Perspectives in nonlinear diffusion: between analysis, physics and geometry

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Abstract. We review some topics in the mathematical theory of nonlinear diffusion. Attention is focused on the porous medium equation and the fast diffusion equation, including logarithmic diffusion. Special features are the existence of free boundaries, the limited regularity of the solutions and the peculiar asymptotic laws for porous medium flows, while for fast diffusions we find the phenomena of finite-time extinction, delayed regularization, nonuniqueness and instantaneous extinction. Logarithmic diffusion with its strong geometrical flavor is also discussed. Connections with functional analysis, semigroup theory, physics of continuous media, probability and differential geometry are underlined.

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1. Introduction

The heat equation, $\partial_t u = \Delta u$, (HE for short) is one of the three classical linear partial differential equations of second order that form the basis of any elementary introduction to the area of partial differential equations. Its success in describing the process of thermal propagation and processes of matter diffusion has witnessed a permanent acceptance since J. Fourier's essay *Théorie Analytique de la Chaleur* was published in 1822, [22], and has motivated the continuous growth of mathematics in the form of Fourier analysis, spectral theory, set theory, operator theory, and so on. Later on, it contributed to the development of measure theory and probability, among other topics.

The prestige of the heat equation has not been isolated. A number of related equations have been proposed both by applied scientists and pure mathematicians as objects of study. In a first extension of the field, the theory of linear parabolic equations was developed, with constant and then variable coefficients. The linear theory enjoyed much progress, but it was soon observed that most of the equations modelling physical phenomena without excessive simplification are nonlinear. However, the mathematical difficulties of building theories for nonlinear versions of the three classical partial differential equations (Laplace's equation, heat equation and wave equation) made it impossible to make significant progress until the 20th century was well advanced. This observation also applies to other important nonlinear PDEs

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or systems of PDEs, like the Navier–Stokes equations and nonlinear Schrödinger equations.

The great development of functional analysis in the decades from the 1930s to the 1960s made it possible for the first time to start building theories for these nonlinear PDEs with full mathematical rigor. This happened in particular in the area of parabolic equations where the theory of linear and quasilinear parabolic equations in divergence form reached a degree of maturity reflected for instance in the classical books of O. Ladyzhenskaya et al. [33] and A. Friedman [23]. A similar evolution for elliptic and parabolic equations in non-divergence form has proceeded at a much slower pace and is now in full bloom.

We will report here on progress in the area of nonlinear diffusion. A quite general form of nonlinear diffusion equation, as it appears in the literature, is

$$\partial_t H(x,t,u) = \sum_{i=1}^{d} \partial_{x_i}(A_i(x,t,u,Du)) + F(x,t,u,Du). \tag{1}$$

Here $Du = (\partial_{x_1} u, \dots, \partial_{x_n} u)$ stands for the spatial gradient of u. Suitable conditions should be imposed to guarantee (a minimum of) parabolicity, e.g., the matrix $(a_{ij}) = (\partial_{u_j} A_i(x,t,u,Du))$ should be positive semi-definite and $\partial_u H(x,t,u) \ge 0$. If we do not want to consider reaction or convection effects, however important they may be in the applications, the last term in the right-hand side should be dropped at the cost of skipping the rich theory for equations of the types

$$\partial_t u = \Delta u \pm u^p, \tag{2}$$

and their numerous variants. Let us also remark at this point that many applications deal with systems of such equations. A theory for equations and systems in such a generality has been in the making during the last decades, but the richness of phenomena that are included in the different examples covered in the general formulation precludes a general theory with detailed enough information. Two main areas of study have focused the attention of researchers in recent years: free boundary problems and blow-up problems. This article deals with the first topic and the associated idea of degenerate parabolic flows with finite propagation.

The study of nonlinear diffusion problems and free boundaries started at an early date. G. Lamé and E. Clapeyron addressed in 1831 the evolution of a two-phase system (water and ice) and were led to a free boundary problem that came to be known as the Stefan problem. There, the evolution of the temperature of the two media, sitting in disjoint domains, and obeying state equations of the heat equation type, has to be coupled with the evolution of the interface or free boundary separating the media. It took more than 120 years until O. Oleinik and S. Kamin gave a complete solution of the problem of existence and uniqueness in the context of weak solutions, a concept originated with J. Leray and S. Sobolev in the 1930s. Together with the obstacle problem, the Hele–Shaw problem and the porous medium equation, it formed a solid basis for the study of free boundary problems, though of course many other examples, like the combustion problems, attracted attention. The combination of functional analysis and geometry has been a key feature of the work in this frontier area that I have followed personally in the contributions of L. Caffarelli and A. Friedman, cf. [13], [24].

I will devote a large part of this exposition to progress in the porous medium equation (shortly, PME), written as

$$\partial_t u = \Delta u^m, \quad m > 1, \tag{3}$$

when nonnegative solutions are considered, as happens in most of the applications. However, when signed solutions are allowed the form $\partial_t u = \Delta(|u|^{m-1}u)$ is used. For the limit value $m = 1$ the classical heat equation is recovered. The PME is in some sense the simplest nonlinear modification of the heat equation in the area of diffusion; this can be easily understood when we write it in the form $\partial_t u = \operatorname{div}(d(u)\nabla u)$ with $d(u) = m|u|^{m-1}$, which means a density-dependent diffusivity. Later on, we will treat the case $m < 1$ that has attracted much attention in recent times and is called the fast diffusion equation (FDE). More generally, we can consider the larger class of *generalized porous medium equations* (GPME),

$$\partial_t u = \Delta\Phi(u) + f, \tag{4}$$

also called *filtration equations*, especially in the Russian literature; here, Φ is an increasing function $\mathbb{R}_+ \mapsto \mathbb{R}_+$, and usually $f = 0$. The diffusion coefficient is now $d(u) = \Phi'(u)$, and the condition $\Phi'(u) \ge 0$ is needed to make the equation parabolic. Whenever $\Phi'(u) = 0$ for some $u \in \mathbb{R}$, we say that the equation degenerates at that u-level, since it ceases to be strictly parabolic. This is the cause for more or less serious departures from the standard quasilinear parabolic theory; such deviations will focus our attention in what follows.

Concentrating on these particular equations will allow us to see the progress of the combination of the methods of functional analysis and geometry in clarifying the novelties and characteristic features of the theory of nonlinear diffusion equations. It also makes possible to describe the peculiarities of the behaviour of the solutions in great detail. It must be said that a somewhat similar and quite impressive progress has been obtained in many connected directions. Thus, an important role in the development of the topic of the filtration equation has been played by the already mentioned *Stefan problem* that can be written as a filtration equation with

$$\Phi(u) = (u-1)_+ \quad \text{for } u \ge 0, \qquad \Phi(u) = u \quad \text{for } u < 0. \tag{5}$$

More generally, we can put $\Phi(u) = c_1(u-L)_+$ for $u \ge 0$, and $\Phi(u) = c_2\,u$ for $u < 0$, where c_1, c_2 and L are positive constants. The Stefan problem and the PME have had a parallel history. When the time derivative term of the Stefan problem is simplified (limit of zero specific heat) we get the Hele–Shaw equation which models the behaviour of viscous fluids in very narrow cells. It can be written in the form

$$\partial_t H(u) = \Delta u + f, \tag{6}$$

with H the Heaviside step function. Different models include convection and/or reaction terms or a different form of nonlinear diffusion called p-Laplacian, where Δu, resp. Δu^m, is replaced by $\nabla \cdot (|\nabla u|^{p-2}\nabla u)$. The limit case $p = 1$ has recently received much attention in connection with image processing, cf. e.g. [1].

All the results to be reported in this exposition can be found explained in more detail in the works [43], [44]. The author takes the opportunity to thank the collaborators he has been lucky to be associated with.

2. Degeneration, free boundaries and geometry

A first difficulty of the theories of nonlinear diffusion including degenerate parabolic cases has been the concept of solution. The degenerate character of the PME as compared with the HE implies that the concept of classical solution that so well suits the latter is not adequate for the former. It was soon realized that the PME has the property of finite speed of propagation of disturbances from the rest level $u = 0$. This is explained in simplest terms when we take as initial data a density distribution given by a nonnegative, bounded and compactly supported function $u_0(x)$. The physical solution of the PME for these data is a continuous function $u(x, t)$ such that for any $t > 0$ the profile $u(\cdot, t)$ is still nonnegative, bounded and compactly supported; the support expands eventually to penetrate the whole space, but it is bounded at any fixed time. The *free boundary* is defined as the boundary of the support,

$$\Gamma_u = \partial S_u, \quad S_u = \text{closure}\,\{(x, t) : u(x, t) > 0\}. \tag{7}$$

Usually, the sections at a fixed time are considered. $\Gamma_u(t)$, $S_u(t)$. From the point of view of analysis, the presence of free boundaries is associated to discontinuities of the first derivatives of the solution. This is quite easy to see in the prototype case $m = 2$ where the equation can be written as

$$\partial_t u = 2u\,\Delta u + 2|\nabla u|^2. \tag{8}$$

It is immediately clear that in the regions where $u \neq 0$ the leading term in the right-hand side is the Laplacian modified by the variable coefficient $2u$; on the contrary, for $u \to 0$, the equation simplifies into $\partial_t u \sim 2|\nabla u|^2$, i.e., the *eikonal equation* (a first-order equation of Hamilton–Jacobi type, that propagates along characteristics). In accordance with this idea, the behaviour near the free boundary is controlled by the law $\mathbf{V} = 2|\nabla u|$, where $\mathbf{V}$ is the advance speed of the front. This is equivalent to $\partial_t u = 2|\nabla u|^2$ on this level line and has been rigorously proved in standard situations. In terms of the application to flows in porous media, this also means that the front speed equals the fluid particle speed which follows Darcy's law, the basic law of fluids in such media. See the typical front propagation in Figure 1 below. A similar calculation can be done for general $m > 1$ after introducing the so-called *pressure variable*, $v = cu^{m-1}$ for some $c \geq 0$. Putting $c = m/(m-1)$ we get

$$\partial_t v = (m-1)\, v\, \Delta v + |\nabla v|^2. \tag{9}$$

This is a fundamental transformation in the theory of the PME that allows us to get similar conclusions about the behaviour of the equation for $u, v \sim 0$ when $m > 1$, $m \neq 2$.

The use of u for functional analysis considerations and v for the dynamical and geometrical aspects is typical "dual thinking" of PME people.

3. Existence and uniqueness: generalized solutions

The non-existence of classical solutions when free boundaries appear delayed the mathematical theory of the PME. When the difficulty was addressed, it became a source of mathematical progress. Existence of solutions in the weak sense is now easy to establish for the PME or the GPME posed in the whole space with bounded initial data or in a bounded domain with zero Neumann or Dirichlet boundary conditions. Work in that direction started with O. Oleinik in 1958 [36]. The concept implies integrating once or twice by parts: in the first case we ask u to be locally integrable, $\Phi(u)$ to also have locally integrable first space derivatives, and finally the identity

$$\iint_{Q_T} \{\nabla\Phi(u)\cdot\nabla\eta - u\eta_t\}\, dxdt = \iint_{Q_T} f\eta\, dxdt \tag{10}$$

must hold for any test function $\eta \in C_c^1(Q_T)$, where $Q_T = \Omega \times (0, T)$ is the space-time domain where the solution is defined. In the case of very weak solutions we ask for a locally integrable function, $u \in L^1_{\rm loc}(Q_T)$, such that $\Phi(u) \in L^1_{\rm loc}(Q_T)$, and the identity

$$\iint_{Q_T} \{\Phi(u)\,\Delta\eta + u\eta_t + f\eta\}\, dxdt = 0 \tag{11}$$

holds for any test function $\eta \in C_c^{2,1}(Q_T)$. A very weak solution is roughly speaking a distributional solution, but all terms appearing in the formulation are required to be locally integrable functions; moreover, the initial and boundary conditions are usually inserted into the formulation (this is reflected as possible new terms in the identity along with a more precise specification of the test function space, see [44], Chapter 5).

These are concepts that have permeated nonlinear analysis in the last century. But the theory of the PME has been strongly influenced by the discovery that it generates a semigroup of contractions in the space $L^1(\Omega)$ and that a solution is most conveniently produced by the method of implicit time discretization using the celebrated result of Crandall–Liggett of the early 1970s [6], [20] that widely extends the scope of the Hille–Yosida generation theorem from linear to nonlinear semigroups. The resulting "numerical solution" obtained in the limit of the time discretization process is called a *mild solution*, a new mathematical object that attracted enormous attention at the time. Moreover, mild solutions form a contraction semigroup in the "natural space" $L^1(\Omega)$, not a common space in analysis since it is not reflexive and has peculiar compactness properties. But note that this space is natural for probabilists.

Now that we have four concepts of solution on the table (classical, weak, very weak and mild), proper relations have to be established among them, what is easy if we can prove a sufficiently strong uniqueness theorem. This is easy for the simplest scenarios, like the PME with nonnegative boundary conditions, but not so easy for more general equations involving general diffusion nonlinearities, variable coefficients and/or lower order effects, specially nonlinear convection. A whole literature has evolved in these years to tackle the issue, for which we refer e.g. to [44], Chapter 10.

On the one hand, the investigations on the properties of mild solutions in semigroup theory led to the interest in examining so-called *strong solutions*, where all derivatives appearing in the differential equation are assumed to exist as locally integrable functions (and satisfy maybe some other convenient requirements) so that the PDE can be interpreted as an abstract evolution $t \mapsto u(t)$ where $u(t)$ lives in a functional Banach space X and satisfies

$$\frac{du(t)}{dt} = Au(t) + f, \tag{12}$$

A being a nonlinear (highly discontinuous) operator on X. Actually, the study of nonlinear diffusion has been a source of examples, counterexamples and new concepts for nonlinear semigroup theory. Concepts from mechanics like blow-up, extinction, initial discontinuity layers, have appeared naturally in these studies.

A very recent trend is the consideration of semigroups in spaces of measures, which is quite natural when we consider diffusion from the point of view of stochastic processes (nonlinear versions of Brownian motion). This has led recently to a flurry of activity concerning the property of contraction of the PME semigroup with respect to the Wasserstein distance defined in the set of nonnegative integrable functions with a fixed mass by the formula

$$d(u,v)^2 = \frac{1}{2} \inf \left\{ \iint |x-y|^2 \, d\mu(x,y) \right\}, \tag{13}$$

the infimum taken over all nonnegative Radon measures μ whose projections (marginals) are $u(x)\,dx$ and $v(y)\,dy$. The PME is then viewed as a gradient flow, cf. F. Otto [37]. The contractivity of the semigroup in this norm is proved by J. A. Carrillo, R. McCann and C. Villani [16].

A very fruitful scenario for the generalization of the PME is the combination of nonlinear diffusion with convection (i.e., with a conservation law) which leads to the need for *entropy conditions* (mainly, of Kruzhkov type) to ensure the selection of a proper kind of solutions that guarantees both uniqueness and accordance with the underlying physics. This is work that counts the names of P. Bénilan, J. Carrillo, P. Wittbold and others, which extends the standard concept of bounded entropy solutions to merely integrable solutions by means of *renormalized entropy solutions* [7].

A still different direction is to focus on the pressure equation, that can be written in general form as

$$\partial_t u = a(u)\Delta u + |\nabla u|^2. \tag{14}$$

This is a non-divergence equation for which the methods of *viscosity solutions* (Crandall–Evans–Lions) should be better suited, [19], [11]. The proof of well-posedness for the PME (case $a(u) = cu$) was done in Caffarelli–Vázquez [14] in the class of continuous and bounded nonnegative viscosity solutions, and extended to more general GPME by Brändle–Vázquez [8]. But proving well-posedness for more general equations does not seem to be easy. In particular, the problem of characterizing bounded signed solutions of the PME is still open for viscosity solutions.

The conclusion is that the theory of nonlinear diffusion needs and benefits from a combination of functional approaches and has contributed in its turn to develop the mathematics of these abstract branches supplying them with problems, ideas and interactions. This is a consequence of the combination of relative simplicity with intrinsic difficulty, something on the other hand typical of all the classical nonlinear PDE models of science.

4. Asymptotic behaviour: nonstandard central limit theorem

This is a subject in which the mathematical investigation of nonlinear diffusion equations (NLDE) has been most active. The study of asymptotic behaviour is made attractive by the combination of analysis and geometry, which is felt in the description in terms of selfsimilar solutions and the formation of patterns.

On a general level, it has been pointed out in many papers and corroborated by numerical experiments that similarity solutions furnish the asymptotic representation for solutions of a wide range of problems in mathematical physics. The books of G. Barenblatt [4], [5] contain a detailed discussion of this subject. Self-similar solutions and scaling techniques play a prominent role in the asymptotic study of our equations.

Even when selfsimilarity does not describe the asymptotics, it usually happens that the whole pattern consists of several pieces which have a selfsimilar form in a more or less disguised way and are tied together by matching. We will not discuss here these more elaborate theories; examples in nonlinear diffusion are abundant, see the book [28] and its references.

The paradigm of long-time behaviour in parabolic equations is the theory of the linear heat equation, $m = 1$, which is the standing reference in diffusion theory.

The asymptotic behaviour of the typical initial and boundary value problems in usual classes of solutions is a well researched subject for the HE. Both the asymptotic patterns and the rates of convergence are known under various assumptions. Thus, the well-known result for the Cauchy problem says that for nonnegative and integrable initial data $u_0 \in L^1(\mathbb{R}^n)$, $u_0 \geq 0$, there is convergence of the solution of the Cauchy problem towards a constant multiple of the Gaussian kernel, $u(x,t) \sim M\,G(x,t)$, where

$$G(x,t) := \frac{1}{(4\pi t)^{N/2}} \exp\{-|x|^2/4t\}, \tag{15}$$

and $M = \int u_0(x)\,dx$ is the mass of the solution (space integration is performed by default in $\mathbb{R}^n$). When the heat equation is viewed as the PDE expression of the basic linear diffusion process in probability theory, the functions $u(\,\cdot\,,t)$ with mass 1 are viewed as the probability distributions of a stochastic process and the formula $u(x,t) \sim M\,G(x,t)$ is a way of formulating the central limit theorem.

We will explore next the analogous of this result for the PME. We start by the investigation of the long-time behaviour of solutions of the PME with data $u_0 \in L^1(\mathbb{R}^n)$ in order to prove that for large t all such solutions can be described in first approximation by the one-parameter family of ZKB solutions (from Zeldovich, Kompanyeets and Barenblatt; often the last name is used in referring to them). They are explicitly given by formulas

$$\mathcal{U}(x,t;C) = t^{-\alpha}\left(C - k\,|x|^2 t^{-2\beta}\right)_+^{\frac{1}{m-1}}, \tag{16}$$

where $(s)_+ = \max\{s,0\}$,

$$\alpha = \frac{n}{n(m-1)+2}, \quad \beta = \frac{\alpha}{n}, \quad k = \frac{\beta(m-1)}{2m}. \tag{17}$$

The constant $C > 0$ is free and can be used to adjust the mass of the solution: $\int_{\mathbb{R}^n} \mathcal{U}(x,t;C)\,dx = M > 0$. Let us write $\mathcal{U}_M$ for the solution with mass M and F_M for its profile. Figure 1 compares the profiles.

This is the precise statement of the asymptotic convergence result.

Theorem 4.1. *Let $u(x,t)$ be the unique weak solution of the Cauchy problem for the PME posed in $Q = \mathbb{R}^n \times (0,\infty)$ with initial data $u_0 \in L^1(\mathbb{R}^N)$, and $\int_{\mathbb{R}^n} u_0\,dx = M > 0$. Let $\mathcal{U}_M$ be the ZKB solution with the same mass as u_0. As $t \to \infty$ the solutions $u(t)$ and $\mathcal{U}_M$ are increasingly close and we have*

$$\lim_{t\to\infty} \|u(t) - \mathcal{U}_M(t)\|_1 = 0. \tag{18}$$

Convergence holds also in L^∞-norm in the proper scale:

$$\lim_{t\to\infty} t^{\alpha}\|u(t) - \mathcal{U}_M(t)\|_\infty = 0 \tag{19}$$

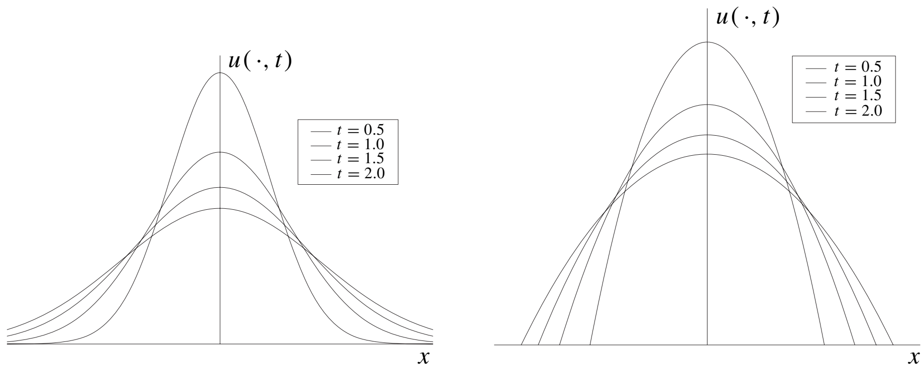

Figure 1. Comparison of the Gaussian profiles with the Barenblatt profiles.

with $\alpha = n/[n(m-1)+2]$. *Moreover, for every* $p \in (1, \infty)$ *we have*

$$\lim_{t\to\infty} t^{\alpha(p-1)/p} \|u(t) - \mathcal{U}_M(t)\|_{L^p(\mathbb{R}^n)} = 0. \tag{20}$$

Let now $\int_{\mathbb{R}^n} u_0\,dx = -M \le 0$. *The same result is true with* $\mathcal{U}_M$ *replaced by* $-\mathcal{U}_M(x.t)$ *when* $M < 0$, *and by* $\mathcal{U}_0(x,t) = 0$ *if* $M = 0$.

This result is one of the highlights of the theory of the PME. When applied to nonnegative solutions with mass 1 *it is the precise statement of the nonlinear central limit theorem for the evolution generated by the PME*. Proof of most of the result in several dimensions appeared in a celebrated paper by A. Friedman and S. Kamin [25]; however, uniform convergence was established only for nonnegative and compactly supported data. The complete proof with uniform convergence for nonnegative data in $L^1(\mathbb{R}^n)$ was done by Vázquez in [41]. In that paper seven different proofs are given. Signed solutions are admitted and even a right-hand side (forcing term) $f \in L^1(Q)$ is allowed and the limit (18) still holds.

Several remarks are in order: (1) According to this result, all the information the solution remembers from the initial configuration after a large time is reduced in first approximation only to the mass M, since the pattern and the rate are supplied by the equation. This is a very apparent manifestation of the equalizing effect of the diffusion.

(2) The PME replaces the Gaussian profile by the ZKB profile as the asymptotic pattern. This pattern has sharp fronts at a finite distance and no space tails, exponential or otherwise. This makes the PME a more realistic physical model than the HE since it avoids the problem of instantaneous propagation of signals at infinite distances.

(3) The standard deviation of the ZKB solution, and more generally of solutions that converge to it as $t \to \infty$, is $O(t^\beta)$ and not $O(t^{1/2})$ as in the heat equation (in fact, the anomalous diffusion exponent β is less than $1/2$ for $m > 1$ and goes to zero as $m \to \infty$ or as $n \to \infty$). This makes the PME qualify as a type of *anomalous diffusion.*

(4) The convergence also takes place for the rescaled version of the equation in the Wasserstein distance d_2 [37]. But this is a different story in some sense.

(5) For a given solution u, there is only one correct choice of the constant $C = C(u_0)$ in these asymptotic estimates, since trying two ZKB solutions with different constants in the above formulas produces non-zero limits. In that sense, the estimates are sharp.

(6) The result is optimal in the sense that we cannot get better convergence rates in the general class of solutions $u_0 \in L^1(\mathbb{R}^n)$, even if we assume $u_0 \ge 0$. Explicit counterexamples settle this question.

(7) It can be further proved that whenever $m > 1$ and we take signed initial data with compact support and the total mass is positive, $M > 0$, then, the solution becomes nonnegative after a finite time. This result is not true if the restriction of compact support is eliminated.

Proof by scaling. The idea is as follows: given a solution $u = u(x,t) \ge 0$ of the PME in the class of strong solutions with finite mass, we obtain a family of solutions

$$\widetilde{u}_\lambda(x,t) = (\mathcal{T}_\lambda u)(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t) \tag{21}$$

with initial data $\widetilde{u}_{0,\lambda}(x) = (\mathcal{T}_\lambda u_0)(x) = \lambda^\alpha u_0(\lambda^\beta x)$. The exponents α and β are related by

$$\alpha(m-1) + 2\beta = 1, \tag{22}$$

so that all the $\widetilde{u}_\lambda$ are again solutions of the PME. This is the scaling formula that we call the λ-scaling or *fixed scaling*. It is in fact a family of scalings with free parameter $\lambda > 0$, that performs a kind of *zoom* on the solution. In the present application we have another constraint that allows to fix both α and β to the desired values (the Barenblatt values). It is the condition of mass conservation. We only need to impose it at $t = 0$,

$$\int_{\mathbb{R}^n} (\mathcal{T}_\lambda u_0)(x)\,dx = \int_{\mathbb{R}^n} u_0(x)\,dx, \tag{23}$$

and we get $\alpha = n\,\beta$. Together with (22), this implies that α and β have the values (17) of the theorem. Note that the source-type solutions are invariant under this mass conserving λ-rescaling, i.e.

$$\mathcal{U}_M(t) = \mathcal{T}_\lambda(\mathcal{U}_M(t)).$$

The proof continues by showing that the family $\{\widetilde{u}_\lambda(t), \lambda > 0\}$, is uniformly bounded and even relatively compact in suitable functional spaces. We then pass to the limit dynamics when $\lambda \to \infty$ and identify the obtained object as the ZKB solution. We refer to [44]; Chapter 18, for complete details on the issue.

4.1. Continuous scaling: Fokker–Planck equation.

A different way of implementing the scaling of the orbits of the Cauchy problem and proving the previous facts consists of using the *continuous rescaling*, which is written in the form

$$\theta(\eta,\tau) = t^\alpha u(x,t), \quad \eta = x\,t^{-\beta}, \quad \tau = \ln t, \tag{24}$$

with α and β the standard similarity exponents given by (17). Then t^α and t^β are called the *scaling factors* (or *zoom factors*), while τ is the *new time*. This version of the scaling technique has a very appealing dynamical flavor. The reader should note that *every asymptotic problem has its corresponding zoom factors that have to be determined as a part of the analysis.* In our case, the rescaled orbit $\theta(\tau)$ satisfies the equation

$$\theta_\tau = \Delta(\theta^m) + \beta\eta \cdot \nabla\theta + \alpha\theta. \tag{25}$$

This is the continuously rescaled dynamics. Since we also have the relation $\alpha = n\,\beta$, we can write the equation in divergence form as

$$\theta_\tau = \Delta(\theta^m) + \beta\,\nabla \cdot (\eta\,\theta). \tag{26}$$

This is a particular case of the so-called *Fokker–Planck equations* which have the general form

$$\partial_t u = \Delta(|u|^{m-1}u) + \nabla \cdot (\mathbf{a}(x)u), \quad \mathbf{a}(x) = \nabla V(x). \tag{27}$$

The last term is interpreted as a confining effect due to a potential V. In our case $V(x) = \beta\,|x|^2/2$, the quadratic potential. The study of Fokker–Planck equations is interesting in itself and not only in connection with the HE and the PME.

This *dynamical systems approach* allows us to see the contents of the asymptotic theorem in a better way: the orbit $\theta(\tau)$ is bounded uniformly in $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$; the source-type (ZKB) solutions transform into the stationary profiles F_M in this transformation, i.e., $F(\eta)$ solves the nonlinear elliptic problem

$$\Delta f^m + \beta\,\nabla \cdot (\eta\,f) = 0; \tag{28}$$

moreover, boundedness of the orbit is established; this and convenient compactness arguments allow to pass to the limit and form the ω-limit, which is the set

$$\omega(\theta) = \{f \in L^1(\Omega) : \text{there exists } \{\tau_j\} \to \infty \text{ such that } \theta(\tau_j) \to f\}. \tag{29}$$

The convergence takes place in the topology of the functional space in question, here any $L^p(\Omega)$, $1 \le p \le \infty$. The end of the proof consists in showing that the ω-limit is just the Barenblatt profile F_M. The argument can be translated in the following way. Corresponding to the sequence of scaling factors λ_n of the previous scaling, we take a sequence of delays $\{s_n\}$ and define $\theta_n(\eta,\tau) = \theta(\eta, \tau + s_n)$. The family $\{\theta_n\}$ is precompact in $L^\infty_{\text{loc}}(\mathbb{R}_+ : L^1(\mathbb{R}^n))$ hence, passing to a subsequence if necessary, we have

$$\theta_n(\eta,\tau) \to \widetilde{\theta}(\eta,\tau). \tag{30}$$

Again, it is easy to see that $\widetilde{\theta}$ is a weak solution of (26) satisfying the same estimates. The end of the proof is identifying it as a stationary solution, $\widetilde{\theta}(\eta,\tau) = F_M(\eta)$, the Barenblatt profile of the same mass.

4.2. The entropy approach: convergence rates. A number of interesting results complement this basic convergence result in the recent literature. Thus, the question of obtaining an estimate of the rate at which rescaled trajectories of the PME stabilize towards the Barenblatt profile has attracted much attention. The main tool of investigation has been the consideration of the so-called *entropy functional*, defined as follows

$$H_\theta(\tau) = \int_{\mathbb{R}^n} \left\{ \frac{1}{m-1}\theta^m + \frac{\beta}{2}\eta^2\,\theta \right\} d\eta, \tag{31}$$

where $\theta = \theta(\eta, \tau)$ is the rescaled solution just defined and $\beta = 1/(n(m-1)+2)$ is the similarity exponent. H_θ represents a measure of the entropy of the mass distribution $\theta(\tau)$ at any time $\tau \geq 0$ which is well adapted to the renormalized PME evolution. Note that the entropy need not be finite for all solutions, a sufficient condition is

$$u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), \qquad \int x^2 u_0(x)\,dx < \infty. \tag{32}$$

These properties will then hold for all times. Note that the restriction of boundedness is automatically satisfied for positive times.

Actually, we can calculate the variation of the entropy in time along an orbit of the Fokker–Planck equation, and under the above conditions on u_0, we find after an easy computation that

$$\frac{dH_\theta}{d\tau} = -I_\theta, \quad \text{where } I_\theta(\tau) = \int \theta \left| \nabla \left(\frac{m}{m-1}\theta^{m-1} + \frac{\beta}{2}\eta^2 \right) \right|^2 d\eta \geq 0. \tag{33}$$

We now pass to the limit along sequences $\theta_n(\tau) = \theta(\tau + s_n)$ to obtain limit orbits $\widetilde{\theta}(\tau)$, on which the Lyapunov function is constant, hence $dH_{\widetilde{\theta}}/d\tau = 0$. The proof of asymptotic convergence concludes in the present instance in a new way, by analyzing when $dH_\theta/d\tau$ is zero. Here is the crucial observation that ends the proof: the second member of (33) vanishes if and only if θ is a Barenblatt profile.

Let us now introduce some notations: the difference $H(\theta|\theta_\infty) = H(\theta) - H(\theta_\infty)$ is called the *relative entropy*. Function I_θ is called the *entropy production*. We can use the entropy functional to improve the convergence result by obtaining rates of convergence. This is done by computing $d^2H_\theta/d\tau^2$, the so-called Bakry–Emery analysis in the heat equation case which has been adapted to the PME by J. A. Carrillo and G. Toscani [17]. The final result of the second derivative computation is

$$\frac{dI}{d\tau} = -2\beta\, I(\tau) - R(\tau), \tag{34}$$

for a certain term $R \geq 0$. Since we know by the previous analysis that $H(\theta|\theta_\infty)$ and I_θ go to zero as $\tau \to \infty$, we conclude the following result.

Theorem 4.2. *Under the assumption that the initial entropy is finite and some regularity assumptions, we have*

$$I(\tau) \leq A\, e^{-2\beta\tau}, \quad 0 \leq H(\theta|\theta_\infty)(\tau) \leq \frac{1}{2\beta} I(\tau), \quad H(\theta|\theta_\infty) \leq B\, e^{-2\beta\tau}. \tag{35}$$

As a consequence, the convergence towards the ZKB stated in Theorem 4.1 *happens with an extra factor* t^γ *in formulas* (18)–(20), *where*

$$\gamma = \beta \quad \text{if } 1 < m \le 2, \quad \gamma = \frac{2\beta}{m} \quad \text{if } m \ge 2, \tag{36}$$

and $\beta = 1/(n(m-1)+2)$.

In dimension $n = 1$, a convergence theorem with sharp rates can be proved by adjusting not only the mass but also the center of mass, cf. [40]. It is an open problem for $n > 1$.

4.3. Eventual concavity. There are many other directions in which the asymptotic behaviour of the PME is made more precise. This is the geometrical result obtained by Lee–Vázquez [34] (Figure 2 shows a practical instance).

Theorem 4.3. *Let u be a solution of the PME in $n \ge 1$ space dimensions with compactly supported initial data* (*and satisfying a certain non-degeneracy condition*). *Then there is $t_c > 0$ such that the pressure $v(x,t)$ is a concave function in $\mathcal{P}(t) = \{x : v(x,t) > 0\}$ for $t \ge t_c$. More precisely, for any coordinate directions x_i, x_j,*

$$\lim_{t\to\infty} t\,\frac{\partial^2 v}{\partial x_i^2} = -\beta, \quad \lim_{t\to\infty} t\,\frac{\partial^2 v}{\partial x_i \partial x_j} = 0, \tag{37}$$

uniformly in $x \in \operatorname{supp}(v)$, $i, j = 1, \ldots, n$, $i \ne j$. Here, $\beta = 1/(n(m-1)+2)$.

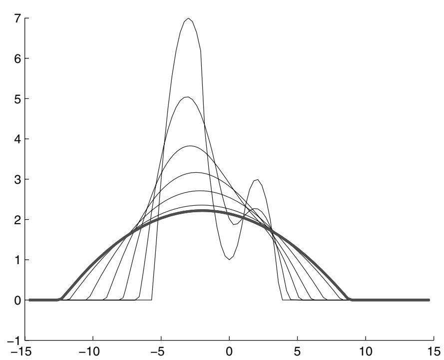

Figure 2. Evolution towards selfsimilarity and eventual concavity in the PME.

4.4. Extensions. In the case of the Cauchy–Dirichlet problem posed in a bounded domain $\Omega \subset \mathbb{R}^N$, it is well known that the asymptotic shape of any solution of the heat equation with nonnegative initial data in $L^2(\Omega)$ approaches one of the special *separated variables solutions*

$$u_1(x,t) = c\,T_1(t)\,F_1(x). \tag{38}$$

Here $T_1(t) = e^{-\lambda_1 t}$, where $\lambda_1 = \lambda_1(\Omega) > 0$ is the first eigenvalue of the Laplace operator in Ω with zero Dirichlet data on $\partial\Omega$, and the space pattern $F_1(x)$ is the corresponding positive and normalized eigenfunction. The constant $c > 0$ is determined as the coefficient of the $L^2(\Omega)$-projection of u_0 on F_1. For the PME a similar result is true with the following differences: $F_1(x)$ is the ground state of a certain nonlinear eigenvalue problem, while $T_1(t) = t^{-1/(m-1)}$, which means power decay in time. For signed solutions there exists a whole sequence of compactly supported profiles $F_k(x)$ as candidates for the separation-of-variables formula (38), but the time factor is always the same $T_1(t) = t^{-1/(m-1)}$; this is a striking difference with the linear case in which the exponentials have an increasing sequence of coefficients.

The analysis applies to the homogeneous Neumann problem and leads to a marked difference between data with positive or negative mass, where the solutions stabilize to the constant state with exponential speed (just as linear flows) and the case of zero mass, where the $t^{-1/(m-1)}$ time factor appears.

Another extension consists of considering the whole space with a number of holes; Dirichlet or Neumann boundary conditions are imposed on the boundary Γ of the holes. If for instance conditions of the form $u = 0$ are chosen on Γ, then the Barenblatt profiles still appear as the asymptotic patterns in dimension $n \geq 3$, but not in dimension $n = 1$ while dimension $n = 2$ is a borderline case [30], [9]. Actually, the influence of the holes is felt in terms of their capacity, which has an interesting physico-geometrical interpretation. Dependence on dimension is very typical of nonlinear diffusion problems, it will appear again in fast diffusion flows and is a very prominent feature in blow-up problems.

5. Some lines of research

The theory of the PME is by now well understood in the case of dimension one, but it still has gaps for $n > 1$ because n dimensional PME flows turned out to be much more complicated. Thus, the regularity of nonnegative solutions is effectively solved for $n = 1$ by stating the best Hölder regularity (the pressure is Lipschitz continuous but not C^1) and showing the worst case situation, that corresponds to the presence of moving free boundaries that always follow the same pattern, the solution locally behaves like a travelling wave. The same question in dimension $n > 1$ is still only partially understood even after intensive work of D. Aronson and coworkers [2], [3] on the so-called focusing problem. They discovered first the radial solutions with limited Hölder regularity and then the nonradial focusing modes with elongated shapes bifurcating from the radial branch as m varies (a typical break of stability in a bifurcation branch, somewhat similar to the Taylor–Couette hydrodynamical instabilities).

After the work of L. Caffarelli et al. [12], [15], the C^∞ regularity of the solutions up to the free boundary and also of the free boundary as a hypersurface was proved for large times by H. Koch [32] under convenient conditions on the data, but further progress is expected.

Even in the theory of nonnegative solutions of the Cauchy problem for the PME, there are still some basic analytical problems, like the following: the theory is very much simplified by the existence of the Aronson–Bénilan pointwise estimate $\Delta u^{m-1} \geq -C/t$. A local version of this estimate is known only in dimension $n = 1$.

On the other hand, the question of obtaining solutions for the widest possible class of initial data has been successfully solved for nonnegative solutions, but only partially understood for signed solutions. See [44], Chapters 12 and 13.

If we replace the exponent 2 by $p \in [1, \infty]$ in the Wasserstein formula (13), we get the Wasserstein non-quadratic distances. The PME semigroup is contractive in all of them in dimension $n = 1$. In particular, contraction in the d_∞ norm allows for fine estimates of the support propagation. Unfortunately, the PME semigroup is not contractive in the distance d_p for p large if $n > 1$, [42]. Actually, a similar negative result happens for the usual L^p norms in all space dimensions, see [44]. Determining the precise ranges of p for these contractivity issues is still an open problem.

6. Fast diffusion equations: physics and geometry

The fast diffusion equation is formally the same as the PME but the exponent is now $m < 1$. It appears in several areas of mathematics and science when the assumptions of linear diffusion are violated in a direction contrary to the PME. Here are some examples.

• Plasma diffusion with the Okuda–Dawson scaling implies a diffusion coefficient $D(u) \sim u^{-1/2}$ in the basic equation $u_t = \nabla \cdot (D(u)\nabla u)$, where u is the particle density. This leads to the FDE with $m = 1/2$. Other models imply exponents $m = 0$, $D(u) \sim 1/u$, or even $m = -1$, $D(u) \sim 1/u^2$.

• A very popular fast diffusion model was proposed by Carleman to study the diffusive limit of kinetic equations. He considered just two types of particles in a one dimensional setting moving with speeds c and $-c$. If the densities are u and v respectively you can write their simple dynamics as

$$\left.\begin{aligned} \partial_t u + c\,\partial_x u &= k(u,v)(v-u) \\ \partial_t v - c\,\partial_x v &= k(u,v)(u-v) \end{aligned}\right\} \tag{39}$$

for some interaction kernel $k(u, v) \geq 0$. Put in a typical case $k = (u + v)^\alpha c^2$. Write now the equations for $\rho = u+v$ and $j = c(u-v)$ and pass to the limit $c = 1/\varepsilon \to \infty$ and you will obtain to first order in powers or $\varepsilon = 1/c$:

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{\rho^\alpha}\frac{\partial \rho}{\partial x}\right), \tag{40}$$

which is the FDE with $m = 1 - \alpha$, cf. [35]. The typical value $\alpha = 1$ gives $m = 0$, a surprising equation that we will find below! The rigorous investigation of the diffusion limit of more complicated particle/kinetic models is an active area of investigation.

• The fast diffusion makes a striking appearance in differential geometry, in the evolution version of the Yamabe problem. The standard Yamabe problem starts with a Riemannian manifold (M, g_0) in space dimension $n \geq 3$ and deals with the question of finding another metric g in the conformal class of g_0 having constant scalar curvature. The formulation of the problem proceeds as follows in dimension $n \geq 3$. We can write the conformal relation as

$$g = u^{4/(n-2)} g_0$$

locally on M for some positive smooth function u. The conformal factor is $u^{4/(n-2)}$. Next, we denote by $R = R_g$ and R_0 the scalar curvatures of the metrics g, g_0 resp. If we write Δ_0 for the Laplace–Beltrami operator of g_0, we have the formula $R = -u^{-N} Lu$ on M, with $N = (n+2)/(n-2)$ and

$$Lu := \kappa \Delta_0 u - R_0 u, \quad \kappa = \frac{4(n-1)}{n-2}.$$

The Yamabe problem becomes then

$$\Delta_0 u - \left(\frac{n-2}{4(n-1)}\right) R_0 u + \left(\frac{n-2}{4(n-1)}\right) R_g u^{(n+2)/(n-2)} = 0. \tag{41}$$

The equation should determine u (hence, g) when g_0, R_0 and R_g are known. In the standard case we take $M = \mathbb{R}^n$ and g_0 the standard metric, so that Δ_0 is the standard Laplacian, $R_0 = 0$, we take $R_g = 1$ and then we get the well-known semilinear elliptic equation with critical exponent.

In the evolution version, the Yamabe flow is defined as an evolution equation for a family of metrics that is used as a tool to construct metrics of constant scalar curvature within a given conformal class. More precisely, we look for a one-parameter family $g_t(x) = g(x, t)$ of metrics solution of the evolution problem

$$\partial_t g = -R\, g, \quad g(0) = g_0 \quad \text{on } M. \tag{42}$$

It is easy to show that this is equivalent to the equation

$$\partial_t (u^N) = Lu, \quad u(0) = 1 \quad \text{on } M.$$

after rescaling the time variable. Let now (M, g_0) be $\mathbb{R}^n$ with the standard flat metric, so that $R_0 = 0$. Put $u^N = v$, $m = 1/N = (n-2)/(n+2) \in (0, 1)$. Then

$$\partial_t v = L v^m, \tag{43}$$

which is a fast diffusion equation with exponent $m_y \in (0, 1)$ given by

$$m_y = \frac{n-2}{n+2}, \quad 1 - m_y = \frac{4}{n+2}.$$

If we now try separate variables solutions of the form $v(x,t) = (T-t)^{\alpha} f(x)$, then necessarily $\alpha = 1/(1-m_y) = (n+2)/4$, and $F = f^m$ satisfies the semilinear elliptic equation with critical exponent that models the stationary version:

$$\Delta F + \frac{n+2}{4} F^{\frac{n+2}{n-2}} = 0. \tag{44}$$

We will stop at this point the motivation of nonlinear diffusion coming from geometry, to return later with the logarithmic diffusion. The influence of PDE techniques is strongly felt in present-day differential geometry, cf. A. Chang's exposition at the 2002 ICM [18].

6.1. Behaviour of fast diffusion flows: the good range. For values $m \sim 1$ the difference with the PME is noticed in the infinite speed of propagation: solutions with nonnegative data become immediately positive everywhere in the space domain (be it finite or infinite). This looks like the heat equation and in some sense it is. When we look at the source solutions we find explicit Barenblatt functions

$$U_m(x,t;M)^{1-m} = \frac{t}{C\,t^{2\alpha/n} + k_1\,x^2}, \tag{45}$$

with $C = a(m,n)M^{2(m-1)\alpha/n}$ and k_1 is an explicit function of m and n. Figure 3 shows the form.

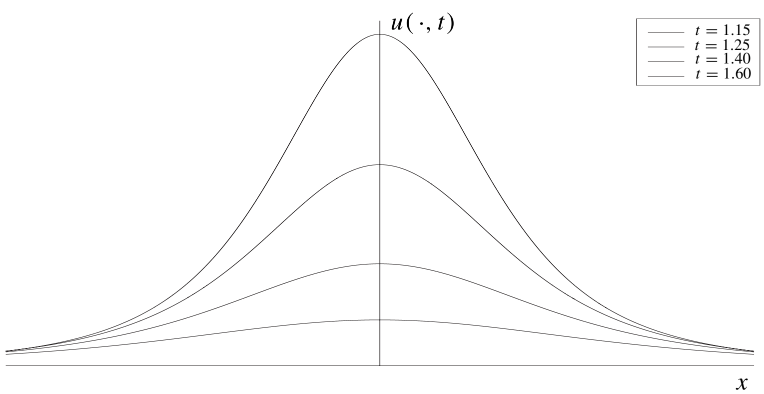

Figure 3. The fast diffusion solution with fat tails.

The main difference in the ZKB profiles is the power-like tail at infinity, which very much differs from the Gaussian (exponential) behaviour. These tails, called fat tails, are currently the object of study in several areas of economy, finance and statistical physics and their properties and role are still not well understood.

The new Barenblatt solutions play the same role they played for $m > 1$ even if they have a different shape. Indeed, the asymptotic behaviour of general solution is still described by Theorem 4.1 if $m_c < m < 1$, where $m_c = (n-2)_+/n$ is called the first critical exponent. Actually, the convergence is better: we have proved in [41] that the relative error $e(x,t) = u(x,t)/U_m(x,t) - 1$ converges to zero uniformly in x as $t \to \infty$, a property that is obviously false for $m > 1$ because of the (slightly) different supports.

6.2. Behaviour of fast diffusion flows: the subcritical range. Significant novelties in the mathematical theory of the FDE appear once we cross the line $m = m_c$ downwards. Then, the picture changes and we enter a realm of strange diffusions. Note that m_c is larger than the Yamabe exponent m_y. Thus, concerning the basic problem of optimal space for existence, H. Brezis and A. Friedman proved in [10] that there can be no solution of the equation if $m \le m_c$ when the initial data is a Dirac mass, so that we lose the source solution, our main example of the range $m > m_c$. M. Pierre [39] extended the non-existence result to measures supported in sets of small capacity if $m < m_c$. But at least solutions exist for all initial data $u_0 \in L^1_{\text{loc}}(\mathbb{R}^n)$ when $0 < m < m_c$, and moreover they are global in time, $u \in C([0,\infty) : L^1_{\text{loc}}(\mathbb{R}^n))$. Such an existence result is not guaranteed when we go further down in exponent, $m \le 0$; then, solutions with initial data in $L^1(\mathbb{R}^n)$ just do not exist.

However, and contrary to the 'upper' fast diffusion range $m_c < m < 1$, it is not true that locally integrable data produce locally bounded solutions at positive times, as in the various examples constructed in [43]. Attention must be paid therefore to the existence and properties of large classes of weak solutions that are not smooth, not even locally bounded. This leads to a main fact to be pointed out about smoothing estimates: there can be no L^1-L^∞ effects.

Extinction and Marcinkiewicz spaces. For $m < m_c$ (even for $m \le 0$) there is an explicit example of solution which completely vanishes in finite time, and has the form

$$U(x,t;T) = c\left(\frac{T-t}{|x|^2}\right)^{1/(1-m)} \tag{46}$$

with $T > 0$ arbitrary, and $c = c(m,n) > 0$ a given constant. Formula (46) produces a weak solution of the FDE that is positive and smooth for $0 < t < T$ and $x \ne 0$; note further that it belongs to the Marcinkiewicz space $M^{p_*}(\mathbb{R}^n)$ with $p_* = n(1-m)/2$, a quantity that is larger than 1 if $m < m_c$.

The positive result that we can derive from comparison with this solution is the following.

Theorem 6.1. *Let $n \ge 3$ and $m < m_c$. For every $u_0 \in M^{p_*}(\mathbb{R}^n)$, there exists a time $T > 0$ such that $u(x,t)$ vanishes for $t \ge T$. It can be estimated as*

$$0 < T \le d\,\|u_0\|^{1-m}_{M^{p_*}} := T_1(u_0) \tag{47}$$

with $d = d(m,n) > 0$. Moreover, for all $0 < t < T$ we have $u(t) \in M^{p_*}(\mathbb{R}^n)$ and $u^*(t) \prec U(t;T)$ where T is chosen so that $U(x,0;T)$ has the same M^{p_*}-norm as u_0, i.e., $M = c_m T^{1-m}$.

The relevant functional spaces $M^p(\mathbb{R}^n)$ are the so-called Marcinkiewicz spaces or weak L^p spaces, defined for $1 < p < \infty$ as the set of locally integrable functions such that

$$\int_K |f(x)|dx \le C|K|^{(p-1)/p},$$

for all subsets K of finite measure. It is further proved that solutions with data in the space $M^p(\mathbb{R}^n)$ with $p > p_*$ become bounded for all positive times and do not necessarily extinguish in finite time, while solutions with data in the space $M^p(\mathbb{R}^n)$ with $1 < p < p_*$ become L^1 functions for all positive times and do not necessarily extinguish in finite time, see [43], Chapter 5. While becoming bounded is a typical feature of the parabolic theory (called the L^∞ smoothing effect), becoming L^1 is a kind of extravaganza that happens for these fast diffusion flows (it was introduced and called *backward effect* in [43]).

Delayed regularity. A further curious effect is proved in Chapter 6 of the same reference: functions in the larger space $M^{p_*}(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ typically preserve their unboundedness for a time $T > 0$ and then become bounded. This is also called *blow-down* in finite time.

Large-time behaviours: selfsimilarity with anomalous exponents. The actual asymptotic behaviour of the solutions of the FDE in the exponent range $0 < m < m_c$ depends on the class of initial data. We are interested in "small solutions" that extinguish in finite time, We will concentrate on solutions that start with initial data in $L^1(\mathbb{R}^n)$, or solutions that fall into this class for positive times previous to extinction by the backward effect. In the range $m > m_c$ the ZKB solutions provide the clue to the asymptotics for all nonnegative solutions with L^1-data. We know that these solutions do not exist in the range $m < m_c$. So we need to look for an alternative. This alternative has to take into account the fact that solutions with M^{p_*}-data disappear (vanish identically) in finite time.

The class of self-similarity solutions of the form

$$U(x,t) = (T-t)^\alpha f(x\,(T-t)^\beta), \tag{48}$$

called selfsimilar of Type II, will provide the clue to the asymptotic behaviour near extinction. We have to find the precise exponents and profile that correspond to a solution of the form (48) and represent the large time behaviour of many other solutions. We know from the start that $\alpha(1-m) = 2\beta + 1$. But there is no conservation law that dictates to us the remaining relation needed to uniquely identify the exponents. This may seem hopeless but the representative solutions exist and have precise exponents and profile. This is a situation that appears with a certain frequency in mechanics and was called by Y. Zel'dovich *self-similarity of the second kind*, the exponents being known as *anomalous exponents.* Anomalous means here that the exponents cannot be obtained from dimensional considerations or conservation laws, as is done in the ZKB case by means of the scaling group. Such kind of solutions is of great interest because of their analytical difficulty.

The selection rule for the anomalous exponents is usually topological, tied to the existence and behaviour of the solutions of a certain nonlinear eigenvalue problem. In the present case, the selection is done through the existence of a special class of self-similar solutions with fast decay at infinity. The construction was performed by J. King, M. Peletier and H. Zhang [31], [38] and is extended and described in detail in [43], Chapter 7. We call them KPZ solutions.

We can then pass to the question of convergence of general classes of solutions towards the KPZ solutions. This is in analogy to the results for the ZKB profiles for $m > m_c$. The results are more complete when $m = m_y$, where the problem has a nice geometrical interpretation. The proof of convergence results towards the KPZ solution is only known in the case of radial solutions when $m \neq m_y$, cf. [26].

FDE with exponent $m = (n-2)/(n+2)$: Yamabe problem. The exponent is special in the sense that when we separate variables and pass to the Emden–Fowler equation for $G = F^m$

$$-\Delta G = G^p, \tag{49}$$

then $p = 1/m$ equals the Sobolev exponent $p_s = (n+2)/(n-2)$, that is known to have a great importance in the theory of semilinear elliptic equations. We may thus call m_y the Sobolev exponent of the FDE. We recall that the FDE is related for this precise exponent to the famous Yamabe flow of Riemannian geometry. As we have said, this flow is used by geometers as a tool to deform Riemannian metrics into metrics of constant scalar curvature within a given conformal class. Returning to the consideration of the family of KPZ solutions with anomalous exponents, this is the "easy case" where $\beta = 0$ (that corresponds to separate variables) so that $\alpha = (n+2)/4$. The corresponding family of self-similar solutions is explicitly known and given by the Loewner–Nirenberg formula

$$F(x,\lambda) = k_n \left(\frac{\lambda}{\lambda^2 + |x|^2}\right)^{(n+2)/2} \tag{50}$$

with $k_n = (4n)^{(n+2)/4}$ and $\lambda > 0$ arbitrary. These patterns represent conformal metrics with constant curvature in the geometrical interpretation. Here is the result of [21] as improved in [43], Chapter 7.

Theorem 6.2. *Let $n \geq 3$ and $m = (n-2)/(n+2)$ and let $u(x,t) \geq 0$ be a solution of the FDE existing for a time $0 < t < T$. Under the assumption that $u_0 \in L^{2n/(n+2)}(\mathbb{R}^n)$ the solution of the FDE is bounded for all $t > 0$. Moreover, there exist λ and $x_0 \in \mathbb{R}^n$ such that*

$$(T-t)^{-(n+2)/4} u(x,t) = F(x - x_0, \lambda) + \theta(x,t) \tag{51}$$

and $\|\theta(t)\|_{L^{p_*}} \to 0$ as $t \to T$. The exponent in the space assumption is optimal.

We just recall that $p_* = 2n/(n+2)$ in the present case.

Exponential decay at the critical end-point. Inspection of the ZKB solutions as m goes down to m_c shows that asymptotic decay happens as $t \to \infty$ with increasing powers of time, i.e., $u \sim t^{-\alpha}$ and $\alpha \to \infty$. Exponential decay holds for the FDE with critical exponent m_c for a large number of initial, but not for all L^1 data. We have

Theorem 6.3. *Let $m = m_c$ and $n \geq 3$. Solutions in $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ decay in time according to the rate*

$$\log(1/\|u(t)\|_\infty) \sim C(n) M^{-2/(n-2)} t^{n/(n-2)} \tag{52}$$

as $t \to \infty$, $M = \|u_0\|_1$. This rate is sharp.

This behaviour is not selfsimilar and has been calculated by V. Galaktionov, L. Peletier and J. L. Vázquez in [27], based on previous formal analysis of J. King [31]. The proof uses a delicate technique of matched asymptotics with an outer region which after a nonlinear transformation undergoes convergence to a selfsimilar profile of convective type. This is said here to show that unexpected patterns can appear as the result of these manipulations. Slower pace of decay happens for non-integrable solutions, as the following selfsimilar example shows

$$u(x,t) = \frac{1}{(bx^2 + ke^{2nbt})^{n/2}}, \quad b, k > 0. \tag{53}$$

7. Logarithmic diffusion

The fast diffusion equation in the limit case $m = 0$ in two space dimensions is a favorite case in the recent literature on fast diffusion equations. The equation can be written as

$$\partial_t u = \text{div}\,(u^{-1}\nabla u) = \Delta \log(u), \tag{54}$$

hence the popular name of *logarithmic diffusion*. The problem has a particular appeal because of its application to differential geometry, since it describes the evolution of surfaces by Ricci flow. More precisely, it represents the evolution of a conformally flat metric g by its Ricci curvature,

$$\frac{\partial}{\partial t} g_{ij} = -2\ \text{Ric}_{ij} = -R\, g_{ij}, \tag{55}$$

where Ric is the Ricci tensor and R the scalar curvature. If g is given by the length expression $ds^2 = u\,(dx^2 + dy^2)$, we arrive at equation (54). This flow, proposed by R. Hamilton in [29], is the equivalent of the Yamabe flow in two dimensions. We remark that what we usually call the mass of the solution (thinking in diffusion terms) becomes here the *total area* of the surface, $A = \iint u\, dx_1 dx_2$.

Several peculiar aspects of the theory are of interest: non-uniqueness, loss of mass, regularity and asymptotics. Maybe the most striking is the discovery that conservation of mass is broken in a very precise way.

• Let us consider integrable data. This is the basic result of the theory.

Theorem 7.1. *For every* $u_0 \in L^1(\mathbb{R}^2)$, *with* $u_0 \geq 0$ *there exists a unique function* $u \in C([0,T) : L^1(\mathbb{R}^2))$, *which is a classical* (C^∞ *and positive*) *solution of* (54) *in* Q_T *and satisfies the area constraint*

$$\int u(x,t)dx = \int u_0(x)dx - 4\pi\, t. \tag{56}$$

Such solution is maximal among the solutions of the Cauchy problem for (54) *with these initial data, and exists for the time* $0 < t < T = \int_{\mathbb{R}^2} u_0(x)\,dx/4\pi$. *Moreover, the solution is obtained as the limit of positive solutions with initial data* $u_{0\varepsilon}(x) = u_0(x) + \varepsilon$ *as* $\varepsilon \to 0$.

The mysterious loss of 4π units of area (mass) has attracted the attention of researchers. In the geometrical application it is quite easy to see it as a form of the Gauss–Bonnet theorem. It is shown that the Gaussian curvature K is given by $K = \Delta u/2u$, and then

$$\int_{\mathbb{R}^2} u_t\, dx = -2\int_M K\, d\,\mathrm{Vol}_g = -4\pi.$$

There are also analytical proofs of this fact, that look a bit mysterious, see [43], Chapter 8.

• One of the peculiar properties of this equation is the existence of multiple solutions with finite area with the same initial data that are characterized by the behaviour at infinity. The situation is completely understood in the radial case and the following general result is proved.

Theorem 7.2. *For every nonnegative radial function* $u_0 \in L^1(\mathbb{R}^2)$ *and for every bounded function* $f(t) \geq 2$, *there exists a unique function* $u \in C([0,T) : L^1(\mathbb{R}^2))$, *which is a radially symmetric and classical solution of* (P_0) *in* Q_T *and satisfies the mass constraint*

$$\int u(x,t)dx = \int u_0(x)dx - 2\pi \int_0^t f(\tau)\,d\tau. \tag{57}$$

It exists as long as the integral in the LHS is positive. The case $f = 0$ *corresponds to the maximal solution of the Cauchy problem. In any case, the solution is bounded for all* $t > 0$.

Thus, not only there are infinitely many solutions for every fixed nontrivial initial function u_0, but also the extinction time can be controlled by means of the flux data f. Moreover, these solutions satisfy the asymptotic spatial behaviour

$$\lim_{r\to\infty} r\,\partial_r(\log u(r,t)) = -f(t) \tag{58}$$

for a.e. $t \in (0,T)$. Non-uniqueness extends to non-integrable solutions. Thus, the stationary profile

$$u(x_1,x_2,t) = Ae^{Bx_1}, \quad A, B > 0,$$

is not the unique solution with such data, it is not even the maximal solution.

• Actually, the geometrical interpretation in the case of data with finite area favors the flow with conditions at infinity $f = 4$, hence the mass loss per unit time equals 8π, which are interpreted as regular closed compact surfaces. Here is the most typical example

$$u(x,t) = \frac{8(T-t)}{(1+x^2)^2}, \tag{59}$$

which represents the evolution of a ball with total area $\int u(x,t)\,dx = 8\pi(T-t)$.

• Coming to the regularity question, solutions with data in $L^1(\mathbb{R}^2)$ are bounded. But the limit when the data tend to a measure, more precisely to a Dirac mass, is not included. Indeed, when we approximate a Dirac delta $M\delta(x)$ by smooth integrable functions $\varphi_n(x)$, solve the problem in the sense of maximal solutions $u_n(x,t)$ and pass to the limit in the approximation, the following result is obtained

$$\lim_{n\to\infty} u_n(x,t) = (M - 4\pi t)_+ \delta(x). \tag{60}$$

In physical terms, it means that logarithmic diffusion is unable to spread a Dirac mass, but somehow it is able to dissipate it in finite time. A delicate thin layer process occurs by which the mass located at $x = 0$ is transferred to $x = \infty$ without us noticing. We explain this phenomenon in the recent paper [45] where the following result is proved: we assume that the initial mass distribution can be written as

$$d\mu_0(x) = f(x)dx + \sum_{i=1}^{k} M_i\,\delta(x - x_i). \tag{61}$$

where $f \geq 0$ is an integrable function in $\mathbb{R}^2$, the x_i, $i = 1, \ldots, k$, are a finite collection of (different) points on the plane, and we are given masses $0 < M_k \leq \cdots \leq M_2 \leq M_1$. The total mass of this distribution is

$$M = M_0 + \sum_{i=1}^{k} M_i, \quad \text{with } M_0 = \int f\,dx. \tag{62}$$

We construct a solution for this problem as the limit of natural approximate problems with smooth data:

Theorem 7.3. *Under the stated conditions, there exists a limit solution of the log-diffusion Cauchy problem posed in the whole plane with initial data* μ_0*. It exists in the time interval* $0 < t < T$ *with* $T = M/2\pi$*. It satisfies the conditions of maximality at infinity.*

More precisely, the solution is continuous into the space of Radon measures, $u \in C([0,T] : \mathcal{M}(\mathbb{R}^2))$*, and it has two components, singular and regular. The* *singular part amounts to a collection of (shrinking in time) point masses concentrated on the points $x = x_i$ of the precise form*

$$u_{sing} = \sum_i (M_i - 4\pi t)_+ \delta(x - x_i). \tag{63}$$

The regular part can be described as follows:

(i) *When restricted to the perforated domain $Q_* = \left(\mathbb{R}^2 - \bigcup_i \{x_i\}\right) \times (0, T)$, u is a smooth solution of the equation, it takes the initial data $f(x)$ for a.e. $x \neq x_i$, and vanishes at $t = T$.*

(ii) *At every time $t \in (0, T)$ the total mass of the regular part is the result of adding to M_0 the inflow coming from the point masses and subtracting the outflow at infinity.*

(iii) *Before each point mass disappears, we get a singular behaviour near the mass location as in the radial case, while later on the solution is regular around that point.*

For complete details on this issue we refer to [45].

The theory of measure-valued solutions of diffusion equations is still in its beginning. A large number of open problems are posed for subcritical fast diffusion.

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