(\mathbf{R}^2)$, $u \geq 0$, $|\nabla u| \in M^2(\mathbf{R}^2)$, $\Delta u \in L^1(\mathbf{R}^2)$ and the mass condition

$$\int \Delta u \, dx = 0 \tag{6.10}$$

if for every $b > 0$ we have

$$\int_{-\infty} \beta(t) \exp(-bt) dt < \infty. \tag{6.11}$$

Moreover, if we replace the hypothesis on β by the weaker condition that (6.11) *holds for all $b > b_0$ then equation* (6.9) *admits solutions with the mass condition $\int \Delta u dx = c$ if $\int f(x)dx \geq c$.*

The case of nonzero Neumann data is well understood for radial data; the complete study of the case of non-radial data remains open.

6.3. Problems in bounded domains. We will skip the detailed discussion of the nonlinear elliptic problems posed in bounded domains, which is a rather classical subject. Here are two appropriate remarks.

(i) For a theory of existence and uniqueness for the Dirichlet problem of our types we refer to Brezis and Strauss [BrS].

(ii) The problem with positive β and zero Dirichlet data admits no solutions.

7 Evolution Equations. Semigroup generation

We turn our attention to evolution problems of nonlinear parabolic type, in the area usually called Nonlinear Diffusion. Following paper [V1], our interest lies in deriving a priori information about the size and distribution of the solutions using the tools of concentration and symmetrization.

Remark about notation. When dealing with evolution equations, we will often use the shortened notation $u(t)$ instead of $u(x,t)$ when emphasis is laid on the t-dependence. Then $u(t)$ is for every t an element of a functional space thanks to its remaining x-dependence.

Before proceeding with the task of deriving estimates, we need to pay some attention to the basics. In this preliminary section we address some questions about the construction of solutions, namely the technique of semigroup generation by time discretization, and apply the results to the typical examples we have in mind.

7.1. Semigroup generation by the ITD method.
The method of Implicit Time Discretization allows to solve abstract Cauchy problems of the form

$$u_t + A(u) = f, \quad u(0) = u_0 \tag{7.1}$$

where A is an operator in a Banach space X, possibly nonlinear and multi-valued[1] by approximate problems as follows: we take a partition $\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_{n-1} \le T < t_n\}$, and pose the discrete problems

$$\frac{u_i - u_{i-1}}{h_i} + Au_i \ni f_i, \ i = 1, 2, \ldots, n \tag{7.2}$$

where $h_i = t_i - t_{i-1}$ is the time step, and f_i, $i = 1, \cdots, n$ is a discretization of f. In numerical analysis this is called in implicit method of time discretization. The resulting equations can be written as

$$u_i = (I + h_i A)^{-1}(u_{i-1} + h_i\, f_i), \tag{7.3}$$

so the problem is treatable in this way if the *resolvent operators*

$$J_\lambda(A) = (I + \lambda A)^{-1}, \lambda > 0, \tag{7.4}$$

have good properties as operators defined in X, or a suitable subspace thereof. When this is the case, solving the evolution problem reduces to

(i) solving the cascade of relations (7.3) to obtain a discrete *approximate solution* $\{u_i\}$ which can be pieced together in various natural forms, like

$$u_{\mathcal{P}}(t) = \frac{t - t_{i-1}}{h_i} u_i + \frac{t_1 - t}{h_i} u_{i-1} \quad \text{if } t_{i-1} \le t \le t_i,$$

to form an *approximate solution* subordinate to the partition, $u_{\mathcal{P}}(t) \in C([0,T] : X)$.

[1] in which case it is identified with a map from X into 2^X.

(ii) The second problem is the convergence of these approximations when the partition is refined. Those problems had a solution with the work of Crandall and Liggett [CL]. Indeed, we work in the class of *accretive operators*, i.e., operators A such that the resolvent $J_\lambda(A) = (I + \lambda A)^{-1}$ is a contraction defined in X for all $\lambda > 0$. Moreover, we say that an accretive operator A in a Banach space X is m-accretive if the range

$$R(I + \lambda A) = X, \quad \text{for all } \lambda > 0. \tag{7.5}$$

Therefore, when A is m-accretive we can solve the "discretized problems" (7.3) for every partition $\mathcal{P}$. Let us put $f = 0$. The convergence is given by the famous result.

Theorem 7.1 (Crandall-Liggett Theorem) *Let A be an m-accretive operator in X. Then, for any $u_0 \in \overline{\mathcal{D}(A)}$*

$$e^{-tA} u_0 = \lim_{n\to\infty} (J_{t/n})^n u_0 \tag{7.6}$$

exists uniformly on compact subsets of $[0, \infty[$. Moreover, the family of operators e^{-tA}, $t > 0$, is a continuous semigroup of contractive mappings of $\overline{\mathcal{D}(A)}$.

Formula (7.6) is called the *Crandall-Liggett exponential formula* for the nonlinear semigroup generated by A. A convergence theorem for the ITD scheme works also when $f \in L^1(Q)$, and the kind of generalized solution of the Abstract Initial-Value Problem obtained in this way has been termed *mild solution* in [BC2]. It is uniquely determined by the scheme independent of the partitions used. A main task of Applied Functional Analysis is to determine the relationship of this concept with the usual concepts of classical, strong, weak (or other types of) solution which are also found in the description of the equations of applied science. There is wide literature on this non-trivial issue but falls out of the scope of the present work.

There is a stronger version of the concept of m-accretive operator, much used by Bénilan, called m-T-accretive operators, cf. [Be1]; it combines the contraction property with the maximum principle typical of elliptic and parabolic equations. It is applicable in spaces X where we can define the positive part of an element $f \in X$, as in the typical Lebesgue function spaces in $\mathbf{R}^n$ or Ω that we have been using. To be precise, an operator A is T-contractive if for every $\lambda > 0$ $J_\lambda(A)$ is a T-contraction, which means that

$$\|(J_\lambda(A)(f_1) - J_\lambda(A)(f_2))_+\|_X| \le \|(f_1 - f_2)_+\|_X$$

where $(\cdot)_+$ denotes the positive part. It is then clear how this is related with comparison arguments: if $f_1 \le f_2$, we have $(f_1 - f_2)_+ = 0$, and the T-contraction implies $J_\lambda(A)(f_1) \le J_\lambda(A)(f_2)$.

7.2. Application to the filtration equation. We take as our main example the equation,

$$u_t = \Delta\varphi(u) + f, \tag{7.7}$$

where φ is a m.m.g. with $0 \in \varphi(0)$ and $f = f(x,t)$ is defined in $Q_T = (0,T) \times \mathbf{R}^n$. This equation generalizes the classical heat equation.

When φ is smooth and φ' is bounded above and below away from zero, i.e., $0 < c < \varphi'(u) < 1/c < \infty$, the equation is a quasilinear parabolic equation and we can apply the standard quasilinear theory, as developed in [LU]. But here we may consider a general setting in which φ is an increasing function or a m.m.g., so that the equation can be singular or degenerate parabolic. Such a generality has interesting applications, but the reader not used or willing to deal with the corresponding complications will do well in assuming that φ is a continuous and strictly monotone function and he will not lose the main action. Note that even if φ is smooth the equation can be degenerate at points where $\varphi'(u) = 0$. This happens in the best-known nonlinear example, the *porous medium equation* (shortly, PME) which is usually written as

$$u_t = \Delta u^m, \quad m > 1, \tag{7.8}$$

when nonnegative solutions are considered; a usual restriction to nonnegative solutions is justified by the applications. For solutions of any sign we replace u^m by $\varphi(u) = |u|^{m-1}u$, a particular case of (1.1). Of course, for $m = 1$ we recover the *classical heat equation*, while for $m < 1$ (7.8) is called the *the fast-diffusion equation.* The data u_0 and $f(x,t)$ satisfy the assumptions

$$u_0 \in L^1(\mathbf{R}^n), \quad f \in L^1(Q_T).$$

The degeneracy of the equation forces us to study the problem in classes of weak solutions, or, as in our approach, mild solutions. A theory of existence and uniqueness of solutions of the Cauchy problem for (7.7) with initial data

$$u(x,0) = u_0(x) \in L^1(\mathbf{R}^n) \tag{7.9}$$

was developed in the early 1970s in the framework of nonlinear semigroups, mainly in the works of Bénilan and Crandall and others, and in fact this example was one of the motivations in the development of that theory. It then happens that the maps $S_t : u_0 \to u(\cdot,t)$ form a semigroup in $L^1(\mathbf{R}^n)$, which turns out to be a very natural space for this problem.

In order to discuss $u_t - \Delta\varphi(u) = f$ within the nonlinear semigroup theory, Ph. Bénilan and M. G. Crandall ([BC1]) associate an m-T-accretive operator A_φ in $L^1(\mathbf{R}^n)$ with the formal expression $A_\varphi = -\Delta\varphi(u)$. This is done via the results of [BBC] discussed in the previous section.

Comparison Results. Via the Crandall-Liggett exponential formula we derive the following evolution version of the concentration result of Proposition 4.1.

Theorem 7.2 *Let φ a m.m.g. in $\mathbf{R}^2$ with $0 \in \varphi(0)$. Let $0 \le u_0$ be a radially symmetric function in $L^1(\mathbf{R}^n)$. If $u(x,t)$ is the mild-solution of problem* (7.7)-(7.9) *with no forcing term, $f = 0$, then*

$$u(.,t) \prec u(.,s) \prec u_0 \quad \text{for} \quad t \ge s \ge 0. \tag{7.10}$$

We see this result as a form of a general principle, that we describe as the law of *decreasing concentration of solutions of nonlinear diffusion*. We next state the main Comparison Result.

Theorem 7.3 *Let u be the mild solution of problem* (7.7)-(7.9) *with data u_0, nonlinearity φ under the above assumptions and second member $f \in L^1(Q)$. Let v be the solution of a similar problem with radially symmetric data $v_0(r) \geq 0$, nonlinearity ψ and second member $g(r,t) \geq 0$. Assume moreover that*

$$\begin{array}{ll} (i) & u_0^* \prec v_0, \\ (ii) & \psi \prec \varphi \quad \text{and} \quad \varphi(0) = \psi(0) = 0, \\ (iii) & f^*(\cdot,t) \prec g(\cdot,t) \quad \text{for every } t \geq 0. \end{array}$$

Then, for every $t \geq 0$

$$u^*(\cdot,t) \prec v(\cdot,t). \tag{7.11}$$

Remark. Note the order reversal in condition (ii). It is quite natural for larger diffusivities to produce solutions that are more spread out and have lesser concentration.

Proof. Using the Crandall-Liggett result we are reduced to comparing the discretization steps, which consist of elliptic problems as those treated in Sections 4 to 6. It is important to realize that comparison of concentrations between the discretized versions of the solutions is inherited in every step of the iteration.

We proceed as follows. In the first step, between $t_0 = 0$ and $t_1 = t/N$, we start from a datum u_0, forcing term f_0, and obtain a solution of the elliptic problem (1.2)

$$-h\Delta\phi(u) + u = u_0 + h\, f_0,$$

which is a form of (6.5). Let us call the solution u_1. We symmetrize it into u_1^* and it becomes a $\prec$-subsolution of the symmetric problem (6.1) with right-hand side $u_0^* + hf_0^*$. Note that this second member is more concentrated than $(u_0 + hf_0)^*$. We compare this solution with the radially symmetric solution of the elliptic equation appearing in the first iteration step with data $v_0 + hg_0$ and with nonlinearity ψ. By Theorem 5.2, we get

$$u_1^* \prec v_1.$$

In the second step we have to solve an elliptic problem three times: the first elliptic equation with data $u_1 + hf_1$ to get the second step of the discretized solution, u_2; the symmetrized version with data $u_1^* + hf_1^*$ to get some radial solution w_2; and the radial version with nonlinearity ψ and data $v_1 + hg_1$ to get a radial solution v_2. We obtain

$$u_2^* \prec w_2 \prec v_2, !`$$

where the first inequality comes from Theorem 5.2 and the second is a comparison of radial functions as in Theorem 4.6. The process is then continued for all the steps. Therefore, the comparison of concentrations works at all levels of the discretized solutions. To end the proof, the limit is taken as the time-step length goes to 0.

Corollary 7.4 *In particular, under the assumptions of Theorem* 7.3*, or every* $t \ge 0$ *and every* $p \in [1, \infty]$ *we have comparison of* L^p *norms,*

$$\|u(\cdot,t)\|_p \le \|v(\cdot,t)\|_p \,. \tag{7.12}$$

Note that the terms of (7.12) can also be infinite for some or all values of p.
Remarks. 1) There is no problem in adapting the main result, Theorem 7.3, to the mixed initial-and-boundary-value problem posed in a bounded domain $\Omega \subset \mathbf{R}^n$ with zero Dirichlet conditions, since the elliptic theory is ready. We leave the easy details to the reader since there is no real difference.

2) We can apply in the evolution setting some comparison even when there is no strict comparison of the data, extending the elliptic result of Corollary 4.5. Here is the easy result

Theorem 7.5 *Let* u, v *be the symmetric mild solutions of problem* (7.7)-(7.9) *with symmetric data* u_0*, nonlinearity* φ *and second members* f, g *resp. under the above assumptions. If there is a constant* $C \ge 0$ *such that*

$$\int_{B_r(0)} f^*(|x|,t)\,dx \le \int_{B_r(0)} g(|x|,t)\,dx + C, \tag{7.13}$$

then, for every $t \ge 0$

$$\int_{B_r(0)} u^*(|x|,t)\,dx \le \int_{B_r(0)} v_{B_r(0)}(|x|,t)\,dx + C. \tag{7.14}$$

8 The p-Laplacian and related equations

We will present here the main facts that we need to develop a theory similar to what we have done for the Filtration Equation in the case of the p-Laplacian evolution equation (PLE for short),

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = 0, \tag{8.1}$$

without forcing term. In diffusion problems the model only reflects gradient-dependent diffusivity. The use of the this kind of dependence in the elliptic term also arises in non-Newtonian fluids, cf. [La], and then we are talking of gradient-dependent viscosity. Exponent p is positive, the case $p = 2$ is the heat equation.

The concentration comparison results have been established for this operator in Section 4. Symmetrization applies, cf. Talenti's works [T1, T2, T3]. ♠

The p-Laplacian operator and the generation of a semigroup. It is a well-known fact that a p-Laplacian operator can be suitably defined so that it is m-accretive in $L^1(\Omega)$ and it has a dense domain. Here, Ω is bounded subdomain of $\mathbf{R}^n$ or the whole space $\mathbf{R}^n$. The basic facts are as follows: for every $p > 1$ a functional defined for $u \in L^2(\Omega)$ by

$$J_p(u) = \frac{1}{p}\int_\Omega |Du|^p\,dx$$

when $Du = \nabla u$ is vector in $(L^p(\Omega))^n$, and as $+\infty$ otherwise. This is a convex, lower semi-continuous and proper functional in $L^2(\Omega)$. Accordingly, its subdifferential A_p is a maximal monotone operator in $L^2(\Omega)$ defined by the formula

$$A_p(u) = -\mathrm{div}\,(|\nabla u|^{p-2}\nabla u)$$

in the domain $D(A_p)$ of functions $u \in L^2(\Omega)$ such that $|\nabla u| \in L^p(\Omega)$, $A_p(u) \in L^2(\Omega)$ and

$$\int_\Omega A_p u.v\,dx = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\,dx$$

for every v in the domain of J_p. It is then proved that A_p is accretive in all $L^p(\Omega)$ spaces, $1 \le p \le \infty$. Indeed, it is T- accretive in Bénilan's terminology, cf. [Be1].

In the application of the ITD to generate a solution of the evolution equation we need a range condition. In order to comply with it, we must close the operator to form a new extended operator $\mathcal{A}_p$ that is m-accretive in $L^p(\Omega)$. We will denote the complete operator $\mathcal{A}_p$ in the usual notation by $-\Delta_p(u)$.

As a conclusion, $\mathcal{A}_p$ generates a semigroup of contractions in $L^p(\Omega)$ by means of the Implicit Discretization Scheme, as explained in Subsection 7.1. It solves the equation

$$u_t - \mathrm{div}\,(|Du|^p Du) = 0, \tag{8.2}$$

in the mild sense.

Doubly nonlinear equation. We can even combine the two types of nonlinearity and consider solving the equation

$$u_t - \mathrm{div}\,(|D\varphi(u)|^p D\varphi(u)) = f. \tag{8.3}$$

The ITD scheme leads us to show that the stationary problem

$$-\Delta_p(\varphi(u)) + u \ni f \tag{8.4}$$

defines a contraction $f \mapsto u$ in $L^1(\Omega)$. A convex subset thereof may be sufficient. This turns out to be the case here if φ is a convenient m.m.g. The Maximum Principle holds, we even have the T-accretivity inequality

$$\|(u(t) - v(t))_+\|_1 \le \|(u(0) - v(0))_+\|_1,$$

valid for any two solutions u, v of equation (8.2) or (8.3).

Symmetrization involving nonlinear Laplacians. The symmetrization considerations we have made in Section 5 for equation (5.1) can be extended to more general nonlinear settings. Thus, we are interested in equations involving gradient-dependent diffusivities, namely, equations of the form

$$-\nabla \cdot (\sigma(|Du|)Du) + b(x,u) = f \tag{8.5}$$

with $\sigma : [0,\infty) \to \mathbf{R}$ continuous and positive (except maybe at $s = 0$), as in (4.6). Following Talenti [T3] we may even assume the form

$$-\sum_i \partial_i a_i(x,u,Du) + b(x,u) = f \tag{8.6}$$

where a_i are measurable functions and there exists a real function $A(r)$ which is positive and convex for all $r > 0$, with $A(r)/r \to 0$ as $r \to 0$, such that

$$\sum_i a_i(x,u,\xi)\xi_i \geq A(|\xi|)$$

for all (x,u,ξ). Assumptions on $b(x,u)$ are: $b(x,s)s \geq \beta(s)s$ for all s, where β is a maximal nondecreasing function with $\beta(0) = 0$. Let us also take as Ω a bounded domain. Write $A(r) = \sigma(r)r^2 = B(r)r$. Note that B is nondecreasing and $\sigma \geq 0$. We take zero boundary conditions on $\partial\Omega$.

Then reference [T3] performs the derivation of the inequalities satisfied by the distribution function $\phi(t)$ of a solution $u(x)$ and arrives at the conclusion that a worst case exists, given by the equation

$$-\nabla \cdot (\sigma(|Du|)Du) = f^*(x), \quad x \in \Omega^*, \tag{8.7}$$

thus disregarding the effect of the term $b(x,t)$. The inequality is

$$n\omega_n^{\frac{1}{n}}\phi(t)^{1-\frac{1}{n}} \leq (-\phi'(t))B^{-1}\left(\frac{\int_{\{u^*>t\}} f^*(x)dx}{n\omega_n^{\frac{1}{n}}\phi(t)^{1-\frac{1}{n}}}\right). \tag{8.8}$$

In terms of the symmetrization of the solution this means that

$$n\omega_n r^{n-1} B(-\frac{du^*}{dr}) \leq \int_{\{u>t\}} (f(x) - b(x,t))\, dx.$$

Dropping the influence of the term $b(x,u)$ Talenti obtains pointwise comparison of the solution $u(x)$ of (8.6) and the radially symmetric solution v of (8.7): $u^*(x) \leq v(x)$ everywhere in Ω^*.

Now, it is easily seen that the influence of the zero-order term can be kept in the form

$$n\omega_n r^{n-1} B(-\frac{du^*}{dr}) + \int_{\{u^*>t\}} \beta(u^*(x))\, dx \leq \int_{\{u^*>t\}} f^*(x)dx. \tag{8.9}$$

This immediately means that $u^*(r)$ is an integral subsolution of the radial equation

$$-\nabla \cdot (\sigma(|Du|)Du) + \beta(u^*(r)) = f^*(r), \quad 0 < r < R, \tag{8.10}$$

in the sense introduced and studied in Section 4. We then have a result completely similar to Theorem 5.2 that we state for the reader's convenience.

Theorem 8.1 *Let $u(x)$ be a solution of equation* (8.7) *under the above assumptions on f and the structure of the equation, and let $v = \beta(u)$. Let $\overline{u}(r)$, $0 < r < R$ be an integral supersolution of the radial problem* (8.10) *with right-hand side $\overline{f}(r)$, with boundary condition $\overline{u}(R) \geq 0$, where $\overline{f}$ is a radial function in $L^1(\Omega^*)$ such that $\overline{f} \succ f^*$, and β_1 is a maximal nondecreasing function such that $\beta_1^{-1} \prec \beta^{-1}$. Under these assumptions, we conclude that*

$$v^*(r) \prec \overline{v}(r). \tag{8.11}$$

Comparison of evolutions involving nonlinear Laplacians. There is no difficulty now in applying ITD plus passage to the limit using the Crandall-Liggett theorem to obtain an evolution comparison result that is completely analogous to Theorems 7.2, 7.3 and 7.5. We ask the reader to fill the details.

9 Appendix on Concentration

We will start this section with an equivalence characterization of the definition of concentration comparison which is valid for rearranged functions.

Lemma 9.1 *Let $f, g \in L^1(\Omega)$ be rearranged functions defined in $\Omega = B_R(0)$ for some $R > 0$ and let $g \to 0$ as $r \to R$. Then, $f \succ g$ if and only if the following property holds: for every $k > 0$*

$$\int_{\Omega \cap \{f > k\}} (f(x) - k)\, dx \geq \int_{\Omega \cap \{g > k\}} (g(x) - k)\, dx\,. \tag{9.1}$$

The result is true if $\Omega = \mathbf{R}^n$ under the assumptions $f, g \in L^1_{loc}(\mathbf{R}^n)$ and $g \to 0$ as $r \to \infty$.

Proof. By approximation we may assume that f and g are continuous, bounded and strictly decreasing. By replacing f by $f + \varepsilon$ we may also assume that the assumed inequalities are strict and $f \geq \varepsilon$ everywhere. We will let $\varepsilon \to 0$ at the end to get the original result. Let

$$I(r) = \int_{B_r(0)} (f - g)\, dx$$

for $0 < r < R$ and

$$J(k) = \int_\Omega (f - k)_+ - \int_\Omega (g - k)_+\, dx.$$

for all $k > 0$. We want to prove that $I(r) \geq 0$ for all r implies $J(k) \geq 0$ for all $k > 0$ and conversely. We consider several possibilities.

(i) If $f(r) \geq g(r)$ for all $r > 0$, then both $I(r)$ and $J(k) \geq 0$ and the result is clear.

(ii) If this is not so, we need to examine what happens in the intervals where $f(r) < g(r)$. Let $K = (a_1, a_2)$ be a maximal interval of this type. It is clear that I is decreasing in K, while J is decreasing as k decreases in the interval $K' = f(K) = g(K)$. Assume also that $0 < a_1 < a_2 < R$. Let $k_i = f(a_i) = g(a_i)$. In this case we have

$$\int_\Omega (f - k_i)_+\, dx = \int_{B_{a_i}(0)} f(x)\, dx - k_i \omega_n\, a_i^n,$$

$$\int_\Omega (g - k_i)_+\, dx = \int_{B_{a_i}(0)} g(x)\, dx - k_i \omega_n\, a_i^n.$$

Therefore, $I(a_i) \geq 0$ is equivalent to $J(a_i) \geq 0$. If $I(r) \geq 0$ for all r, then $I(a_2) = J(k_2) \geq 0$ and by the monotonicity of J we get $J(k) > 0$ for $k_1 < k < k_2$. On the converse, if $J(k) \geq 0$ for all k, then from $I(a_2) = J(k_2) \geq 0$ and the monotonicity of I we conclude that $I(r) > 0$ for $a_1 < r < a_2$. Therefore, under either assumption both I and J are positive in such an interval where $f < g$.

We still have to exclude the possibility $a_2 = R$. This is not possible since $g \to 0$ as $r \to R$ while $f \geq \varepsilon$.

(iii) The argument at the intervals where $f > g$ is easier, since both I and J are increasing on those intervals. We argue from the starting point of the interval. Now we need to notice that $a_1 > 0$. Therefore, the proof is complete.

The integrals in (9.1) have an immediate relation with the distribution function. Indeed,

$$\int_\Omega (f(x) - k)_+ \, dx = -\int_k^\infty (s-k) d\mu_f(s) = \int_k^\infty \mu_f(s)\, ds. \tag{9.2}$$

This is maybe the reason why the late Prof. Bénilan preferred to use this reformulation as the comparison tool in his works, cf. [AB], [BB].

This result is the basis for the proof of the more powerful result stated in Lemma 3.2. Indeed, the functions of the form $\Phi(s) = (s-k)_+$ are particular cases of convex nondecreasing functions. On the other hand, they form a basis that generates the rest of such functions Φ, using the formula

$$\Phi(u) - \Phi(0) = \int_0^\infty \Phi''(k)(u-k)_+ \, dk. \tag{9.3}$$

Generalization of the comparison. In view of these results, there is a natural extension of the concept of concentration comparison that applies for non-radial functions in $\mathcal{L}_0$ and follows the ideas of Hardy-Littlewood-Pólya. We propose the following definition in this general context:

Definition 9.2 *Given two functions $f, g \in \mathcal{L}_0(\Omega)$, $f, g \geq 0$, f is more concentrated than g, $f \succ g$, iff f^* is more concentrated than g^* in the sense of Definition 1 for radial functions. This can be formulated as saying that for every $k > 0$*

$$\int_{\{f>k\}} f(x)\ dx \geq \int_{\{g>k\}} g(x)\ dx. \tag{9.4}$$

In view of the preceding lines the definition is equivalent to the inequalities

$$\int_\Omega \Phi(f(x))dx \geq \int_\Omega \Phi(g(x))dx. \tag{9.5}$$

being valid for all convex nondecreasing functions Φ such that $\Phi(0) = 0$.

We must remark that this generalization does not allow to prove comparison theorems for solutions of elliptic or parabolic equations, unless f is already rearranged, in which case we recover the theorems we have already established in our paper. Note also that it does not coincide with Definition 3.1 used in this paper for radial functions f and g unless f is rearranged, so it seems not very promising in our context. There is however an interesting result that does hold and comes in very handy as a final touch for our notes.

Proposition 9.3 *The solutions of the elliptic and parabolic equations considered in this paper with nonnegative data in $L^1(\Omega)$ are less concentrated than their data.*

The equations we refer to are the elliptic equations of Sections 4, 5, and the evolution equations of Sections 7 and 8. Cf. this result with the more specialized results of Proposition 4.1 and Theorem 7.2.

Proof. Let us take for instance equation (4.1) and assume that the solution has smooth level sets. Then we have

$$\int_{\{u>k\}} \Delta u \, dx = \int_{\partial\{u>k\}} \nabla u \cdot \mathbf{n} \, dS = \int_{\{u=k\}} \frac{\partial u}{\partial n} \, dS \leq 0.$$

(we denote by $\mathbf{n}$ the exterior normal, hence the normal derivative is nonpositive). Subtracting k from the equation, we get $-\Delta u + (u - k) = f - k$; integrating in $\{u > k\}$ and using this inequality, we get

$$\int_\Omega (u-k)_+ \, dx = \int_{\{u>k\}} (u-k) \, dx \leq \int_{\{u>k\}} (f-k) \, dx \leq \int_\Omega (f-k)_+ \, dx$$

for all $k > 0$, which concludes in this case. For more general solutions or elliptic equations we will have to use approximation. The adaptation to the parabolic case is immediate, either adapting the proof or using the hereditary property just proved for the elliptic case in the ITD.

The result can be found in the version (9.5) the result is found in the papers of the late Philippe Bénilan, and the proofs derive it as a consequence of the accretivity and boundedness properties of such equations, see e.g. [Be2]. Let this final note be a tribute to the memory of a scholar from which the author has learned so much of the basic mathematics of nonlinear diffusion, nonlinear operators and semigroups.

10 Part II, comments and extensions

Part II. In the continuation of this work, [V7], we will try and show the strength of the machinery introduced so far by applying it to the study of time decay for the power-like subcases of the Filtration Equation, namely the Porous Medium Equation and the Fast Diffusion Equation. A very important additional tool will be the use of *worst-case strategies* to cast our estimates as optimal problems for mappings in suitable Banach spaces. We will apply similar techniques to the study of two further models of nonlinear diffusion, the p-Laplacian and the Doubly Nonlinear Diffusion Equation.

General doubly-nonlinear equations and other extensions. A generalization to equations with more complicated structure like

$$u_t - \operatorname{div}\left(Q(|D\psi(u)|)D\psi(u)\right) = f \tag{10.1}$$

is immediate. Diaz [Di1, Di2] has studied more general cases like

$$b(u)_t - \operatorname{div}(Q(|Du|)Du) + \beta(u) = f, \tag{10.2}$$

under appropriate assumptions on b, Q and β. We refer to his work for further details.

Among the extensions that can be done reasonably well, the first ones concern the presence of zero-order terms, which leads to reaction-diffusion problems. The techniques of symmetrization and comparison can be used only under suitable assumptions on the reaction term.

There are applications using other types of nonlinear equations let us mention the obstacle problem in the stationary case, or the Stefan problem, mean curvature equations in the stationary and evolution forms, and some fluids like Bingham fluids in evolution equations. Systems of nonlinear equations have also been considered, but only for particular structure.

Other types of problems. Not much has been studied about elliptic problems in bounded domains with Neumann boundary data, to which the methods of this paper are not well adapted. An early work is due to Maderna and Salsa [MS] in the elliptic case. See also [AB] for a very interesting result with mixed boundary conditions.

Symmetrization has been much used in the study of variational inequalities, see for instance [BM].

The method is not easy to apply to equations with first order terms (convective terms), see e.g. Giarrusso and Nunziante [GN] and Ferone et al. [FPV]. Higher order equations have been treated recently treated by Ferone and Kawohl [FK].

We will not pursue further these quite interesting issues which represent difficult applications of the techniques of symmetrization and rearrangement.

Other types of symmetrization for PDEs. Other types of symmetrization are being discussed in the literature, like Steiner symmetrization, cf. [Kw, Fk] or circular symmetrization, [B1, B2]. There are many issues being explored, see [vSW]. The use such types of symmetrization for purposes similar to those of the present work has to be explored further. Moreover, there are other geometrical techniques, like the moving plane method, [GNN] that are related to the topics of Symmetry and the Maximum Principle. The connection could be worth exploring.

11 Comments on the literature

Mass-concentration comparison is a simple way of expressing the comparison properties that underly the symmetrization process and its use seems to have been neglected by most of the authors interested in symmetrization, probably because, as we have pointed out, it is more natural to parabolic than to elliptic problems. Moreover, it allows for a more general comparison for one-dimensional or radially symmetrical problems. We contend that it is a quite simple and at times powerful method to combine with symmetrization in obtaining a priori estimates.

We include below some additional information on the contents of the different sections as an orientation for the reader. In doing that we have tried to be concise and fair, and not stray far away from the main subject. We ask for apologies if unintended omissions of important topics arise.

SECTION 2. Symmetrization is covered in a number of monographs, like Pólya and Szegö [PS], Payne [P], Bandle [Ba3], Kawohl [Kw] and Mossino [Mo]. Some recent reviews of symmetrization are due to Talenti [T6] and Trombetti [Tr]. All of them provide extensive lists of references. Specially interesting as motivation for our development is the application to obtain sharp constants in Sobolev-type inequalities done by Talenti [T2] and Lieb [Lb].

In her work, Bandle treats linear and also nonlinear parabolic equations with linear diffusion terms, cf. [Ba2, Ba3]. In comparison with her treatment, our use of the ITD method allows for quite general solutions and equations. Mossino and Rakotoson [MR] proved later comparison results similar to Bandle's for weak solutions of linear parabolic equations. See below.

Our symmetrization result in the context of elliptic equations needs a refinement to take into account the effect of the lower-order term, which is crucial in the iteration process. As we have seen, it consists in keeping the lower-order term in the symmetrized problem and replacing comparison of point values by comparison of integrals in balls. This type of refinement has a precedent in the work by Chiti [Ch], 1979, concerning linear equations of the form

$$\sum_{i,j} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j}) + c(x)u = f. \tag{11.1}$$

His work had completely different motivation. Symmetrization results for elliptic and parabolic equations with lower order terms are numerous in the recent literature, cf. e.g. [ATL, T4, TV].

SECTION 3. Most of the definitions are standard. The notations of concentration comparison were introduced into the study of a priori estimates for nonlinear parabolic PDEs in [V1]. We recall that our definition does not coincide with Hardy-Littlewood's in the sense that we do not rearrange the radial functions before taking integrals.

Maximal monotone operators in Hilbert spaces were thoroughly studied in Brezis' monograph, [Br]. Marcinkiewicz spaces are used by [BBC] in the study of the elliptic problem $-\Delta u + \beta(u) = f$, which plays a role in solving the PME. Talenti [T3] uses them in symmetrization.

SECTION 4. Most of the contents of the section is new, developing the ideas of [V1]. In using comparison theorems for integrals we were inspired in the use of distribution functions is one-dimensional probability, which we developed for the Porous Medium Equation in the form of Shifting Comparison Principle, cf. [V3]. Mass Concentration Comparison is a radially symmetric version of Shifting Comparison.

The concentration comparison of solutions of elliptic equations with different nonlinearities seems new, but it has precedents in parabolic equations. Thus, Bouillet [Bo] made pointwise comparisons for the solutions of Stefan problems with different diffusivities in $n = 1$, and Abourjaily and Bénilan [AB] obtained comparison for general filtration equations with their concentration version. We developed this idea in 1997 to have an alternative proof of the mass-loss effect in logarithmic diffusion of [REV]. The proof of this phenomenon is explained in [V7].

A convenient way of understanding the comparison results for solutions $u(r,t)$ of equation (4.7) is viewing them in terms of the differential equation satisfied by the integral $V(r) = \int_0^r r^{n-1} v(r)\, dr$ which is a reformulation of (4.9):

$$-r^{n-1}\sigma(|u'(r)|)u'(r) + V(r) = F(r), \quad u(r) = \beta(V'(r)r^{1-n}). \tag{11.2}$$

which is formally elliptic, possibly degenerate or singular elliptic, of the type that is called doubly nonlinear and generalizes the p-Laplacian equations of Section 8.

Section 5. Symmetrization for elliptic equations is a large topic. Our proof of Theorem 5.1 follows Talenti [T1]. It is possible to avoid the use of the isoperimetric inequality, as in Lions [Li], see also [AB]. Several developments in the elliptic theory can be followed in the books mentioned above. Let us just mention that Alvino, Trombetti and P.L. Lions consider problems with first-order terms, see [AT, ALT, TV]. An adaptation of the mass comparison technique is needed to deal with zero-order terms. Related work on symmetrization for parabolic equations includes Bénilan and Berger [BB] who introduce another quite interesting symmetrization approach, further developed in [AB] and the method is applicable to a quite general class of quasilinear boundary value problems.

Section 6. The main reference in this section is [BBC], a paper that has had a strong influence on the author. The singular cases have been studied by the author and collaborators as indicated in the text.

Section 7. The theory of nonlinear semigroups is taken from the works of Bénilan, Crandall and Evans, cf. [Be1], [Cr] and [Ev]. Unfortunately, the book [BCP], that contains a wealth of results on the subject, has not been published.

The semigroup approach was used in the 1970's by Bénilan [Be2] and Véron [Ve] to derive smoothing effects and decay rates. Our contribution with respect to them lies in the obtention of best constants, and also in the simplicity of the worst-case copy strategy proposed in [V1]. See the detail in [V7]. In taking this approach we were inspired by Talenti's work on Sobolev inequalities, [T2].

The porous medium equation owes its name to the modeling of the flow of gases in porous media, [Le], [Mu]. The use in high-temperature Physics is documented in [ZR]. General references for the Porous Medium Equation are [Ar, Pe, V5].

There are a number of works on symmetrization for nonlinear parabolic equations appeared in these years. We have already mentioned [MR] and [Di1, Di2]. This author improves on our method by establishing a number of results for general equations, in particular interesting stability properties. Let us point out the important paper of Abourjaily and Bénilan, [BB], who treat general nonlinear equations of parabolic type by a beautiful new technique that we cannot describe for reasons of space, but deserves attention.

Section 8. The complete characterization of the closed p-Laplacian operator A_p for all $p > 1$ is done in [B6] by means of the concept of *entropy solution*. The statements in the section are proved there. See also see also [V6]. An alternative characterization has been proposed in terms of renormalized solutions by P. L. Lions and F. Murat [D4]. Note that when acting on $W^{1,p}$ functions, we do not need the extended definition of A_p.

Section 9. The definition of concentration comparison of this section is modeled on the Hardy-Littlewood-Pólya relation, as described in [BS], Section 2.3. Indeed, the work of these authors always refers to one space dimension and the comparison is stated in terms of the *maximal function* f^{**}, which is defined for measurable real functions as

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds.$$

Then $f \prec g$ is defined iff $f^{**}(t) \le g^{**}(t)$ for all $t > 0$. We give a proof of Lemma 3.2 passing through the integrals $\int (f-k)_+ dx$ that are important for their own sake. The result originates with a result of Littlewood and Pólya for decreasing functions of one variable in $[0,1]$, [HLP]. The adaptation to several dimensions is easy and is mentioned by [Ba3].

Final Comment. We have stated that our main interest in the comparison results studied in this paper concerns parabolic problems. The elliptic presentation of the technique has a two-fold purpose: on the one hand, the elliptic results may be interesting for their own sake; on the other hand, we think that they may also serve as a natural basis for obtaining estimates of numerical approximations to the proposed equations.

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